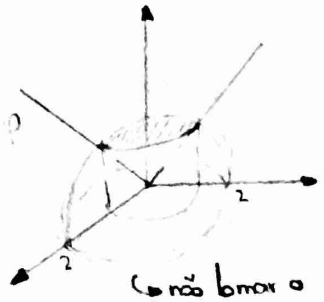
Lista 3

1) a) Coordenadas Esféricas:

$\phi: \begin{cases} x = \rho\cos\theta\,\mathrm{sen}\,\varphi \\ y = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi \\ z = \rho\cos\varphi \end{cases}$



(não tomar o volume só a superfície)

$\Delta = \{\rho\cos\varphi \geq \rho\,\mathrm{sen}\,\varphi,\ \rho = 2,\ 0 \leq \theta \leq 2\pi\} =$

$= \{\mathrm{tg}\,\varphi \leq 1,\ \rho = 2,\ 0 \leq \theta \leq 2\pi\} =$

$= \{0 \leq \varphi \leq \pi/4,\ \rho = 2,\ 0 \leq \theta \leq 2\pi\}$

Parâmetros: $u = \theta$ e $v = \varphi$

$\sigma(u,v) = (2\cos u\,\mathrm{sen}\,v,\ 2\,\mathrm{sen}\,u\,\mathrm{sen}\,v,\ 2\cos v)$

$\sigma_u = (-2\,\mathrm{sen}\,u\,\mathrm{sen}\,v,\ 2\cos u\,\mathrm{sen}\,v,\ 0)$

$\sigma_v = (2\cos u\cos v,\ 2\,\mathrm{sen}\,u\cos v,\ -2\,\mathrm{sen}\,v)$

$\vec{N} = \sigma_u \wedge \sigma_v = -4\,\mathrm{sen}\,v\,(\cos u\,\mathrm{sen}\,v,\ \mathrm{sen}\,u\,\mathrm{sen}\,v,\ \cos v)$

$\|\vec{N}\| = \sqrt{16\,\mathrm{sen}^2 v} = 4\,\mathrm{sen}\,v$

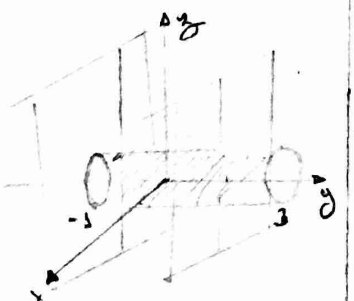
$A = \iint_{\Delta} 1\,d\sigma = \iint_{\Omega} 4\,\mathrm{sen}\,v\,du\,dv = \int_0^{2\pi}\left(\int_0^{\pi/4} 4\,\mathrm{sen}\,v\,dv\right)du =$

$= 4\int_0^{2\pi} -\cos v\Big|_0^{\pi/4} du = -4\int_0^{2\pi}\left(\frac{\sqrt{2}}{2} - 1\right)du = 4\int_0^{2\pi}\left(1 - \frac{\sqrt{2}}{2}\right)du =$

$= 4 - 2\sqrt{2}\Big|_0^{2\pi} = 2\pi(4 - 2\sqrt{2}) = 4\pi(2 - \sqrt{2})$

b) Coordenadas cilíndricas:

$\phi: \begin{cases} x = \rho\cos\theta \\ y = y \\ z = \rho\,\mathrm{sen}\,\theta \end{cases}$



$\Delta = \{-1 \leq y \leq 3,\ \rho = 1,\ 0 \leq \theta \leq 2\pi\}$

Parâmetros: $u = y$, $v = \theta$

$\Omega = \{-1 \leq u \leq 3,\ 0 \leq v \leq 2\pi\}$

$\sigma(u,v) = (\cos v,\ u,\ \mathrm{sen}\,v)$

$\sigma_u = (0,1,0)$; $\sigma_v = (-\mathrm{sen}\,v,\ 0,\ \cos v)$

$\vec{N} = \sigma_u \wedge \sigma_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\mathrm{sen}\,v & 0 & \cos v \end{vmatrix} = \cos v\,\vec{i} + \mathrm{sen}\,v\,\vec{k}$

$\|\vec{N}\| = \sqrt{\cos^2 v + \mathrm{sen}^2 v} = 1$

$A = \iint_{\Delta} 1\,d\sigma = \iint_{\Omega} du\,dv = \int_0^{2\pi}\left(\int_{-1}^{3} du\right)dv = \int_0^{2\pi} 4\,dv = 8\pi$

c) Parâmetros: $u = x$, $v = y$

$\sigma(u,v) = (u, v, 2u + 3v)$

$\sigma_u = (1,0,2)$; $\sigma_v = (0,1,3)$

$\vec{N} = (-2,-3,1) \therefore \|\vec{N}\| = \sqrt{4+9+1} = \sqrt{14}$

$\Omega = \{u^2 + v^2 \leq 16\}$ no xy

$A = \iint_{\Delta} d\sigma = \iint_{\Omega} \sqrt{14}\,du\,dv$

Passando para coordenadas polares:

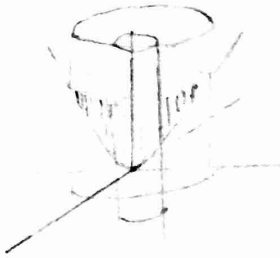
$\begin{cases} u = \rho\cos\theta \\ v = \rho\,\mathrm{sen}\,\theta \end{cases} \therefore J\phi = \rho$

$\Omega_{\rho\theta} = \{\rho^2 \leq 16,\ 0 \leq \theta \leq 2\pi\} = \{0 \leq \rho \leq 4,\ 0 \leq \theta \leq 2\pi\}$

$A = \iint_{\Omega_{\rho\theta}} \rho\sqrt{14}\,d\rho\,d\theta = \int_0^{2\pi}\left(\int_0^{4} \rho\sqrt{14}\,d\rho\right)d\theta =$

$= \int_0^{2\pi} \frac{\rho^2\sqrt{14}}{2}\Big|_0^4 d\theta = \int_0^{2\pi} 8\sqrt{14}\,d\theta = 16\pi\sqrt{14}$

d) Parâmetros: $u = x$, $v = y$



$\sigma(u,v) = (u, v, v^2 - u^2)$

$\sigma_u = (1,0,-2u)$

$\sigma_v = (0,1,2v)$

$\vec{N} = \sigma_u \wedge \sigma_v = (2u, -2v, 1) \therefore \|\vec{N}\| = \sqrt{4u^2 + 2v^2 + 1}$

$\Omega = \{1 \leq u^2 + v^2 \leq 4\}$

$A = \iint_{\Delta} d\sigma = \iint_{\Omega} \sqrt{4u^2 + 4v^2 + 1}\,du\,dv$

Passando para coordenadas polares:

$\begin{cases} u = \rho\cos\theta \\ v = \rho\,\mathrm{sen}\,\theta \end{cases}$; $J\phi = \rho$

$\Omega_{\rho\theta} = \{1 \leq \rho^2 \leq 4,\ 0 \leq \theta \leq 2\pi\} = \{1 \leq \rho \leq 2,\ 0 \leq \theta \leq 2\pi\}$

$A = \iint_{\Omega_{\rho\theta}} \rho\cdot\sqrt{4\rho^2 + 1}\,d\rho\,d\theta = \int_0^{2\pi}\left(\int_1^2 \rho\sqrt{4\rho^2+1}\,d\rho\right)d\theta =$

$= \int_0^{2\pi} \frac{\sqrt{(4\rho^2+1)^3}}{12}\Big|_1^2 d\theta = \int_0^{2\pi} \frac{17\sqrt{17}}{12} - \frac{5\sqrt{5}}{12} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$

h) Coordenadas Esféricas.

$$\begin{cases} x = \rho\cos\theta\,\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\,\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$\Omega = \left\{ \rho \le 2,\ \rho\cos\varphi \ge \sqrt{\frac{\rho^2\operatorname{sen}^2\varphi}{3}} \right\} =$$

$$= \left\{ \rho \le 2,\ \rho\cos\varphi \ge \frac{\rho\operatorname{sen}\varphi}{3} \right\} = \left\{ \rho \le 4,\ \operatorname{tg}\varphi \le 3 \right\} =$$

$$= \left\{ \rho \le 2,\ 0 \le \varphi \le \operatorname{arctg} 3,\ 0 \le \theta \le 2\pi \right\}$$

Parâmetros: $u = \theta$; $v = \varphi$

$$\sigma(u,v) = (2\cos u\operatorname{sen}v,\ 2\operatorname{sen}u\cos v,\ 2\cos v)$$

$$\sigma_u = (-2\operatorname{sen}u\operatorname{sen}v,\ 2\cos u\operatorname{sen}v,\ 2\cos v)$$

$$\sigma_v = (2\cos u\cos v,\ 2\operatorname{sen}u\cos v,\ -2\operatorname{sen}v)$$

$$\vec{N} = \sigma_u \wedge \sigma_v = -4\operatorname{sen}v\,(\cos u\operatorname{sen}v,\ \operatorname{sen}u\cos v,\ \cos v)$$

$$\|\vec{N}\| = \sqrt{16\operatorname{sen}^2 v} = 4\operatorname{sen}v$$

$$A = \iint_{\Omega} d\sigma = \int_0^{2\pi}\left(\int_0^{\operatorname{arctg}3} 4\operatorname{sen}v\,dv\right)du =$$

$$= 4\int_0^{2\pi} -\cos v\,\Big|_0^{\operatorname{arctg}3}\,du = 4\int_0^{2\pi} -\frac{1}{\sqrt{10}} + 1\,du = 8\pi\left(1 + \frac{1}{\sqrt{10}}\right)$$

3) a) Parâmetros: $u = x$; $v = y$; $u + v^2 = z$

$$\sigma(u,v) = (u, v, u + v^2)$$

$$\sigma_u = (1,0,1) \text{ e } \sigma_v = (0,1,2v)$$

$$\sigma_u \wedge \sigma_v = (-1, -2v, 1)$$

$$\|\sigma_u \wedge \sigma_v\| = \sqrt{1 + 4v^2 + 1} = \sqrt{2 + 4v^2}$$

Domínio: $\Omega = \{0 \le u \le 1,\ 0 \le v \le 2\}$

$$\iint_{\Omega} y\,d\sigma = \int_0^1\left(\int_0^2 v\sqrt{2 + 4v^2}\,dv\right)du =$$

$$= \int_0^1 \frac{\sqrt{(2+4v^2)^3}}{12}\Big|_0^2\,du = \int_0^1 \frac{54\sqrt{2} - 2\sqrt{2}}{12}\,du =$$

$$= \int_0^1 \frac{52\sqrt{2}}{12}\,du = \int_0^1 \frac{13\sqrt{2}}{3}\,du = \frac{13\sqrt{2}}{3}$$

b) Coordenadas Esféricas:

$$\begin{cases} x = \rho\cos\theta\,\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\,\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$\rho^2 = 1 \Rightarrow \rho = 1$$

Parâmetros: $u = \theta$, $v = \varphi$

$$\sigma(u,v) = (\cos u\operatorname{sen}v,\ \operatorname{sen}u\operatorname{sen}v,\ \cos v)$$

$$\sigma_u = (-\operatorname{sen}u\operatorname{sen}v,\ \cos u\operatorname{sen}v,\ 0)$$

$$\sigma_v = (\cos u\cos v,\ \operatorname{sen}u\cos v,\ -\operatorname{sen}v)$$

$$\sigma_u \wedge \sigma_v = \begin{vmatrix} i & j & k \\ -\operatorname{sen}u\operatorname{sen}v & \cos u\operatorname{sen}v & 0 \\ \cos u\cos v & \operatorname{sen}u\cos v & -\operatorname{sen}v \end{vmatrix} =$$

$$= (-\cos u\operatorname{sen}^2 v,\ -\operatorname{sen}u\cdot\operatorname{sen}^2 v,\ -\operatorname{sen}^2 u\operatorname{sen}v\cos v + -\cos^2 u\cos v\operatorname{sen}v) =$$

$$= -\operatorname{sen}v\,(\cos u\operatorname{sen}v,\ \operatorname{sen}u\operatorname{sen}v,\ \cos v)$$

$$\|\sigma_u \wedge \sigma_v\| = \operatorname{sen}v$$

Domínio: $\Omega = \{0 \le u \le 2\pi,\ 0 \le v \le \pi\}$

$$\iint_{\Omega} x^2\,d\sigma = \iint_{\Omega} \cos^2 u\cdot\operatorname{sen}^2 v\cdot\operatorname{sen}v\cdot dv\,du =$$

$$= \int_0^{2\pi}\left(\int_0^{\pi} \cos^2 u\cdot\operatorname{sen}^3 v\,dv\right)du =$$

$$= \int_0^{2\pi}\cos^2 u\left(\int_0^{\pi}\operatorname{sen}^3 v\,dv\right)du - \int_0^{2\pi}\cos^2 u\left[\frac{\cos^3 v}{3} - \cos v\right]_0^{\pi}du =$$

$$= \frac{4}{3}\int_0^{2\pi}\cos^2 u\,du = \frac{2}{3}\int_0^{2\pi}\cos 2u + 1\,du =$$

$$= \frac{2}{3}\left[\frac{\operatorname{sen}2u}{2} + u\right]_0^{2\pi} = \frac{2}{3}\left[2\pi\right] = \frac{4\pi}{3}$$

c) Parâmetros: $u = x$, $v = y$, $v + 3 = z$

$$\sigma(u,v) = (u, v, v + 3)$$

$$\sigma_u = (1,0,0) \text{ e } \sigma_v = (0,1,1)$$

$$\sigma_u \wedge \sigma_v = (0,-1,1) \quad \|\sigma_u \wedge \sigma_v\| = \sqrt{2}$$

Domínio: $\Omega = \{u^2 + v^2 \le 1\}$

$$I = \iint_{\Omega} yz\,d\sigma = \iint_{\Omega} v^2 + 3v\,du\,dv$$

Passando para coordenadas polares:

$$\begin{cases} u = \rho\cos\theta \\ v = \rho\operatorname{sen}\theta \end{cases} \qquad J = \rho$$

$$\Omega_{\rho\theta} = \{\rho^2 \le 1;\ 0 \le \theta \le \pi\} = \{0 \le \rho \le 1,\ 0 \le \theta \le \pi\}$$

$$I = \int_0^{2\pi}\left(\int_0^1 (\rho^2 \operatorname{sen}^2\theta + 3\rho\operatorname{sen}\theta)\,\rho\,d\rho\right)d\theta =$$

$$= \int_0^{2\pi}\left(\int_0^1 \rho^3\operatorname{sen}^2\theta + 3\rho^2\operatorname{sen}\theta\,d\rho\right)d\theta =$$

$$= \int_0^{2\pi}\frac{\operatorname{sen}^2\theta}{4}\,d\theta + \int_0^{2\pi}\operatorname{sen}\theta\,d\theta = \frac{1}{8}\int_0^{2\pi}\cos 2\theta + 1\,d\theta +$$

$$- \cos\theta\Big|_0^{2\pi} = \frac{1}{8}\left[\frac{-\operatorname{sen}2\theta}{2} + \theta\right]_0^{2\pi} = \frac{\pi}{4}$$

d) Coordenadas cilíndricas:

$$\begin{cases} x = \rho\cos\theta \\ y = y \\ z = \rho\operatorname{sen}\theta \end{cases}$$

$\rho^2 = 1 \Rightarrow \rho = 1$

Parâmetros: $u = \theta$, $v = y$

$\sigma(u,v) = (\cos u, v, \operatorname{sen}u)$

$\sigma_u = (-\operatorname{sen}u, 0, \cos u)$ e $\sigma_v = (0,1,0)$

$$\sigma_u \wedge \sigma_v = \begin{vmatrix} i & j & k \\ -\operatorname{sen}u & 0 & \cos u \\ 0 & 1 & 0 \end{vmatrix} = (-\cos u, 0, -\operatorname{sen}u)$$

$\|\sigma_u \wedge \sigma_v\| = \sqrt{\cos^2 u + \operatorname{sen}^2 u} = 1$

Domínio: $Q = \{0 \le v \le 2 - \cos u\,;\ 0 \le u \le 2\pi\}$

$$I = \iint_S xyz\,d\sigma = \iint_Q v\cos u\,du\,dv = \int_0^{2\pi}\left(\int_0^{2-\cos u} v\cos u\,dv\right)du =$$

$$= \int_0^{2\pi}\frac{v^2\cos u}{2}\Big|_0^{2-\cos u}du = \frac{1}{2}\int_0^{2\pi}\cos u(4 - 4\cos u + \cos^2 u)\,du =$$

$$= \frac{1}{2}\int_0^{2\pi}\cos^3 u - 4\cos^2 u + 4\cos u\,du =$$

$$= -\frac{1}{2}\left[\operatorname{sen}u - \frac{\operatorname{sen}^3 u}{3}\right]_0^{2\pi} - \left[\frac{\operatorname{sen}2u}{2} + u\right]_0^{2\pi} + \frac{5}{2}[\operatorname{sen}2u]_0^{2\pi} =$$

$$= -2\pi$$

5) Coordenadas esféricas:

$$\begin{cases} x = \rho\cos\theta\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$\rho^2 = 1 \Rightarrow \rho = 1$

Parâmetros: $u = \theta$, $v = \varphi$

$\sigma(u,v) = (\cos u\operatorname{sen}v, \operatorname{sen}u\operatorname{sen}v, \cos v)$

$\sigma_u = (-\operatorname{sen}u\operatorname{sen}v, \cos u\operatorname{sen}v, 0)$

$\sigma_v = (\cos u\cos v, \operatorname{sen}u\cos v, -\operatorname{sen}v)$.

$\sigma_u \wedge \sigma_v = -\operatorname{sen}v(\cos u\operatorname{sen}v, \operatorname{sen}u\operatorname{sen}v, \cos v)$

$\|\sigma_u \wedge \sigma_v\| = \operatorname{sen}v$

Domínio: $Q = \{\rho\cos\varphi \ge \rho\operatorname{sen}\varphi\} = \{\operatorname{tg}\varphi \le 0\} =$

$= \{0 \le v \le \frac{\pi}{4},\ 0 \le u \le 2\pi\}$

$$I = \iint_S xyz\,d\sigma = \iint_Q \cos u\operatorname{sen}u\operatorname{sen}^2 v\cos v\,du\,dv =$$

$$= \int_0^{2\pi}\cos u\operatorname{sen}u\left(\int_0^{\pi/4}\operatorname{sen}^2 v\cos v\,dv\right)du = \frac{1}{3}\int_0^{2\pi}\operatorname{sen}u\cos u\operatorname{sen}^3 v\Big|_0^{\pi/4}du =$$

$$= 3\,\frac{2\sqrt{2}}{8}\int_0^{2\pi}\frac{\operatorname{sen}2u}{2}\,du = \frac{3\sqrt{2}}{4}\left[\frac{-\cos 2u}{4}\right]_0^{2\pi} = 0$$

5) a) Parâmetros: $u = x$, $v = y$, $9 - u^2 - v^2 = z$

$\sigma(u,v) = (u, v, 9 - u^2 - v^2)$

$\sigma_u = (1, 0, -2u)$ e $\sigma_v = (0, 1, -2v)$

$\sigma_u \wedge \sigma_v = (2u, 2v, 1)$

$\vec{F}(\sigma(u,v)) = (u^2 v, -3uv^2, 4v^3)$

Domínio: $Q = \{9 - u^2 - v^2 \ge 0\} = \{u^2 + v^2 \le 9\}$

$$\phi = \oint\!\!\!\oint_S \vec{F}\cdot\vec{N}\,d\sigma = \iint_Q \vec{F}\cdot\sigma_u \wedge \sigma_v\,du\,dv =$$

(+) mesma orientação

$$= \iint_Q 2u^3 v - 6uv^3 + 4v^3\,du\,dv = \iint_Q 2uv(u^2 - 3v^2) + 4v^3\,du\,dv$$

Passando para polares: $\begin{cases} u = \rho\cos\theta \\ v = \rho\operatorname{sen}\theta \end{cases}$ $\therefore J = \rho$

$Q_{r\theta} = \{\rho^2 \le 9,\ 0 \le \theta \le 2\pi\} = \{0 \le \rho \le 3,\ 0 < \theta < 2\pi\}$

$$\phi = \int_0^{2\pi}\left(\int_0^3 [\rho^2\operatorname{sen}2\theta(\rho^2\cos^2\theta - 3\rho^2\operatorname{sen}^2\theta) + 4\rho^2\operatorname{sen}^3\theta]\cdot\rho\,d\rho\right)d\theta = \int_0^{2\pi}\int_0^3 \rho^5\operatorname{sen}2\theta\cos^2\theta - 3\rho^5\operatorname{sen}2\theta\operatorname{sen}^2\theta +$$

$$+ 4\rho^4\operatorname{sen}^3\theta\,d\rho\,d\theta =$$

$$= \int_0^{2\pi}\left[\frac{\rho^6\operatorname{sen}\theta\cos^3\theta}{6} - \frac{\rho^6\operatorname{sen}^2\theta\cos\theta}{2} + \frac{4\rho^5\operatorname{sen}^3\theta}{5}\right]_0^3 d\theta =$$

$$= \frac{243}{2}\left[\frac{-\cos^4\theta}{4}\right] - \frac{729}{2}\left[\frac{\operatorname{sen}^4\theta}{4}\right]_0^{2\pi} + \frac{972}{5}\int_0^{2\pi}\operatorname{sen}^3\theta\,d\theta = 0$$

b) Parâmetros: $x=u,\ y=v,\ z=6-3u-2v$

$\sigma(u,v)=(u,v,6-3u-2v)$

$\sigma_u=(1,0,-3)$ e $\sigma_v=(0,1,-2)$

$\sigma_u \wedge \sigma_v=(3,2,1)$ → mesma orientação

$\vec{F}(\sigma(u,v))=(u,uv,6u-3u^2-2uv)$

Domínio: $Q=\{u^2+v^2\le 1\}$

$$\phi=\iint_S \vec{F}\cdot\vec{N}\,dS=\iint_Q 3u+2uv+6u-3u^2-2uv\,du\,dv=$$

$$=\iint_Q 9u-3u^2\,du\,dv$$

Passando para polares: $\begin{cases} u=\rho\cos\theta \\ v=\rho\operatorname{sen}\theta \end{cases}$, $J=\rho$

$Q_{\rho\theta}=\{\rho^2\le 1,\ 0\le\theta\le 2\pi\}$

$$\phi=\int_0^{2\pi}\left(\int_0^1 9\rho^2\cos\theta-3\rho^3\cos^2\theta\,d\rho\right)d\theta=$$

$$=\int_0^{2\pi} 3\rho^3\cos\theta-\frac{3}{4}\rho^4\cos^2\theta\,\Big|_0^1\,d\theta=$$

$$=\int_0^{2\pi} 3\cos\theta-\frac{3}{4}\cos^2\theta\,d\theta=3\int_0^{2\pi}\cos\theta-\frac{1-\cos 2\theta}{8}d\theta=$$

$$=3\left[\operatorname{sen}\theta-\frac{\theta}{8}-\frac{\operatorname{sen}2\theta}{16}\right]_0^{2\pi}=-\frac{3\pi}{4}$$

c) Parâmetros: $x=u,\ y=v,\ z=\sqrt{u^2+v^2}$

$\sigma(u,v)=(u,v,\sqrt{u^2+v^2})$

$\sigma_u=\left(1,0,\frac{u}{\sqrt{u^2+v^2}}\right)$; $\sigma_v=\left(0,1,\frac{v}{\sqrt{u^2+v^2}}\right)$

$\sigma_u\wedge\sigma_v=\left(-\frac{u}{\sqrt{u^2+v^2}},-\frac{v}{\sqrt{u^2+v^2}},1\right)$ → orientação diferente

$\vec{F}(\sigma(u,v))=(-u,-v,u^2+v^2)$

Domínio: $Q=\{1\le\sqrt{u^2+v^2}\le 2\}$

$$\phi=-\iint_Q(-u,-v,u^2+v^2)\left(-\frac{u}{\sqrt{u^2+v^2}},-\frac{v}{\sqrt{u^2+v^2}},1\right)du\,dv=$$

$$=-\iint_Q \frac{u^2+v^2}{\sqrt{u^2+v^2}}+u^2+v^2\,du\,dv$$

Passando para polares:

$\begin{cases} u=\rho\cos\theta \\ v=\rho\operatorname{sen}\theta \end{cases}$, $J=\rho$

$Q_{\rho\theta}=\{1\le\rho\le 2;\ 0\le\theta\le 2\pi\}$

$$\phi=-\int_0^{2\pi}\int_1^2\left(\frac{\rho^2}{\rho}+\rho^2\right)\rho\,d\rho\,d\theta=$$

$$=-\int_0^{2\pi}\int_1^2 \rho^2+\rho^3\,d\rho\,d\theta=-\int_0^{2\pi}\frac{\rho^3}{3}+\frac{\rho^4}{4}\Big|_1^2\,d\theta=$$

$$=-\int_0^{2\pi}\frac{8}{3}+\frac{16}{4}-\frac{1}{3}-\frac{1}{4}\,d\theta=-\int_0^{2\pi}\frac{7}{3}+\frac{15}{4}\,d\theta=$$

$$=-\frac{73}{12}\int_0^{2\pi}d\theta=-\frac{73\pi}{6}$$

e) Parâmetros: $x=u,\ y=v,\ z=\sqrt{16-u^2-v^2}$

$\sigma(u,v)=(u,v,\sqrt{16-u^2-v^2})$

$\sigma_u=\left(1,0,\frac{-u}{\sqrt{16-u^2-v^2}}\right)$; $\sigma_v=\left(0,1,\frac{-v}{\sqrt{16-u^2-v^2}}\right)$

$\sigma_u\wedge\sigma_v=\left(\frac{u}{\sqrt{16-u^2-v^2}},\frac{v}{\sqrt{16-u^2-v^2}},1\right)$ → mesma orientação

$\vec{F}(\sigma(u,v))=(-v,u,3\sqrt{16-u^2-v^2})$

Domínio: $Q=\{z(u,v)=0\}=\{u^2+v^2=16\}$

$$\phi=\iint_Q \frac{-uv}{\sqrt{16-u^2-v^2}}+\frac{uv}{\sqrt{16-u^2-v^2}}+3\sqrt{16-u^2-v^2}\,du\,dv=$$

$$=\iint_Q 3\sqrt{16-u^2-v^2}\,du\,dv$$

Passando para polares:

$\begin{cases} u=\rho\cos\theta \\ v=\rho\operatorname{sen}\theta \end{cases}$, $J=\rho$

$Q_{\rho\theta}=\{0\le\rho\le 4,\ 0\le\theta\le 2\pi\}$

$$\phi=\int_0^{2\pi}\int_0^4 3\rho\sqrt{16-\rho^2}\,d\rho\,d\theta=3\int_0^{2\pi}\int_0^4\rho\sqrt{16-\rho^2}\,d\rho\,d\theta$$

$$\Phi = \frac{3}{3}\int_0^{2\pi} -\sqrt{(16-\rho^2)^3}\Big|_0^4 d\theta = 2^6\int_0^{2\pi} d\theta = 128\pi$$

d) Coordenadas esféricas

$$\begin{cases} x = \rho\cos\theta \operatorname{sen}\varphi \\ y = \rho\cos\theta \operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$\Omega = \{\rho^2 = 9,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

Parâmetros: $u = \theta$ e $v = \varphi$

$\sigma(u,v) = (3\cos u \operatorname{sen} v,\ 3\operatorname{sen} u \operatorname{sen} v,\ 3\cos v)$

$\sigma_u = (-3\operatorname{sen} u \operatorname{sen} v,\ 3\cos u \operatorname{sen} v,\ 0)$

$\sigma_v = (3\cos u \cos v,\ 3\operatorname{sen} u \cos v,\ -3\operatorname{sen} v)$

$\sigma_u \wedge \sigma_v = (-9\cos u \operatorname{sen}^2 v,\ -9\operatorname{sen} u \operatorname{sen}^2 v,$
$-9\operatorname{sen}^2 v \cos u \operatorname{sen} v - 9\cos^2 u \cos v \operatorname{sen} v)$

$\vec{F}(\sigma(u,v)) = (3\cos u \operatorname{sen} v,\ 3\operatorname{sen} u \operatorname{sen} v,\ 3\cos v)$

Domínio: $\Omega = \{0 \le u \le 2\pi,\ 0 \le v \le \pi\}$

$$\Phi = -\int_0^{2\pi}\int_0^{\pi} (3\cos u \operatorname{sen} v,\ 3\operatorname{sen} u \operatorname{sen} v,\ 3\cos v)\cdot(-9\cos u \operatorname{sen}^2 v,$$
$$-9\operatorname{sen}\operatorname{sen}^2 v,\ \frac{9\operatorname{sen} 2v}{2})\, dv\,du = 108\pi$$

7) $\vec{F} = (P, Q, R)$

$\sigma(x,y) = (x, y, f(x,y)),\ \forall x,y \in \Omega.$

$\sigma_x = (1, 0, \frac{\partial f}{\partial x})$ e $\sigma_y = (0, 1, \frac{\partial f}{\partial y})$

$\sigma_x \wedge \sigma_y = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$

$$\iint_\Omega \vec{F}\vec{N}\, d\sigma = \iint_\Omega (P,Q,R)\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) dx\,dy =$$

$$= \iint_\Omega -P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R\, dx\,dy.$$

10) a) $\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ xz & 2xy & 3xy \end{vmatrix} = (\partial_y 3xy - \partial_z 2xy,$

$\partial_z xz - \partial_x 3xy,\ \partial_x 2xy - \partial_y xz) =$

$= (3x, x - 3y, 2y)$

$\Omega = \{z = 3 - 3x - y\}$

Parâmetros: $x = u$, $y = v$, $z = 3 - 3u - v$

$\sigma(u,v) = (u, v, 3 - 3u - v)$

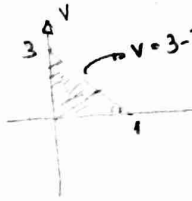
$\sigma_u = (1, 0, -3)$ e $\sigma_v = (0, 1, -1)$

$\sigma_u \wedge \sigma_v = (3, 1, 1)$

γ é percorrida no sentido anti-horário

Tomamos $\vec{n}$ tal que γ seja orientada como no enunciado $\Rightarrow \sigma_u \wedge \sigma_v$ e $\vec{n}$ têm o mesmo sentido

Domínio: $\Omega = \{0 \le v \le 3 - 3u,\ 0 \le u \le 1\}$



Pelo Teorema de Stokes:

$$\iint_\Omega \operatorname{rot}\vec{F}\,\vec{n}\, d\sigma = \int_\gamma \vec{F}\,d\vec{r} \Rightarrow \int_\gamma \vec{F}\,d\vec{r} =$$

$$= \iint_\Omega (3u, u - 3v, 2v)\cdot(3,1,1)\, dv\,du =$$

$$= \int_0^1\int_0^{3-3u} 9u + u - 3v + 2v\, dv\,du = \int_0^1\int_0^{3-3u} 10u - v\, dv\,du =$$

$$= \int_0^1 10uv - \frac{v^2}{2}\Big|_0^{3-3u} du = \int_0^1 30u - 30u^2 - \frac{9}{2} + 18u - \frac{9u^2}{2} du =$$

$$= \int_0^1 39u - \frac{69u^2}{2} - \frac{9}{2}\, du = \left[\frac{39u^2}{2} - \frac{69u^3}{6} - \frac{9u}{2}\right]_0^1 =$$

$$= \left[\frac{39}{2} - \frac{23}{2} - \frac{9}{2}\right] = \frac{7}{2}$$

c) $\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2z + \operatorname{sen} e^{x^3} & 4x & 5y + \operatorname{sen}(\operatorname{sen} z^3) \end{vmatrix} =$

$= (5 - 0, 2 - 0, 4 - 0) = (5, 2, 4)$

$\Omega = \{z = x + 4\}$

Parâmetros: $x = u$, $y = v$, $z = u + 4$

$\sigma(u,v) = (u, v, u + 4)$

$\sigma_u = (1, 0, 1)$ e $\sigma_v = (0, 1, 0)$

$\sigma_u \wedge \sigma_v = (-1, 0, 1)$

γ é percorrido no sentido anti-horário. Tomamos $\vec{n}$ para que γ seja orientado como no enunciado.

Então $\vec{n}$ e $\sigma_u \wedge \sigma_v$ têm o mesmo sentido.

$\text{rot}\,\vec{F}(\sigma(u,v)) = (5, 2, 4)$

Domínio: $Q = \{u^2 + v^2 \leq 4\}$

Pelo Teorema de Stokes:

$$\iint_Q \text{rot}\,\vec{F}\,\vec{n}\,d\sigma = \int_\gamma \vec{F}\,d\vec{r} \Rightarrow \int_\gamma \vec{F}d\vec{r} =$$

$$= \iint_Q -5 + 4\,du\,dv = -\iint_Q du\,dv$$

Passando para polares: $\begin{cases} u = \rho\cos\theta \\ v = \rho\sin\theta \end{cases}$ $|J| = \rho$

Domínio: $Q = \{\rho^2 \leq 4,\ 0 \leq \theta \leq 2\pi\} =$

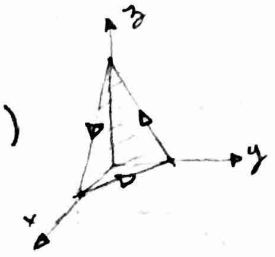
$= \{0 \leq \rho \leq 2,\ 0 \leq \theta \leq 2\pi\}$

$$\int_\gamma \vec{F}d\vec{r} = -\int_0^{2\pi}\int_0^2 \rho\,d\rho\,d\theta = -\int_0^{2\pi} \frac{\rho^2}{2}\Big|_0^2 = -4\pi$$

b) $\text{rot}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z^2 + e^{x^2} & y^2 - \ln(1+y^4) & xy + \operatorname{sen} z^3 \end{vmatrix} =$

$= (x - 0, 2z - y, 0)$

$\vec{N_1} = (1, -1, 0)$ e $\vec{N_2} = (1, 0, -2)$



$\vec{N} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -2 \end{vmatrix} = (2, 2, 1)$

substituir $(1,0,0)$

$2x + 2y + z + d = 0 \quad 2 + d = 0 \Rightarrow d = -2$

$\Omega = \{z = 2 - 2x - 2y\}$

Parâmetros: $x = u$, $y = v$, $z = 2 - 2u - 2v$

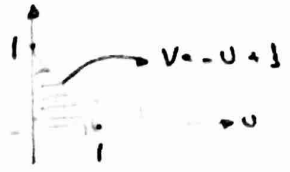
$\sigma(u,v) = (u, v, 2 - 2u - 2v)$

$\sigma_u = (1, 0, -2)$ e $\sigma_v = (0, 1, -2)$

$\sigma_u \wedge \sigma_v = (2, 2, 1)$

$\text{rot}\,\vec{F}(\sigma(u,v)) = (u, 4 - 4u - 4v - v, v) = (u, 4 - 4u - 5v)$

Domínio:



$Q = \{0 \leq u \leq 1,\ 0 \leq v \leq -u + 1\}$

Pelo Teorema de Stokes:

$$\int_\gamma \vec{F}d\vec{r} = \iint_Q (u, 4 - 4u - 5v, 0)\,(2, 2, 1)\,du\,du =$$

$$= \iint_Q 2u + 8 - 8u - 10v\,du\,du = \iint_R -6u - 10v + 8\,du\,du$$

$$= \int_0^1 \int_0^{1-u} -6u - 10v + 8\,du\,du = \int_0^1 -6uv - 5v^2 + 8v\Big|_0^{1-u} du =$$

$$= \int_0^1 -6u + 6u^2 - 5(1 - 2u + u^2) + 8 - 8u\,du =$$

$$= \int_0^1 u^2 - 4u + 3\,du = \left[\frac{u^3}{3} - 2u^2 + 3u\right]_0^1 =$$

$$= \frac{1}{3} - 2 + 3 = \frac{1 - 6 + 9}{3} = \frac{4}{3}$$

11) b) $\vec{V} = \vec{V_1} + \vec{V_2} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) +$

$+ \left(0, 0, \frac{z^3}{1+z^2}\right)$

(I) $\text{rot}\,\vec{V_1} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} =$

$$= \left(0, 0, \frac{1(x^2+y^2) - 2x^2}{(x^2+y^2)^2} + \frac{1(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right) =$$

$= (0, 0, 0)$

$\text{dom}\,\vec{V_1} = \mathbb{R}^3 - \{(0, 0, z)\}$

Acrescentamos uma curva α: $x^2 + y^2 = r$ com r suficientemente pequeno de forma que os cilindros se tornem disjuntos.

Seja Σ a superfície de interseção do plano com os cilindros

Tomamos $\vec{n}$ que orienta γ como no enunciado

Portanto, $\vec{n}$ induz sobre γ o sentido oposto, i.e., ©

anti-horário (que é o mesmo "induzido" por $\vec{n}$)

Pelo Teorema de Stokes:

$$\iint_{\Delta} \operatorname{rot}\vec{V_1}\cdot\vec{n}\,d\sigma = \int_{\gamma}\vec{V_1}\,d\vec{r} \overset{(*)}{=} \int_{\alpha}\vec{V_1}\,d\vec{r} \Rightarrow$$

$$\Rightarrow \int_{\gamma}\vec{V_1}\,d\vec{r} = -\int_{\alpha}\vec{V_1}\,d\vec{r}$$

$\alpha(t) = (r\cos(t), r\operatorname{sen}(t), z(t)),\ t\in[0,2\pi]$

$\alpha'(t) = (-r\operatorname{sen}(t), r\cos(t), z'(t))$

$\vec{V_1}(\alpha(t)) = \left(\frac{-r\operatorname{sen}t}{r^2}, \frac{r\cos t}{r^2}, 0\right)$

$$\int_{\gamma}\vec{V_1}\,d\vec{r} = -\int_0^{2\pi}\left(-\frac{r\operatorname{sen}t}{r^2}, \frac{r\cos t}{r^2}, 0\right)(-r\operatorname{sen}t, r\cos t, z'(t))\,dt$$

$$= -\int_0^{2\pi}\cos^2 t + \operatorname{sen}^2 t\,dt = -2\pi$$

(II) $\operatorname{rot}\vec{V_2} = \vec{0}$

$\operatorname{dom}\vec{V_2} = \mathbb{R}^3 \therefore$ é simplesmente conexo $\Rightarrow$ $\vec{V_2}$ é conservativo.

Como γ é fechado $\Rightarrow \int_{\gamma}\vec{V_2}\,d\vec{r} = 0$.

$$\therefore \int_{\gamma}\vec{V}\,d\vec{r} = 2\pi.$$

f) $\vec{V} = \underbrace{\left(0, \frac{-z}{y^2+z^2}, \frac{y}{y^2+z^2}\right)}_{\vec{V_1}} + \underbrace{(\cos(1+x^2), e^{y^4}, 0)}_{\vec{V_2}}$

(I) $\operatorname{rot}\vec{V_1} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{-z}{y^2+z^2} & \frac{y}{y^2+z^2} \end{vmatrix} =$

$$= \left(\frac{(y^2+z^2)-2y^2}{(y^2+z^2)^2} + \frac{(z^2+y^2)-2z^2}{(y^2+z^2)^2}, 0, 0\right) =$$

$= (0,0,0)$

$= \operatorname{dom}\vec{V_1} = \mathbb{R}^3 - \{(x,0,0)\}$

Vamos escolher a superfície que se refere a casca do cilindro limitada superiormente por $x+y+z$ e inferiormente por $x=0$. Assim, teremos a superfície

$$\Delta = \{y^2+z^2=4,\ 0\le x\le y+z\}$$

Vamos introduzir uma curva $\alpha: y^2+z^2=4$ no plano yz de forma a limitar a superfície

Seja $\vec{n}$ a orientação de Δ que induz em γ o sentido do enunciado (sentido anti-horário). Portanto, $\vec{n}$ induz sobre α o sentido oposto (sentido horário)

$\alpha(t) = (x', 2\operatorname{sen}t, 2\cos t),\ t\in[0,2\pi]$

$\alpha'(t) = (x', 2\cos t, -2\operatorname{sen}t) \therefore \vec{V_1}(\alpha(t)) = \left(0, \frac{-2\cos t}{4}, \frac{2\operatorname{sen}t}{4}\right)$

Pelo Teorema de Stokes:

$$\iint_{\Delta}\operatorname{rot}\vec{V_1}\cdot\vec{n}\,d\sigma = \int_{\gamma}\vec{V_1}\,d\vec{r} + \int_{\alpha}\vec{V_1}\,d\vec{r} \Rightarrow \int_{\gamma}\vec{V_1}\,d\vec{r} = -\int_{\alpha}\vec{V_1}\,d\vec{r}$$

$$\Rightarrow \int_{\gamma}\vec{V_1}\,d\vec{r} = -\int_0^{2\pi}\left(0, \frac{-2\cos t}{4}, \frac{2\operatorname{sen}t}{4}\right)(x', 2\cos t, -2\operatorname{sen}t)\,dt$$

$$\Rightarrow \int_{\gamma}\vec{V_1}\,d\vec{r} = -\int_0^{2\pi}-\cos^2 t - \operatorname{sen}^2 t\,dt = 2\pi.$$

(III) $\operatorname{rot}\vec{V_2} = (0,0,0)$

$\operatorname{dom}\vec{V_2} = \mathbb{R}^3 \therefore$ é simplesmente conexo $\Rightarrow \vec{V_2}$ é conservativo. Como γ é fechado $\Rightarrow \int_{\gamma}\vec{V_2}\,d\vec{r} = 0$.

$$\therefore \int_{\gamma}\vec{V}\,d\vec{r} = 2\pi.$$

a) $\operatorname{rot}\vec{V} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & xz+\ln(1+y^4) & yz \end{vmatrix} =$

$= (z-x, y, z-z) = (z-x, y, 0)$

$\Delta: \{z = 2x+3\}$

Parâmetros: $x=u,\ y=v,\ z=2u+3$

$\sigma(u,v) = (u, v, 2u+3)$

$\sigma_u = (1,0,2)$ e $\sigma_v = (0,1,0)$

$\sigma_u \wedge \sigma_v = (-2,0,1)$

Seja $\vec{n}$ a orientação de Δ que induz em γ o sentido do enunciado.

$\text{rot}\,\vec{v}(\sigma(u,v)) = (u+3, v, 0)$

Domínio: $D = \{u^2 + v^2 \leq 4\}$

Passando para polares: $\begin{cases} u = \rho\cos\theta \\ v = \rho\operatorname{sen}\theta \end{cases}$, $J = \rho$

Domínio: $D_{\rho\theta} = \{\rho^2 \leq 4,\ 0 \leq \theta \leq 2\pi\} =$

$= \{0 \leq \rho \leq 2,\ 0 \leq \theta \leq 2\pi\}$

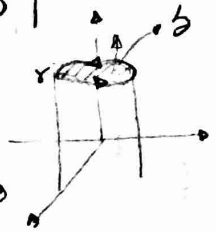
Pelo Teorema de Stokes:

$$\int_\gamma \vec{v}\,d\vec{r} = \int_0^{2\pi}\int_0^2 -2\rho(\rho\cos\theta - 3)\,|J|\,d\rho\,d\theta = \int_0^{2\pi} \left[-\frac{2\rho^3}{3}\cos\theta - 3\rho^2\right]_0^2 d\theta =$$

$$= \int_0^{2\pi} -\frac{16}{3}\cos\theta - 12\,d\theta = -24\pi$$

c) $\text{rot}\,\vec{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2x+3y & x^2y^2 & 3x^2z^2 \end{vmatrix} =$

$= (0-0,\ 6xz^2 - 6xz^2,\ 2xy^2 - 1)$



$\Sigma = \{z = \operatorname{sen} y + 10\}$

Parâmetros: $x = u$, $y = v$, $z = \operatorname{sen} v + 10$

$\sigma(u,v) = (u, v, \operatorname{sen} v + 10)$, $\text{rot}\,\vec{v}(\sigma(u,v)) = (0, 0, 2uv^2 - 1)$

$\sigma_u = (1, 0, 0)$ e $\sigma_v = (0, 1, \cos v)$

$\sigma_u \wedge \sigma_v = (0, -\cos v, 1)$

Domínio: $D = \{u^2 + v^2 \leq 16\}$

Passando para polares: $\begin{cases} u = \rho\cos\theta \\ v = \rho\operatorname{sen}\theta \end{cases}$, $J = \rho$

$D_{\rho\theta} = \{0 \leq \rho \leq 4,\ 0 \leq \theta \leq 2\pi\}$

Pelo Teorema de Stokes:

$$\int_\gamma \vec{v}\cdot d\vec{r} = \int_0^{2\pi}\int_0^4 \rho(2\rho\cos\theta\,\rho^2\operatorname{sen}^2\theta - 1)\,d\rho\,d\theta =$$

$$= \int_0^{2\pi}\int_0^4 2\rho^4\cos\theta\operatorname{sen}^2\theta - \rho\,d\rho\,d\theta =$$

$$= \int_0^{2\pi} \frac{2\rho^5}{5}\cos\theta\operatorname{sen}^2\theta - \frac{\rho^2}{2}\Big|_0^4 d\theta =$$

$$= \int_0^{2\pi} \frac{2048}{5}\cos\theta\operatorname{sen}^2\theta - 8\,d\theta = \left[\frac{2048}{5}\operatorname{sen}^3\theta - 8\theta\right]_0^{2\pi} =$$

$$= -16\pi$$

13) $\vec{F}(x,y,z) = (bz - cy)dx + (cx - az)dy + (ay - bx)dz$

Pelo Teorema de Stokes:

$$\int_\gamma \vec{F}d\vec{r} = \iint_\Delta \text{rot}\vec{F}\cdot\vec{N}\,d\sigma =$$

$$= \iint_\Delta (a+a, b+b, c+c)\cdot(a,b,c)\,d\sigma =$$

$$= \iint_\Delta 2(a^2+b^2+c^2)\,d\sigma = 2\iint_\Delta 1\,d\sigma \Rightarrow A = \frac{1}{2}\int_\gamma \vec{F}d\vec{r}$$

19) a) $\text{div}\,\vec{F} = -z + 3y^2 - z + 2z = 3y^2$

Como Δ é fechado vamos escolher $\Omega = \text{Int}\,\Delta$

$\Omega \cup \partial\Omega$ C dom $\vec{F} = \mathbb{R}^3$

A normal $\vec{n_c}$ do complementar de Ω tem mesmo sentido da normal do enunciado.

$$\Omega = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

Passando para coordenadas cilíndricas:

$$\begin{cases} x = a\rho\cos\theta\operatorname{sen}\varphi \\ y = b\rho\operatorname{sen}\theta\operatorname{sen}\varphi \\ z = c\rho\cos\varphi \end{cases}, \quad J = abc\rho^2\operatorname{sen}\varphi$$

$D_{\rho\theta\varphi} = \{0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi\}$

Pelo Teorema de Gauss:

$$\iint_\Delta \vec{F}\cdot\vec{n}\,d\sigma = \iiint_\Omega \text{div}\vec{F}\,dx\,dy\,dz \Rightarrow$$

$$\Rightarrow \iint_\Delta \vec{F}\cdot\vec{n}\,d\sigma = \iiint_{D_{\rho\theta\varphi}} 3b^2\rho^2\operatorname{sen}^2\theta\operatorname{sen}^2\varphi)\,d\rho\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3ab^3c\,\rho^4\operatorname{sen}^2\theta\operatorname{sen}^3\varphi\,d\rho\,d\varphi\,d\theta =$$

$$= 3ab^3c\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\operatorname{sen}^2\theta\operatorname{sen}^3\varphi\Big|_0^1 d\varphi\,d\theta =$$

$$= \frac{3ab^3c}{5}\int_0^{2\pi}\int_0^{\pi} \operatorname{sen}^2\theta\operatorname{sen}^3\varphi\,d\varphi\,d\theta =$$

$$= \frac{3ab^3c}{5}\int_0^{2\pi} \operatorname{sen}^2\theta \int_0^{\pi} (1-\cos^2\varphi)\operatorname{sen}\varphi\,d\varphi\,d\theta$$

$$= \frac{2ab^3c}{5} \int_0^{2\pi} \operatorname{sen}^2\theta \left[\frac{\cos 3\varphi}{3} - \cos\varphi \right]_0^{\pi} d\theta =$$

$$= \frac{2ab^3c}{5} \int_0^{2\pi} \frac{1-\cos 2\theta}{2} \cdot \frac{4}{3} d\theta - \frac{4ab^3c}{5} \int_0^{2\pi} \frac{1}{2} - \cos 2\theta \, d\theta =$$

$$= \frac{4ab^3c}{5} \left[\frac{\theta}{2} - \frac{\operatorname{sen} 2\theta}{4} \right]_0^{2\pi} = \frac{4\pi ab^3c}{5}$$

b) $\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3$

Σ é uma superfície fechada então $R = \operatorname{Int} \Sigma$

$R \cup \partial R \in \operatorname{dom} \vec{F} = \mathbb{R}^3$

$$R = \{ z \geq \sqrt{1-x^2-y^2},\ z \leq \sqrt{4-x^2-y^2},\ z \geq 0 \}$$

$$R = \left\{ 1 - x^2 - y^2 \leq z^2 \leq 4 - x^2 - y^2,\ z \geq 0 \right\}$$

Passando para coordenadas esféricas:

$$\begin{cases} x = \rho \cos\theta \operatorname{sen}\varphi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = \rho \cos\varphi \end{cases} \quad | J = \rho^2 \operatorname{sen}\varphi$$

$$R = \{ 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq \pi/2,\ 0 \leq \theta \leq 2\pi \}$$

$$\iint_{\Sigma} \vec{F}\,\vec{n}\, d\sigma = \iiint_{R} \operatorname{div} \vec{F}\, dx\,dy\,dz \Rightarrow$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 3(\rho^2 \operatorname{sen}^2\varphi + 1)(\rho^2 \operatorname{sen}\varphi)\, d\rho\, d\varphi\, d\theta =$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^4 \operatorname{sen}^3\varphi + \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^5}{5} \operatorname{sen}^3\varphi + \frac{\rho^3}{3} \operatorname{sen}\varphi \Big|_1^2 d\varphi\, d\theta =$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \frac{31}{5} \operatorname{sen}^3\varphi + \frac{7}{3} \operatorname{sen}\varphi \, d\varphi\, d\theta =$$

$$= 3 \int_0^{2\pi} \frac{31}{5} \left[\frac{\cos 3\varphi}{3} - \cos\varphi \right]_0^{\pi/2} - \frac{7}{3} \cos\varphi \Big|_0^{\pi/2} d\theta =$$

$$= 3 \int_0^{2\pi} \frac{31}{5} - \frac{31}{15} + \frac{7}{3} \, d\theta = \frac{97}{5} \int_0^{2\pi} d\theta = \frac{194\pi}{5}$$

Exercícios de prova

Q1 P3 de 2013)

a) Parametrização: $x = u$, $y = v$, $z = u^2 + v^2 + 2uv$

$\sigma(u,v) = (u, v, u^2 + v^2 + 2uv)$

$\sigma_u = (1, 0, 2u + 2v)$

$\sigma_v = (0, 1, 2v + 2u)$

$\therefore \sigma_u \wedge \sigma_v = (-2u - 2v, -2u - 2v, 1)$ $\quad \|\sigma_u \wedge \sigma_v\| = \sqrt{(2u+2v)^2 + 1} =$

$= \sqrt{2(4u^2 + 8uv + 4v^2 + 1)} = \sqrt{8u^2 + 16uv + 8v^2 + 1}$

$\delta(\sigma(u,v)) = \dfrac{2u^2 + 3v^2}{\sqrt{8u^2 + 16uv + 8v^2 + 1}}$; $Q = \{u^2 + v^2 \leq 2\}$

$$m = \iint_Q \delta(x,y,z)\, d\sigma = \iint_Q 2u^2 + 3v^2 \, du\,dv$$

Passando para coordenadas polares: $\begin{cases} u = \rho\cos\theta \\ v = \rho\sin\theta \end{cases}$, $J = \rho$ $\therefore Q_{r\theta} = \{0 \leq \rho \leq \sqrt{2},\ 0 \leq \theta \leq 2\pi\}$

$$m = \int_0^{2\pi}\int_0^{\sqrt{2}} 2\rho^3\cos^2\theta + 3\rho^3\sin^2\theta \, d\rho\,d\theta = \int_0^{2\pi} \frac{\rho^4}{2}\cos^2\theta + \frac{3\rho^4}{4}\sin^2\theta \Big|_0^{\sqrt{2}} d\theta =$$

$$= \int_0^{2\pi} 2\cos^2\theta + 3\sin^2\theta \, d\theta = \int_0^{2\pi} 1 + \cos 2\theta + \frac{3}{2} - 3\cos 2\theta \, d\theta = \int_0^{2\pi} \frac{5}{2} - 2\cos 2\theta \, d\theta =$$

$$= \left[\frac{5\theta}{2} - \sin 2\theta\right]_0^{2\pi} = 5\pi$$

b) Parametrização: $x = \sin u + \cos v$, $y = u$, $z = v$

$\sigma(u,v) = (\sin u + \cos v, u, v)$

$\sigma_u = (\cos u, 1, 0)$

$\sigma_v = (-\sin v, 0, 1)$

$$\therefore \sigma_u \wedge \sigma_v = \begin{vmatrix} i & j & k \\ \cos u & 1 & 0 \\ -\sin v & 0 & 1 \end{vmatrix} = (1, -\cos u, \sin v)$$

$Q = \{-1 \leq u \leq 1,\ 0 \leq v \leq 2\}$; $\vec{F}(\sigma(u,v)) = (\cos^2 v, \cos^2 u, \sin v)$

Orientação: $\sigma_u \wedge \sigma_v$ está no sentido oposto de $\vec{N}$

$$\iint_Q \vec{F}\cdot\vec{N}\cdot d\sigma = \iint_Q (\cos^2 v, \cos^2 u, \sin v)(1, -\cos u, \sin v)\, du\,dv = \int_0^2\int_{-1}^{1} \cos^2 v - \cos^3 u + \sin^2 v \, du\,dv =$$

$$= -\int_0^2\int_{-1}^{1} 1 - \cos^3 u \, du\,dv = -\int_0^2\int_{-1}^{1} 1 - \cos u(1 - \sin^2 u)\, du\,dv = \int_0^2 1 - \sin 1 + \frac{\sin^3 1}{3} + 1 - \sin 1 + \frac{\sin^3 1}{3}\, dv$$

$$= \int_0^2 2 - 2\operatorname{sen} 1 + \frac{2\operatorname{sen}^3 1}{3}\, dv = -4 + 4\operatorname{sen} 1 - \frac{4\operatorname{sen}^3 1}{3}$$

$$\begin{vmatrix} i & j & k \\ 1 & c & 1 \\ c & 1 & 1 \end{vmatrix} = (-1, -1, 1)$$

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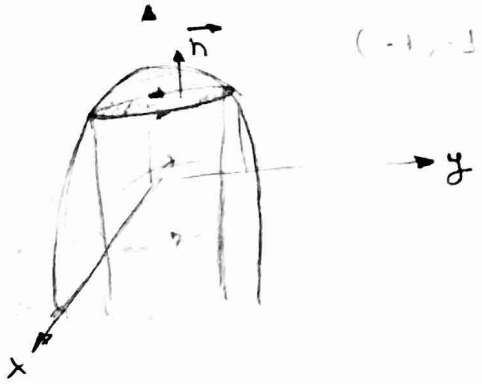
a) $\operatorname{dom} \vec{F} = \mathbb{R}^3$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x-3+\frac{y^2}{2+\operatorname{sen} y} & y \end{vmatrix} =$$



$= (1+1, 0, 2) = (2, 0, 2)$

Note que podemos tomar qualquer superfície. Se escolhermos o paraboloide teremos que fecha-lo com o plano $z = 6$ para que γ esteja no bordo da superfície. Se escolhermos o cilindro teremos que fecha-lo com o plano $y = 0$, pelo mesmo motivo. Portanto, vamos escolher a superfície $\Sigma = \{z = 6, x^2 + y^2 \leq 6\}$

Parametrizando Σ: $x = u$, $y = v$, $z = 6$ $\therefore \sigma(u,v) = (u, v, 6)$

$\sigma_u = (1, 0, 0)$ e $\sigma_v = (0, 1, 0)$ $\quad \sigma_u \wedge \sigma_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$

Domínio $Q = \{u^2 + v^2 \leq 6\}$

Vamos escolher $\vec{n}$ para Σ de modo que $\vec{n}$ induz em γ a orientação do enunciado

$\sigma_u \wedge \sigma_v$ tem mesmo sentido de $\vec{n}$

Pelo Teorema de Stokes: $\iint_\Sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, d\sigma = \int_\gamma \vec{F}\, d\vec{r} \Rightarrow \int_\gamma \vec{F}\, d\vec{r} = \iint_Q (2,0,2)(0,0,1)\, du\, dv =$

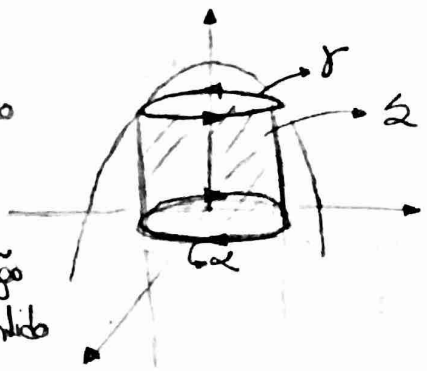
$= 2 \iint_Q du\, dv = 2\pi\, (\sqrt{6})^2 = 12\pi$

b) $\operatorname{dom} \vec{F} = \mathbb{R}^3 - \{(0,0,z)\}$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & \frac{\operatorname{sen} z}{e+z^4} \end{vmatrix} = \left(0, 0, \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}\right) = (0,0,0)$$

Devemos escolher uma superfície que não é interceptada pelo eixo z e que γ esteja no

bordo para aplicarmos o Teorema de Stokes.



Vamos escolher $\Sigma = \{x^2+y^2-6,\ 0 \le z \le 6\}$. Note que o bordo de Σ é $\partial\Sigma = \gamma \cup \alpha$, em que $\alpha: x^2+y^2=6,\ z=0$

Tomemos $\vec{n}$ para Σ de modo que $\vec{n}$ induza sobre γ a orientação do enunciado (anti-horário). Dessa forma $\vec{n}$ vai induzir em α o sentido contrário (horário).

Parametrizando α (sentido horário): $\alpha(t) = (6\sin t, 6\cos t, 0),\ t \in [0, 2\pi]$

$\alpha'(t) = (6\cos t, -6\sin t, 0)$

$$\vec{F}(\alpha(t)) = \left(\frac{6\cos t}{36}, \frac{6\sin t}{36}, 0\right)$$

Pelo Teorema de Stokes:

$$\iint_{\Sigma} \operatorname{rot}\vec{F}\cdot\vec{n}\,d\sigma = \int_{\gamma}\vec{F}\cdot d\vec{r} \oplus \int_{\alpha}\vec{F}\,d\vec{r} \Rightarrow \int_{\gamma}\vec{F}\,d\vec{r} = -\int_0^{2\pi}\left(\frac{-6\cos t}{36}, \frac{6\sin t}{36}, 0\right)(6\cos t, -6\sin t, 0)\,dt =$$

$$= \int_0^{2\pi} \cos^2 t + \sin^2 t\,dt = 2\pi$$

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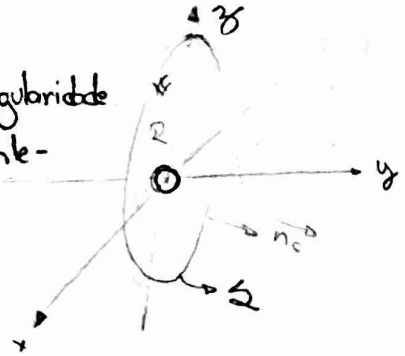
$\vec{F}(x,y,z) = \vec{F_1}(x,y,z) + \vec{F_2}(x,y,z)$, em que $\vec{F_1} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$ e $\vec{F_2} = \frac{z^3}{3}\vec{k}$

a) $\operatorname{div}\vec{F_1} = \frac{(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} +$

$+ \frac{(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} - \frac{3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3} = 0$

$\operatorname{dom}\vec{F_1} = \mathbb{R}^3 - \{(0,0,0)\}$



Para podermos aplicar o Teorema de Gauss devemos isolar a singularidade. Vamos adicionar a superfície $T = \{x^2+y^2+z^2 = r\}$ com r suficientemente pequeno. Definimos $R = \operatorname{Int}\Sigma \cup \operatorname{Ext} T$. Temos que $\partial R \cup RC \operatorname{dom}\vec{F_1}$.

Orientação:

Em Σ, $\vec{n_c}$ tem o sentido de $\vec{n}$

Em T, $\vec{n_c} = \vec{n}_{\text{interior}}$

Pelo Teorema de Gauss:

$$\iiint_R \operatorname{div}\vec{F_1}\,dx\,dy\,dz = \iint_{\Delta} \vec{F_1}\cdot\vec{n_c}\,d\sigma + \iint_{T} \vec{F_1}\cdot\vec{n_c}\,d\sigma = \iint_{\Delta} \vec{F_1}\,\vec{N}\,d\sigma + \iint_{T} \vec{F_1}\,\vec{n}_{interior}\,d\sigma \Rightarrow$$

$$\Rightarrow \iint_{\Delta} \vec{F_1}\,\vec{N}\cdot d\sigma = -\iint_{T} \vec{F_1}\,\vec{n}_{interior}\,d\sigma$$

Em coordenadas esféricas $\begin{cases} x = \rho\cos\theta\,\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\,\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$ $\therefore \rho = r$

$Dep = \{0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$

Parâmetros $\theta = u$, $\varphi = v$

$\sigma(u,v) = (r\cos u\,\operatorname{sen} v,\ r\operatorname{sen} u\,\operatorname{sen} v,\ r\cos v)$

$\sigma_u = (-r\operatorname{sen} u\operatorname{sen} v,\ r\cos u\operatorname{sen} v,\ 0)$

$\sigma_v = (r\cos u\cos v,\ r\operatorname{sen} u\cos v,\ -r\operatorname{sen} v)$

$$\sigma_u \wedge \sigma_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\operatorname{sen} u\operatorname{sen} v & r\cos u\operatorname{sen} v & 0 \\ r\cos u\cos v & r\operatorname{sen} u\cos v & -r\operatorname{sen} v \end{vmatrix} = (-r^2\cos u\operatorname{sen}^2 v,\ -r^2\operatorname{sen} u\operatorname{sen}^2 v,\ -r^2\operatorname{sen} v\cos v) =$$

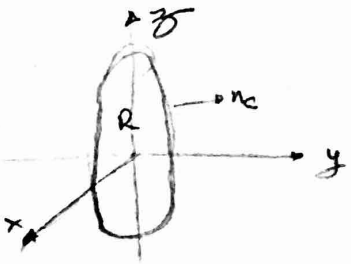
$$= -r^2\operatorname{sen} v\,(\cos u\operatorname{sen} v,\ \operatorname{sen} u\operatorname{sen} v,\ \cos v)$$

Orientação: $\vec{n}_{interior}$ e $\sigma_u \wedge \sigma_v$ tem mesmo sentido.

$$\vec{F}(\sigma(u,v)) = \left(\frac{-r\cos u\operatorname{sen} v}{r^3},\ \frac{r\operatorname{sen} u\operatorname{sen} v}{r^3},\ \frac{r\cos v}{r^3}\right)$$

$$\iint_{\Delta} \vec{F_1}\,\vec{N}\,d\sigma = \int_0^{2\pi}\int_0^{\pi} \cos^2 u\operatorname{sen}^3 v + \operatorname{sen}^2 u\operatorname{sen}^3 v + \operatorname{sen} v\cos^2 v\,dv\,du =$$

$$= -\int_0^{2\pi}\int_0^{\pi} \operatorname{sen}^3 v + \operatorname{sen} v\cos^2 v\,dv\,du = \int_0^{2\pi} -\cos v\Big|_0^{\pi} du = 4\pi$$



II) Seja $R = Int\,\Delta$. Então $R \cup \partial R \subset Dom\,\vec{F_2} = \mathbb{R}^3$

$\operatorname{div}\vec{F_2} = z^2$

Orientação:

Em Δ, $\vec{n_c} = \vec{N}$

$R = \{x^2 + \frac{y^2}{4} + \frac{z^2}{9} \le 1\}$

Pelo Teorema de Gauss:

$$\iiint_{\mathcal{Q}} \operatorname{div}\vec{F}\,dx\,dy\,dz = \iint_{\partial\mathcal{Q}} \vec{F}\cdot\vec{N}\cdot d\sigma \rightarrow \iint_{\partial\mathcal{Q}} \vec{F}\cdot\vec{N}\cdot d\sigma = \iiint_{\mathcal{Q}} z^2\,dx\,dy\,dz$$

Passando para coordenadas esféricas: $\begin{cases} x = \rho\cos\theta\,\operatorname{sen}\varphi \\ y = \rho\operatorname{sen}\theta\,\operatorname{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$, $J = 6\rho^2\operatorname{sen}\varphi$

$$\mathcal{Q}_{\rho\theta\varphi} = \{0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$$

$$\iint_{\partial\mathcal{Q}} \vec{F}\cdot\vec{N}\,d\sigma = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 54\rho^4\cos^2\varphi\operatorname{sen}\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{54\rho^5}{5}\cos^2\varphi\operatorname{sen}\varphi\Big|_0^1 d\varphi\,d\theta =$$

$$= \frac{54}{5}\int_0^{2\pi} -\frac{\cos^3\varphi}{3}\Big|_0^{\pi} d\theta = \frac{54}{5}\cdot\frac{2}{3}\cdot 2\pi = \frac{72\pi}{5}$$

$$\therefore \iint_{\partial\mathcal{Q}} \vec{F}\vec{N}\cdot d\sigma = 4\pi + \frac{72\pi}{5} = \frac{82\pi}{5}$$

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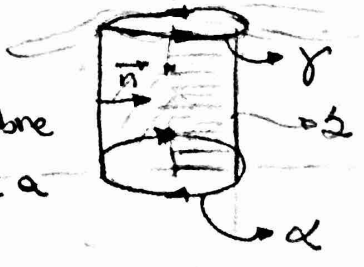
$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{xz}{x^2+y^2} + y & \frac{yz}{x^2+y^2} & e^{\partial^4} \end{vmatrix} = \left(0 - \frac{y}{x^2+y^2},\ 0 + \frac{x}{x^2+y^2},\ -1\right) -$$

$$-\left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ -1\right)$$

$\operatorname{dom}\vec{F} = \mathbb{R}^3 - \{(0,0,z)\}$

Devemos escolher uma superfície que não seja interceptada pelo eixo z e que a curva γ esteja no bordo. Vamos escolher $\mathcal{Q} = \{x^2+y^2 = 1,\ 0 \le z \le \operatorname{arctg}(5+x^4+y^6)\}$



O bordo de $\mathcal{Q}$ é $\partial\mathcal{Q} = \gamma \cup \alpha$, em que $\alpha: x^2+y^2 = 1$

Orientação: vamos escolher $\vec{n}$ para $\mathcal{Q}$ de forma que $\vec{n}$ induza sobre γ o sentido do enunciado (anti-horário). Assim $\vec{n}$ induz sobre α a orientação no sentido horário.

Pelo Teorema de Stokes:

$$\iint_{\Sigma} \operatorname{rot} \vec{F}\,\vec{n}\,d\sigma = \int_{\gamma} \vec{F}\,d\vec{r} + \int_{\alpha} \vec{F}\,d\vec{r} \Rightarrow \int_{\gamma} \vec{F}\,d\vec{r} = \iint_{\Sigma} \operatorname{rot} \vec{F}\,\vec{n}\,d\sigma - \int_{\alpha} \vec{F}\,d\vec{r}$$

(I) (II)

I) Paramotrizando Σ:

Em cilíndricas $\begin{cases} x = \cos\theta \\ y = \operatorname{sen}\theta \\ z = z \end{cases}$

Parâmetros $u = \theta$, $v = z$

$\sigma(u,v) = (\cos u, \operatorname{sen} u, v)$

$\sigma_u = (-\operatorname{sen} u, \cos u, 0)$

$\sigma_v = (0, 0, 1)$

se z não gerasse ambiguidade no domínio de integração $\Sigma = \{0 \le u \le 2\pi,\ 0 \le v \le \sup(z)\}$

$\therefore$ z vai gerar, pois, a normal é perpendicular ao eixo z

$$\sigma_u \wedge \sigma_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\operatorname{sen} u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$= (\cos u, \operatorname{sen} u, 0)$

Orientação: $\sigma_u \wedge \sigma_v$ e $\vec{n}$ têm sentidos opostos

$$-\iint_{\Sigma} (-\operatorname{sen} u, \cos u, -1)(\cos u, \operatorname{sen} u, 0)\,du\,dv = -\iint_{\Sigma} -\operatorname{sen} u \cos u + \operatorname{sen} u \cos u\,du\,dv = 0$$

II) $\alpha(t) = (\operatorname{sen} t, \cos t, 0)$, $t \in [0, 2\pi]$

$\alpha'(t) = (\cos t, -\operatorname{sen} t, 0)$

$\vec{F}(\alpha(t)) = (\cos t, 0, 1)$

$$\int_{\alpha} \vec{F}(\alpha(t))\,\gamma'(t)\,dt = \int_0^{2\pi} -\cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = \frac{\theta}{2} + \frac{\operatorname{sen} 2t}{4}\Big|_0^{2\pi} = \pi$$

$$\int_{\gamma} \vec{F}\,d\vec{r} = -\pi$$