FUJA DO NABO - CÁLCULO 4

1) Variáveis separadas:

$P(x)\,dx + Q(y)\,dy = 0$

$\rightarrow \int P(x)\,dx + \int Q(y)\,dy = 0$

Ex: $y' = -\frac{x}{y} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx$

$\int y\,dy = -\int x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C \rightarrow y^2 = -x^2 + C_1$

$\boxed{y = \pm\sqrt{C_1 - x^2}}$, $C_1 \in \mathbb{R}$

2) Exatas:

$P(x,y)\,dx + Q(x,y)\,dy = 0 \Rightarrow$ não dá p/ separar x de y!

$\rightarrow$ verificar: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\rightarrow$ Existe uma função $\varphi(x,y)$ tal que:

$\frac{\partial \varphi}{\partial x} = P(x,y)$ e $\frac{\partial \varphi}{\partial y} = Q(x,y)$

$\rightarrow$ A solução geral é: $\varphi(x,y) = C$

Ex: $(\underbrace{\operatorname{sen} xy + xy\cos xy}_{P(x,y)})\,dx + (\underbrace{x^2\cos xy}_{Q(x,y)})\,dy = 0$

$\frac{\partial P}{\partial y} = 2x\cos xy + x^2y(-\operatorname{sen} xy)$

$\frac{\partial Q}{\partial x} = 2x\cos xy + x^2y(-\operatorname{sen} xy)$

é exata

$\frac{\partial \varphi}{\partial y} = Q = x^2\cos xy \xRightarrow{\text{integrando}} \varphi = x\operatorname{sen} xy + g(x)$

$\frac{\partial \varphi}{\partial x} = \operatorname{sen} xy + xy\cos xy + g'(x) = P = \operatorname{sen} xy + xy\cos xy$

$\therefore g'(x) \Rightarrow g(x) = 0$

$\therefore$ Solução geral: $\boxed{x\operatorname{sen} xy = C}$

3) Não Exata $\Rightarrow$ fator integrante

$P(x,y)\,dx + Q(x,y)\,dy = 0$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$\rightarrow h(x) = \dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} \Rightarrow \mu(x) = e^{\int h(x)\,dx}$ é fator integrante

ou

$\rightarrow g(y) = \dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P} \Rightarrow \mu(y) = e^{\int -g(y)\,dy}$ é fator integrante

$\rightarrow \mu \cdot (P\,dx + Q\,dy) = 0 \Rightarrow$ Se torna exata

Ex: $(1-xy)\,y' = y^2$

$$(1-xy)\frac{dy}{dx} = y^2 \Rightarrow \underbrace{-y^2}_{P}dx + \underbrace{(1-xy)}_{Q}dy = 0$$

$$\frac{\partial P}{\partial y} = -2y \neq \frac{\partial Q}{\partial x} = -x$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = -y \xrightarrow{\div P} g(y) = 1/y$$

$$\mu(y) = e^{-\int g(y)\,dy} = e^{-\int 1/y\,dy} = e^{-\ln|y|} \Rightarrow \frac{1}{|y|}$$

Voltando p/ a equação original:

$$\left(\frac{1}{y} - x\right)dy + (-y)\,dx = 0 \Rightarrow \text{é exata!}$$

$\rightarrow$ Resolver pelo mesmo método do item anterior

4) Linear (ordem 1)

$$y'(x) + p(x)\,y(x) = q(x)$$

$$\rightarrow y(x) = e^{-\int p(x)dx} \cdot \left[\int e^{\int p(x)dx} \cdot q(x)dx + C \right]$$

Ex: $y' + \frac{y}{x} = x^4$

$p(x) = \frac{1}{x}$, $q(x) = x^4$

$$y(x) = e^{-\int \frac{1}{x}dx} \left(\int e^{\int \frac{1}{x}dx} \cdot x^4 dx + C \right)$$

$$y(x) = \frac{1}{|x|} \left(\int |x| \cdot x^4 dx + C \right)$$

$$\boxed{y(x) = \frac{C}{x} + \frac{x^5}{6}} \quad , \quad C \in \mathbb{R}$$

5) EDOs ordem n

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \ldots + p_1(x)\,y' + p_0(x)\,y = q(x)$$

Homogênea: $q(x) = 0$

→ Coef. ctes:

Equação característica: $\lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0 = 0$

- α raiz real, multiplicidade K:
  $y_1(x) = e^{\alpha x}$, $y_2(x) = x \cdot e^{\alpha x}$, ... , $y_K(x) = x^{K-1} e^{\alpha x}$

- α imaginária ($\alpha = a \pm bi$), multiplicidade K:
  $y_1(x) = e^{\alpha x} \cos bx$
  $y_2(x) = e^{\alpha x} \operatorname{sen} bx$

Ex: $y''' - 3y'' + 4 = 0$

$\lambda^3 - 3\lambda^2 + 4 = 0$

$\lambda = -1$ é solução (por inspeção)

	1	-3	0	4
-1	1	-4	4	0

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \Delta = 0 \Rightarrow \lambda = 2$ (multiplicidade 2)

Base da solução: $\{e^{-x}, e^{2x}, xe^{2x}\}$

Solução geral: $y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$

Ex: $y''' + 2y'' + 2y' = 0$

$\lambda^3 + 2\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 + 2\lambda + 2) = 0$ $\begin{cases} \alpha = 0 \\ \alpha = -1 \pm i \end{cases}$

Base: $\{1;\ e^{-x}\cos x;\ e^{-x}\operatorname{sen} x\}$

Sol. geral: $y(x) = c_1 + c_2 e^{-x}\cos x + c_3 e^{-x}\operatorname{sen} x$

Eq. não homogênea:

- $q(x) = P_n(x)$

  $y_p(x) = x^k(a_n x^n + \dots + a_1 x + a_0)$ $\qquad \lambda = 0$, mult. k

- $q(x) = P_n(x) \cdot e^{\alpha x}$

  $y_p(x) = x^k(a_n x^n + \dots + a_1 x + a_0) \cdot e^{\alpha x}$

  $\lambda = 0$, mult. k

- $q(x) = P_n \cdot e^{\alpha x} \cdot \cos\beta x$ ou $\operatorname{sen}\beta x$

  $y_p(x) = x^k(a_n x^n + \dots + a_1 n + a_0) e^{\alpha x}\cos\beta x +$
  $+ x^k(\quad " \quad) e^{\alpha x}\operatorname{sen}\beta x$

  $\lambda = \alpha \pm \beta i$

Ex: $y''' - 7y'' + 15y' - 9y = e^x$

→ solução homogênea:

$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

$\lambda = 1$ é raiz

$(\lambda - 1)(\lambda^2 - 6\lambda + 9) = 0 \quad \begin{cases} \lambda = 1 \\ \lambda = 3 \text{ (multip. 2)} \end{cases}$

Bases $\{e^x;\ e^{3x};\ xe^{3x}\}$

$y_h(x) = c_1 e^x + c_2 e^{3x} + c_3 x e^{3x}$

→ solução particular:

$$\left.\begin{aligned} y_p &= Ae^x \cdot x \\ y'_p &= Ae^x + Axe^x \\ y''_p &= 2Ae^x + Axe^x \\ y'''_p &= 2Ae^x + Axe^x \end{aligned}\right\} \text{substituir na equação}$$

$\cancel{e^x}(3A + \cancel{Ax} - 14A - \cancel{7Ax} + 15A + \cancel{15Ax} - \cancel{9Ax}) = \cancel{e^x}$

$4A = 1 \Rightarrow A = \frac{1}{4} \Rightarrow y_p(x) = \frac{1}{4} x e^x$

→ solução geral: $y(x) = y_p + y_h$

$$\boxed{y(x) = \frac{1}{4} x e^x + c_1 e^x + c_2 e^{3x} + c_3 x e^{3x}}$$

→ Método de variação dos parâmetros

→ usado quando $q(x)$ for um diferentão ($\ln$, $tg\,x$, $sec\,x$...)

$1 \cdot y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = q(x)$

↑ precisa ser 1

Base da homogênea: $\{y_1, y_2, \ldots, y_n\}$

$y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$

$$\begin{cases} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0 \\ \vdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = q(x) \end{cases}$$

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \qquad W_1, W_2, \cdots W_n$$

$$u_1' = \frac{W_1}{W} \rightarrow u_1 = \int u_1' \, dx$$

$$u_2' = \frac{W_2}{W}$$

$$\vdots$$

$$u_n' = \frac{W_n}{W}$$

Ex: $y'' - 4y' + 4y = e^{2x} \ln x$

1) Homogênea: $\{e^{2x}; x e^{2x}\}$

2) Particular: $y_p = u_1 e^{2x} + u_2' x e^{2x}$

$$\begin{cases} u_1' e^{2x} + u_2' x e^{2x} = 0 \\ u_1' 2e^{2x} + u_2' (2xe^{2x} + e^{2x}) = e^{2x} \ln x \end{cases}$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2xe^{2x} + e^{2x}) \end{vmatrix} \overset{\div e^{2x}}{=} \begin{vmatrix} 1 & x \\ 2 & (2x+1) \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & x \\ \ln x & (2x+1) \end{vmatrix} = x \ln x$$

$$u_1' = \frac{W_1}{W} = -x \ln x \Rightarrow u_1 = -\int x \ln x \, dx = \frac{x^2}{4} - \frac{x^2}{2} \ln x$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 2 & \ln x \end{vmatrix} = \ln x = u_2'$$

$$u_2 = \int \ln x \, dx = x \ln x - x$$

$$y_p(x) = e^{2x}\left(\frac{x^2}{4} - \frac{x^2}{2}\ln x - x^2 \ln x - x^2\right)$$

3) Solução geral: $\boxed{y(x) = e^{2x}\left(A + Bx + \frac{x^2 \ln x}{2} - \frac{3x^2}{4}\right)}$