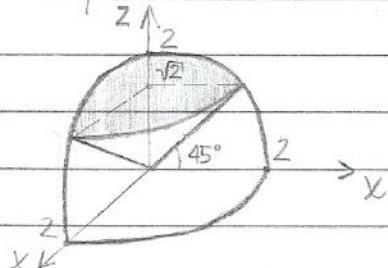


MAT2453 - LISTA 3

1. a) S : parte da superfície $x^2 + y^2 + z^2 = 4$ interior ao cone $z \geq \sqrt{x^2 + y^2}$



$$\begin{cases} x = 2 \sin \varphi \cos \theta \\ y = 2 \sin \varphi \sin \theta \\ z = 2 \cos \varphi \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/4 \end{cases}$$

$$\chi(\varphi, \theta) = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 \cos \varphi), \quad 0 \leq \varphi \leq \pi/4, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial \varphi} = (2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi) = X_\varphi$$

$$\frac{\partial \chi}{\partial \theta} = (2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, 0) = X_\theta$$

$$E = X_\varphi \cdot X_\theta = 4 \cos^2 \varphi \sin^2 \theta + 4 \cos^2 \varphi \cos^2 \theta + 4 \sin^2 \varphi = 4$$

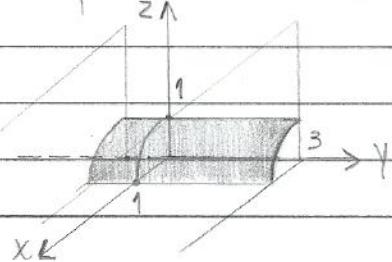
$$F = X_\varphi \cdot X_\theta = 4 \sin^2 \varphi \cos^2 \theta + 4 \sin^2 \varphi \sin^2 \theta = 4 \sin^2 \varphi$$

$$G = X_\varphi \cdot X_\theta = \sin 2\varphi \cdot \sin 2\theta - \sin 2\varphi \cdot \sin 2\theta = 0$$

$$\|X_\varphi \wedge X_\theta\| = \sqrt{EF - G^2} = \sqrt{16 \sin^2 \varphi} = 4 \sin \varphi$$

$$A_S = \iint_S dA = \iint_S \|X_\varphi \wedge X_\theta\| d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/4} 4 \sin \varphi d\varphi d\theta = 4 \int_0^{2\pi} [-\cos \varphi]_0^{\pi/4} d\theta = 4(-\sqrt{2} + 1) \int_0^{2\pi} d\theta = 4\pi(2 - \sqrt{2})$$

b) S : parte do cilindro $x^2 + z^2 = 1$ entre os planos $y = -1$ e $y = 3$



$$\begin{cases} x = \cos \theta \\ y = \sin \theta \\ z = v \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ -1 \leq v \leq 3 \end{cases}$$

$$\chi(\theta, v) = (\cos \theta, \sin \theta, v), \quad 0 \leq \theta \leq 2\pi, \quad -1 \leq v \leq 3$$

$$\frac{\partial \chi}{\partial \theta} = (-\sin \theta, \cos \theta, 0) = \chi_\theta, \quad \frac{\partial \chi}{\partial u} = (0, 0, 1) = \chi_v$$

$$E = \chi_\theta \cdot \chi_\theta = \sin^2 \theta + \cos^2 \theta = 1$$

$$F = \chi_v \cdot \chi_v = 1$$

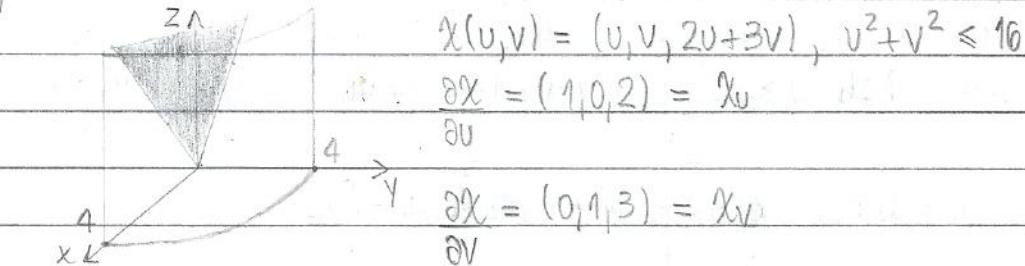
$$G = \chi_\theta \cdot \chi_v = 0$$

$$\|\chi_\theta \wedge \chi_v\| = \sqrt{EF - G^2} = 1$$

$$A_s = \iint_S dA = \iint_S \|\chi_\theta \wedge \chi_v\| d\theta du = -1 \int_0^{2\pi} \int_0^\pi d\theta du = 8\pi$$

c) S: parte do plano $z = 2x + 3y$ interior ao cilindro

$$x^2 + y^2 = 16$$



$$E = \chi_u \cdot \chi_u = 5, \quad F = \chi_u \cdot \chi_v = 10, \quad G = \chi_v \cdot \chi_v = 6$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{14}$$

$$A_s = \iint_S dA = \iint_S \|\chi_u \wedge \chi_v\| dudv = \iint_S \sqrt{14} dudv$$

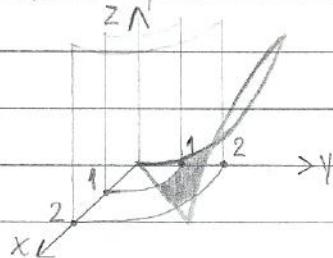
$$\begin{cases} u = r \cos \theta \\ 0 \leq \theta \leq 2\pi \end{cases} \quad J_{uv} = r$$

$$\begin{cases} v = r \sin \theta \\ 0 \leq r \leq 4 \end{cases}$$

$$\therefore A_s = \int_0^{2\pi} \int_0^4 \sqrt{14} r dr d\theta = \sqrt{14} \int_0^{2\pi} [r^2]_0^4 d\theta =$$

$$8\sqrt{14} \int_0^{2\pi} d\theta = 16\pi\sqrt{14}$$

d) S : parte do parabolóide hiperbólico $z = y^2 - x^2$ entre os cilindros $x^2 + y^2 = 1$ e $x^2 + y^2 = 4$



$$\chi(u, v) = (u, v, -u^2 + v^2), \quad 1 \leq u^2 + v^2 \leq 4$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -2u) = \chi_u$$

$$\frac{\partial \chi}{\partial v} = (0, 1, 2v) = \chi_v$$

$$E = \chi_u \cdot \chi_u = 1 + 4u^2$$

$$F = \chi_v \cdot \chi_v = 1 + 4v^2$$

$$G = \chi_u \cdot \chi_v = -4uv$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{1+4v^2+4u^2+16u^4v^4-16u^4v^4} = \sqrt{4u^2+4v^2+1}$$

$$A_S = \iint_S dA = \iint_S \|\chi_u \wedge \chi_v\| du dv = \iint_S \sqrt{4u^2+4v^2+1} du dv$$

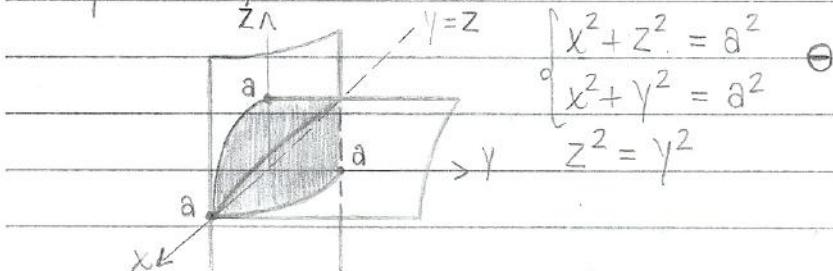
$$\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 1 \leq r \leq 2 \end{cases} \quad J_{uv} = r$$

$$\therefore A_S = \int_0^{2\pi} \int_1^2 \sqrt{4r^2+1} \cdot r dr d\theta = \frac{1}{8} \int_0^{2\pi} \int_1^2 \sqrt{4r^2+1} \cdot 8r dr d\theta =$$

$$\frac{1}{8} \int_0^{2\pi} \left[\frac{2(4r^2+1)^{3/2}}{3} \right]_1^2 d\theta = \frac{1}{12} (17^{3/2} - 5^{3/2}) \int_0^{2\pi} d\theta = \frac{\pi}{6} (17^{3/2} - 5^{3/2})$$

e) S : parte do cilindro $x^2 + z^2 = a^2$ no interior do cilindro

$$x^2 + y^2 = a^2, \quad a > 0$$



$$\chi(u, v) = (-\sqrt{a^2 - v^2}, u, v), \quad 0 \leq u, v \leq a$$

(Obs: χ está parametrizando metade da área hachurada)

$$\frac{\partial \chi}{\partial u} = (0, 1, 0) = \chi_u, \quad \frac{\partial \chi}{\partial v} = \left(-\frac{v}{\sqrt{a^2-v^2}}, 0, 1 \right) = \chi_v$$

$$E = \chi_u, \chi_u = 1$$

$$F = \chi_v, \chi_v = \frac{v^2}{a^2-v^2} + 1$$

$$G = \chi_u, \chi_v = 0$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{\frac{v^2}{a^2-v^2} + 1} = \frac{a}{\sqrt{a^2-v^2}}$$

$$A_S = 16 \iint_S dA = 16 \iint_S \|\chi_u \wedge \chi_v\| du dv = 16 \cdot 2 \int_0^a a \frac{dv}{\sqrt{a^2-v^2}} du$$

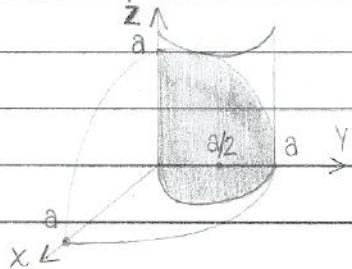
$$= 16 a \int_0^a \int_0^a \frac{1}{a\sqrt{1-(v/a)^2}} dv du = 16 \int_0^a \frac{1}{\sqrt{1-(v/a)^2}} dv du$$

$$\int \frac{1}{\sqrt{1-(v/a)^2}} dv \stackrel{v/a = \cos t}{=} \int \frac{\sin t}{\sqrt{1-\cos^2 t}} dt =$$

$$-at = -a \left(\arccos \left(\frac{v}{a} \right) \right) + K, K \in \mathbb{R}$$

$$\therefore A_S = 16 \int_0^a \left[-a \cdot \arccos \left(\frac{v}{a} \right) \right]_0^a dv = 16 \int_0^a \frac{\pi a}{2} dv = 8\pi a^2$$

b) S : parte da esfera $x^2 + y^2 + z^2 = a^2$ no interior do cilindro
 $x^2 + y^2 = ax$, $a > 0$



$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{a^2 - r^2}), 0 \leq r \leq a \cos \theta, 0 \leq \theta \leq 2\pi$$

(χ parametriza metade de S)

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, -\frac{r}{\sqrt{a^2 - r^2}}) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) = \chi_\theta$$

$$E = \chi_r \cdot \chi_r = \cos^2 \theta + \sin^2 \theta + \frac{r^2}{a^2 - r^2} = 1 + \frac{r^2}{a^2 - r^2} = \frac{a^2}{a^2 - r^2}$$

$$F = \chi_\theta \cdot \chi_r = r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2$$

$$G = \chi_r \cdot \chi_\theta = -\frac{r \sin 2\theta}{2} + \frac{r \cos 2\theta}{2} = 0$$

$$\|\chi_r \wedge \chi_\theta\| = \sqrt{E - G^2} = \frac{ar}{\sqrt{a^2 - r^2}}$$

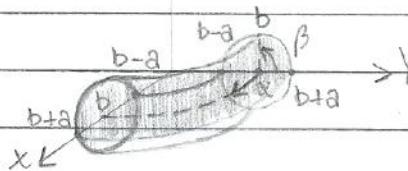
$$A_S = 2 \iint_S dA = 2 \iint_S \|\chi_r \wedge \chi_\theta\| dr d\theta = 2 \int_0^{2\pi} \int_0^{a \cos \theta} ar \frac{dr d\theta}{\sqrt{a^2 - r^2}}$$

$$= -a \int_0^{2\pi} \int_0^{a \cos \theta} -2r \frac{dr d\theta}{\sqrt{a^2 - r^2}} = -a \int_0^{2\pi} [2\sqrt{a^2 - r^2}]_0^{a \cos \theta} d\theta =$$

$$-2a \int_0^{2\pi} a |\sin \theta| - a d\theta = -2a^2 (\int_0^{\pi} \sin \theta d\theta - \int_{\pi}^{2\pi} \sin \theta d\theta) + 2a^2 \int_0^{2\pi} d\theta = 2a^2 [\cos \theta]_0^{\pi} - 2a^2 [\cos \theta]_{\pi}^{2\pi} + 4\pi a^2 = -8a^2 + 4\pi a^2 = -4a^2(\pi - 2)$$

g) S: toro obtido pela rotação de $(x-b)^2 + z^2 = a^2$, $a < b$, em torno do eixo z

2A



$$\begin{cases} x = (b+a\cos\beta)\sin\alpha \\ y = (b+a\cos\beta)\cos\alpha \\ z = a\sin\beta \end{cases} \quad \begin{cases} 0 \leq \beta \leq 2\pi \\ 0 \leq \alpha \leq 2\pi \end{cases}$$

$$\chi(\beta, \alpha) = ((b+a\cos\beta)\sin\alpha, (b+a\cos\beta)\cos\alpha, a\sin\beta), 0 \leq \alpha, \beta \leq 2\pi$$

$$\frac{\partial \chi}{\partial \beta} = (-a\sin\beta\sin\alpha, -a\sin\beta\cos\alpha, a\cos\beta) = \chi_\beta$$

$$\frac{\partial \chi}{\partial \alpha} = ((b+a\cos\beta)\cos\alpha, -(b+a\cos\beta)\sin\alpha, 0) = \chi_\alpha$$

$$E = \chi_\beta \cdot \chi_\beta = a^2 \sin^2\beta \sin^2\alpha + a^2 \sin^2\beta \cos^2\alpha + a^2 \cos^2\beta = a^2$$

$$F = \chi_\alpha \cdot \chi_\alpha = (b+a\cos\beta)^2 \cos^2\alpha + (b+a\cos\beta)^2 \sin^2\alpha = (b+a\cos\beta)^2$$

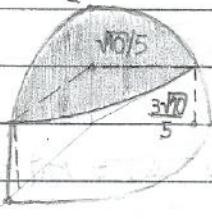
$$G = \chi_\beta \cdot \chi_\alpha = -a(b+a\cos\beta)\sin 2\alpha + a(b+a\cos\beta)\frac{\sin 2\alpha}{2} = 0$$

$$\|\chi_\beta \wedge \chi_\alpha\| = \sqrt{EF - G^2} = a(b+a\cos\beta)$$

$$A_S = \iint_S dA = \iint_S \|\chi_\beta \wedge \chi_\alpha\| d\beta d\alpha = \int_0^{2\pi} \int_0^{2\pi} ab + a^2 \cos\beta d\beta d\alpha = \int_0^{2\pi} [ab\beta + a^2 \sin\beta]_0^{2\pi} d\alpha = 2\pi ab \int_0^{2\pi} d\alpha = 4\pi^2 ab$$

h) S: parte da superfície $x^2 + y^2 + z^2 = 4$ com $z \geq \frac{\sqrt{x^2 + y^2}}{3}$

2A



$$\begin{cases} x = 2\sin\varphi\sin\theta \\ y = 2\sin\varphi\cos\theta \\ z = 2\cos\varphi \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \arctan 3 \end{cases}$$

$$\chi(\varphi, \theta) = (2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 2\cos\varphi), 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \arctan 3$$

$$\frac{\partial \mathbf{x}}{\partial \psi} = (2\cos\psi \sin\theta; 2\cos\psi \cos\theta, -2\sin\psi) = \mathbf{x}_\psi$$

$$\frac{\partial \mathbf{x}}{\partial \theta} = (2\sin\psi \cos\theta, -2\sin\psi \sin\theta, 0) = \mathbf{x}_\theta$$

$$E = \mathbf{x}_\psi \cdot \mathbf{x}_\psi = 4\cos^2\psi \sin^2\theta + 4\cos^2\psi \cos^2\theta + 4\sin^2\psi = 4$$

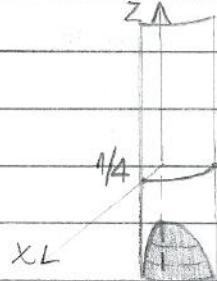
$$F = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = 4\sin^2\psi \cos^2\theta + 4\sin^2\psi \sin^2\theta = 4\sin^2\psi$$

$$G = \mathbf{x}_\psi \cdot \mathbf{x}_\theta = \sin 2\psi \cdot \sin 2\theta - \sin 2\psi \cdot \sin 2\theta = 0$$

$$\|\mathbf{x}_\psi \wedge \mathbf{x}_\theta\| = \sqrt{EF - G^2} = 4\sin\psi$$

$$A_S = \iint_S dA = \iint_S \|\mathbf{x}_\psi \wedge \mathbf{x}_\theta\| d\psi d\theta = \int_0^{2\pi} \int_0^{\arctg^3} 4\sin\psi d\psi d\theta = \\ 4 \int_0^{2\pi} [-\cos\psi]_{0}^{\arctg^3} d\theta = 4 \int_0^{2\pi} 1 - \frac{1}{\sqrt[3]{10}} d\theta = 4\pi(10 - \sqrt[3]{10})$$

i) S: parte do paraleloide elíptico $z = 1 - 2x^2 - y^2$ limitada pelo cilindro elíptico $16x^2 + 4y^2 = 1$



$$\mathbf{x}(u, v) = (u, v, 1 - 2u^2 - v^2), 0 \leq 16u^2 + 4v^2 \leq 1$$

$$\frac{\partial \mathbf{x}}{\partial u} = (1, 0, -4u) = \mathbf{x}_u, \quad \frac{\partial \mathbf{x}}{\partial v} = (0, 1, -2v) = \mathbf{x}_v$$

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 16u^2 + 1$$

$$F = \mathbf{x}_v \cdot \mathbf{x}_v = 4v^2 + 1$$

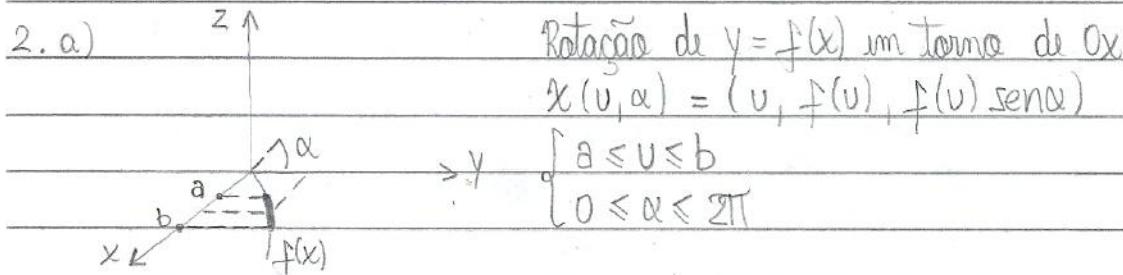
$$G = \mathbf{x}_u \cdot \mathbf{x}_v = 8uv$$

$$\|\mathbf{x}_u \wedge \mathbf{x}_v\| = \sqrt{EF - G^2} = \sqrt{64u^2v^2 + 16u^2 + 4v^2 + 1 - 64u^2v^2} = \\ \sqrt{16u^2 + 4v^2 + 1}$$

$$A_S = \iint_S dA = \iint_S \|\mathbf{x}_u \wedge \mathbf{x}_v\| du dv$$

$$\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{cases} \quad J = \frac{r}{8}$$

$$\therefore A_S = \iint_S dA = \int_0^{2\pi} \int_0^1 \sqrt{2} \cdot r \ dr d\theta = \sqrt{2} \int_0^{2\pi} d\theta = \pi \sqrt{2}$$



$$\frac{\partial \chi}{\partial u} = (1, f'(u), f'(u) \operatorname{sen} \alpha) = \chi_u, \quad \frac{\partial \chi}{\partial \alpha} = (0, 0, f(u) \operatorname{cos} \alpha)$$

$$E = \chi_u \cdot \chi_u = 1 + f'(u)^2 + f'(u)^2 \operatorname{sen}^2 \alpha$$

$$F = \chi_u \cdot \chi_\alpha = f(u)^2 \operatorname{cos}^2 \alpha$$

$$G = \chi_\alpha \cdot \chi_\alpha = f'(u) f(u) \operatorname{sen} 2\alpha$$

$$\|\chi_u \wedge \chi_\alpha\| = \sqrt{EF - G^2} = \left[(f(u) \operatorname{cos} \alpha)^2 + (f(u) f'(u) \operatorname{cos} \alpha)^2 + (f(u) f'(u) \operatorname{sen} \alpha \operatorname{cos} \alpha)^2 - (f(u) f'(u) \operatorname{sen} \alpha \operatorname{cos} \alpha)^2 \right]^{1/2} =$$

$$f(u) |\operatorname{cos} \alpha| \sqrt{1 + f'(u)^2}$$

$$A = \iint_S dA = \iint_S \|\chi_u \wedge \chi_\alpha\| du d\alpha = a \int_a^b \int_0^{2\pi} f(u) |\operatorname{cos} \alpha| \sqrt{1 + f'(u)^2} du d\alpha$$

$$= a \int_a^b \left[-\frac{1}{2} \int_{-\pi/2}^{\pi/2} f(u) \operatorname{cos} \alpha \sqrt{1 + f'(u)^2} du - \frac{1}{2} \int_{\pi/2}^{3\pi/2} f(u) \operatorname{cos} \alpha \sqrt{1 + f'(u)^2} du \right] du$$

$$= a \int_a^b f(u) \sqrt{1 + f'(u)^2} \left([\operatorname{sen} \alpha]_{-\pi/2}^{\pi/2} - [\operatorname{sen} \alpha]_{\pi/2}^{3\pi/2} \right) du = 4a \int_a^b f(u) \sqrt{1 + f'(u)^2} du$$

b) Rotação de $y = f(x)$ em torno de Oy

$$\chi(u, \alpha) = (f(u), u, u \operatorname{sen} \alpha), \quad a \leq u \leq b, \quad 0 \leq \alpha \leq 2\pi$$

Analogamente ao item a), tem-se $A = \iint_S dA = 4a \int_a^b u \sqrt{1 + f'(u)^2} du$

3. a) $\iint_S y \, dA$

$$S: z = x + y^2, 0 \leq x \leq 1, 0 \leq y \leq 2$$

$$\chi(u, v) = (u, v, u + v^2), 0 \leq u \leq 1, 0 \leq v \leq 2$$

$$\frac{\partial \chi}{\partial u} = (1, 0, 1) = \chi_u, \frac{\partial \chi}{\partial v} = (0, 1, 2v) = \chi_v$$

$$E = \chi_u \cdot \chi_u = 2, F = \chi_v \cdot \chi_v = 1 + 4v^2, G = \chi_u \cdot \chi_v = 2v$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EG - F^2} = \sqrt{2 + 8v^2 - 4v^2} = \sqrt{4v^2 + 2}$$

$$\iint_S y \, dA = \iint_S y \|\chi_u \wedge \chi_v\| \, du \, dv = \int_0^1 \int_0^2 y \sqrt{4v^2 + 2} \, dv \, du =$$

$$\int_0^1 \int_0^2 8v \sqrt{4v^2 + 2} \, dv \, du = \int_0^1 [2(4v^2 + 2)^{3/2}]_0^2 \, du =$$

$$\frac{1}{12} (54\sqrt{2} - 2\sqrt{2}) \int_0^1 du = \frac{52\sqrt{2}}{12} = \frac{13\sqrt{2}}{3}$$

b) $\iint_S x^2 \, dA$

$$S: x^2 + y^2 + z^2 = 1$$

$$\chi(\ell, \theta) = (\operatorname{sen} \ell \operatorname{sen} \theta, \operatorname{sen} \ell \cos \theta, \cos \ell), 0 \leq \ell \leq \pi, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial \ell} = (\cos \ell \operatorname{sen} \theta, \cos \ell \cos \theta, -\operatorname{sen} \ell) = \chi_e$$

$$\frac{\partial \chi}{\partial \theta} = (\operatorname{sen} \ell \cos \theta, -\operatorname{sen} \ell \operatorname{sen} \theta, 0) = \chi_o$$

$$E = \chi_e \cdot \chi_e = \cos^2 \ell \operatorname{sen}^2 \theta + \cos^2 \ell \cos^2 \theta + \operatorname{sen}^2 \ell = 1$$

$$F = \chi_e \cdot \chi_o = \operatorname{sen}^2 \ell \cos^2 \theta + \operatorname{sen}^2 \ell \operatorname{sen}^2 \theta = \operatorname{sen}^2 \ell$$

$$G = \chi_e \cdot \chi_o = \operatorname{sen}^2 \ell \operatorname{sen} 2\theta - \operatorname{sen} 2\ell \operatorname{sen} 2\theta = 0$$

$$\|\chi_e \wedge \chi_o\| = \sqrt{EG - F^2} = \operatorname{sen} \ell$$

$$\iint_S x^2 \, dA = \iint_S \operatorname{sen}^2 \ell \operatorname{sen}^2 \theta \|\chi_e \wedge \chi_o\| \, d\ell \, d\theta =$$

$$\int_0^{2\pi} \int_0^\pi \operatorname{sen}^3 \ell \operatorname{sen}^2 \theta \, d\ell \, d\theta = \int_0^{2\pi} \operatorname{sen}^2 \theta \int_0^\pi \operatorname{sen}^3 \ell \, d\ell \, d\theta$$

$$\int \operatorname{sen}^3 \ell \, d\ell = \int \operatorname{sen} \ell (1 - \cos^2 \ell) \, d\ell = \int \operatorname{sen} \ell - \operatorname{sen} \ell \cdot \cos^2 \ell \, d\ell$$

$$= -\cos \ell - \int \operatorname{sen} \ell \cdot \cos^2 \ell \, d\ell \stackrel{t = \cos \ell}{=} -\cos \ell + \int t^2 \, dt =$$

$$-\cos \ell + \frac{t^3}{3} = -\cos \ell + \cos^3 \ell + K, K \in \mathbb{R}$$

$$\therefore \iint_S x^2 dA = \int_0^{2\pi} \sin^2 \theta [-\cos \theta + \cos^3 \theta]_0^{\pi} d\theta =$$

$$\frac{4}{3} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{2}{3} [\theta - \sin \theta \cos \theta]_0^{2\pi} = \frac{4\pi}{3}$$

c) $\iint_S yz dA$

S : parte do plano $z = y+3$ limitada pelo cilindro $x^2+y^2=1$

$$x(u, v) = (u, v, v+3), \quad 0 \leq u^2 + v^2 \leq 1$$

$$\frac{\partial x}{\partial u} = (1, 0, 0) = x_u, \quad \frac{\partial x}{\partial v} = (0, 1, 1) = x_v$$

$$E = x_u \cdot x_v = 1, \quad F = x_v \cdot x_v = 2, \quad G = x_u \cdot x_v = 0$$

$$\|x_u \wedge x_v\| = \sqrt{EF - G^2} = \sqrt{2}$$

$$\iint_S yz dA = \iint_S (v^2 + 3v) \|x_u \wedge x_v\| du dv = \sqrt{2} \iint_S v^2 + 3v du dv$$

$$\begin{cases} u = r \cos \theta & 0 \leq \theta \leq 2\pi \\ v = r \sin \theta & 0 \leq r \leq 1 \end{cases} \quad J\ell = r$$

$$\therefore \iint_S yz dA = \int_0^{2\pi} \int_0^1 (r^2 \sin^2 \theta + 3r \sin \theta) \cdot r dr d\theta =$$

$$\int_0^{2\pi} \int_0^1 r^3 \sin^2 \theta + 3r^2 \sin \theta dr d\theta = \int_0^{2\pi} [r^4 \sin^2 \theta + r^3 \sin \theta]_0^1 d\theta =$$

4

$$\int_0^{2\pi} \int_0^1 \sin^2 \theta + \sin \theta d\theta = [\theta - \sin \theta \cos \theta - \cos \theta]_0^{\pi} = \frac{\pi}{4}$$

d) $\iint_S xy dA$

S : bordo da região limitada pelo cilindro $x^2+z^2=1$

e pelos planos $y=0$ e $x+y=2$

$$x(\theta, v) = (\cos \theta, v, \sin \theta), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq v \leq 2 - \cos \theta$$

$$\frac{\partial x}{\partial \theta} = (-\sin \theta, 0, \cos \theta) = x_\theta, \quad \frac{\partial x}{\partial v} = (0, 1, 0) = x_v$$

$$E = x_\theta \cdot x_v = \sin^2 \theta + \cos^2 \theta = 1$$

$$F = x_v \cdot x_v = 1$$

$$G = x_\theta \cdot x_v = 0$$

$$\|x_\theta \wedge x_v\| = \sqrt{EF - G^2} = 1$$

$$\iint_S xy \, dA = \iint_S u \cos \theta \|x_0 \wedge x_0\| \, du \, d\theta = \int_0^{2\pi} \int_0^{2-\cos \theta} u \cos \theta \, du \, d\theta \\ = \int_0^{2\pi} \cos \theta \cdot (2 - \cos \theta)^2 \, d\theta = 1 \cdot \int_0^{2\pi} \cos \theta (\cos^2 \theta - 4 \cos \theta + 4) \, d\theta =$$

$$\frac{1}{2} \int_0^{2\pi} \cos^3 \theta - 4 \cos^2 \theta + 4 \cos \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \cos \theta (1 - \sin^2 \theta) - 4 \cos^2 \theta$$

$$+ 4 \cos \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} -\cos \theta \cdot \sin^2 \theta - 4 \cos^2 \theta + 5 \cos \theta \, d\theta =$$

$$\frac{-1}{2} \int_0^{2\pi} \cos \theta \cdot \sin^2 \theta \, d\theta - [\theta + \sin \theta \cos \theta]_0^{2\pi} + \frac{5}{2} [\sin \theta]_0^{2\pi} =$$

$$\frac{-1}{2} \left[\sin^3 \theta \right]_0^{2\pi} - 2\pi = -2\pi$$

$$1) \iint_S z(x^2 + y^2) \, dA$$

$$S: \text{hemisfério } x^2 + y^2 + z^2 = 4, z \geq 0$$

$$x(\varphi, \theta) = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 \cos \varphi), 0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial x}{\partial \varphi} = (2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi) = x_\varphi$$

$$\frac{\partial x}{\partial \theta} = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 0) = x_\theta$$

$$E = x_\varphi \cdot x_\varphi = 4 \cos^2 \varphi \sin^2 \theta + 4 \cos^2 \varphi \cos^2 \theta + 4 \sin^2 \varphi = 4$$

$$F = x_\theta \cdot x_\theta = 4 \sin^2 \varphi \cos^2 \theta + 4 \sin^2 \varphi \sin^2 \theta = 4 \sin^2 \varphi$$

$$G = x_\varphi \cdot x_\theta = \cancel{\sin 2\varphi \sin 2\theta} - \cancel{\sin 2\varphi \sin 2\theta} = 0$$

$$\|x_\varphi \wedge x_\theta\| = \sqrt{E - F^2} = 4 \sin \varphi$$

$$\iint_S z(x^2 + y^2) \, dA = \iint_S 2 \cos \varphi \cdot 4 \sin \varphi \cdot \|x_\varphi \wedge x_\theta\| \, d\varphi \, d\theta =$$

$$0 \int_0^{2\pi} \int_0^{\pi/2} 32 \cos \varphi \cdot \sin^3 \varphi \, d\varphi \, d\theta = 32 \cdot 2\pi \left[\frac{\sin^4 \varphi}{4} \right]_0^{\pi/2} \, d\theta = 8 \cdot 2\pi \, d\theta = 16\pi$$

f) $\iint_S xyz \, dA$

S : parte da esfera $x^2 + y^2 + z^2 = 1$ interior ao cone

$$z = \sqrt{x^2 + y^2}$$

$$\chi(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi), 0 \leq \varphi \leq \pi/4, 0 \leq \theta \leq 2\pi$$

$$\partial \chi = (\cos \varphi \cos \theta, \cos \varphi \sin \theta, -\sin \varphi) = \chi_\varphi$$

$$\partial \chi = (\sin \varphi \cos \theta, -\sin \varphi \sin \theta, 0) = \chi_\theta$$

$$E = \chi_\varphi \cdot \chi_\varphi = \cos^2 \varphi \sin^2 \theta + \cos^2 \varphi \cos^2 \theta + \sin^2 \varphi = 1$$

$$F = \chi_\theta \cdot \chi_\varphi = \sin^2 \varphi \cos^2 \theta + \sin^2 \varphi \sin^2 \theta = \sin^2 \varphi$$

$$G = \chi_\theta \cdot \chi_\theta = \cancel{\sin^2 \varphi} \frac{\sin 2\theta}{4} - \cancel{\sin^2 \varphi} \frac{\sin 2\theta}{4} = 0$$

$$\|\chi_\varphi \wedge \chi_\theta\| = \sqrt{EF - G^2} = \sin \varphi$$

$$\iint_S xyz \, dA = \iint_S \sin^2 \varphi \cos \varphi \cdot \sin 2\theta \, \|\chi_\varphi \wedge \chi_\theta\| \, d\varphi \, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \sin^3 \varphi \cos \varphi \cdot \sin 2\theta \, d\varphi \, d\theta = \int_0^{2\pi} \frac{\sin 2\theta}{2} [\sin^4 \varphi]_0^{\pi/4} \, d\theta =$$

$$\frac{1}{16} \int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

g) $\iint_S \sqrt{\frac{2x^2 + 2y^2 - 2}{2x^2 + 2y^2 - 1}} \, dA$

S : parte do hiperbolóide $x^2 + y^2 - z^2 = 1, 1 \leq z \leq 3$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{r^2 - 1}), \sqrt{2} \leq r \leq \sqrt{10}, 0 \leq \theta \leq 2\pi$$

$$\partial \chi = (\cos \theta, \sin \theta, \frac{r}{\sqrt{r^2 - 1}}) = \chi_r$$

$$\partial \chi = (-r \sin \theta, r \cos \theta, 0) = \chi_\theta$$

$$E = \chi_r, \chi_r = \cos^2\theta + \sin^2\theta + \frac{r^2}{r^2-1} = 2r^2-1$$

$$F = \chi_\theta, \chi_\theta = r^2 \sin^2\theta + r^2 \cos^2\theta = r^2$$

$$G = \chi_r, \chi_\theta = -r \cancel{\sin 2\theta} + r \cancel{\sin 2\theta} = 0$$

$$\|\chi_r \wedge \chi_\theta\| = \sqrt{EF - G^2} = r \sqrt{\frac{2r^2-1}{r^2-1}}$$

$$\iint_S \sqrt{2x^2+2y^2-2} \, dA = \iint_S \sqrt{\frac{2r^2-2}{2r^2-1}} \|\chi_r \wedge \chi_\theta\| \, dr \, d\theta =$$

$$\int_{2\pi}^{2\pi} \int_{\sqrt{2}}^{\sqrt{10}} \sqrt{\frac{2(r^2-1)}{2r^2-1}} \cdot r \sqrt{\frac{2r^2-1}{r^2-1}} \, dr \, d\theta = \sqrt{2} \int_{2\pi}^{2\pi} \int_{\sqrt{2}}^{\sqrt{10}} r \, dr \, d\theta =$$

$$\sqrt{2} \int_{2\pi}^{2\pi} \left[\frac{r^2}{2} \right]_{\sqrt{2}}^{\sqrt{10}} \, d\theta = 4\sqrt{2} \int_{2\pi}^{2\pi} \, d\theta = 8\pi\sqrt{2}$$

h) $\iint_S x+1 \, dA$

S : parte do cone $z = \sqrt{x^2+y^2}$ limitada pelo cilindro

$$x^2+y^2=2y$$

$$\chi(r, \theta) = (r \cos\theta, r \sin\theta, r), 0 \leq r \leq 2\sin\theta, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, 1) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin\theta, r \cos\theta, 0) = \chi_\theta$$

$$E = \chi_r, \chi_r = \cos^2\theta + \sin^2\theta + 1 = 2$$

$$F = \chi_\theta, \chi_\theta = r^2 \sin^2\theta + r^2 \cos^2\theta = r^2$$

$$G = \chi_r, \chi_\theta = -r \cancel{\sin 2\theta} + r \cancel{\sin 2\theta} = 0$$

$$\|\chi_r \wedge \chi_\theta\| = \sqrt{EF - G^2} = r\sqrt{2}$$

$$\iint_S x+1 \, dA = \iint (r \cos\theta + 1) \cdot \|\chi_r \wedge \chi_\theta\| \, dr \, d\theta =$$

$$\int_{2\pi}^{2\pi} \int_0^{2\sin\theta} \sqrt{2} (r^2 \cos\theta + r) \, dr \, d\theta = \sqrt{2} \int_{2\pi}^{2\pi} \left[r^3 \cos\theta + r^2 \right]_0^{2\sin\theta} \, d\theta =$$

$$\frac{\sqrt{2} \cdot \int_0^{2\pi} 8 \sin^3 \theta \cos \theta + 2 \sin^2 \theta \, d\theta}{3} = \frac{8\sqrt{2}}{3} [\sin^4 \theta]_0^{2\pi}$$

$$+ \sqrt{2} [\theta - \sin \theta \cos \theta]_0^{2\pi} = 2\pi\sqrt{2}$$

4. a) S: parte do plano $3x + 2y + z = 6$ contida no primeiro octante

$$\delta(x, y, z) = y$$

$$\chi(u, v) = (u, v, -3u - 2v + 6), \quad 0 \leq u \leq 2, \quad 0 \leq v \leq -\frac{3u}{2} + 3$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -3) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -2) = \chi_v$$

$$E = \chi_u \cdot \chi_v = 10, \quad F = \chi_u \cdot \chi_v = 5, \quad G = \chi_u \cdot \chi_v = 6$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{14}$$

$$m = \iint_S \delta(x, y, z) \, dA = \iint_S v \|\chi_u \wedge \chi_v\| \, du \, dv = \int_0^2 \int_{-\frac{3u}{2}+3}^{\frac{3u}{2}+3} \sqrt{14} v \, dv \, du = \sqrt{14} \int_0^2 [v^2]_{-\frac{3u}{2}+3}^{\frac{3u}{2}+3} \, du = \sqrt{14} \int_0^2 9u^2 - 9u + 9 \, du = \frac{9\sqrt{14}}{2} [u^3 - u^2 + u]_0^2 = 3\sqrt{14}$$

b) S: triângulo com vértices $(1, 0, 0), (1, 1, 1), (0, 0, 2)$

$$\delta(x, y, z) = xz$$

$$\vec{a} = (1, 1, 1) - (1, 0, 0) = (0, 1, 1)$$

$$\vec{b} = (0, 0, 2) - (1, 0, 0) = (-1, 0, 2) \rightarrow \text{determinam } S$$

$$P = (1, 0, 0)$$

$$S: \begin{vmatrix} 0 & -1 & x-1 \\ 1 & 0 & y \\ \cancel{x} & 2 & \cancel{z} \end{vmatrix} = 0 \rightarrow 2x - 2 - y + z = 0$$

$$2x - y + z - 2 = 0$$

$$\chi(u, v) = (u, v, -2u + v + 2), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1-u$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -2) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, 1) = \chi_v$$

$$E = \chi_u \cdot \chi_v = 5, \quad F = \chi_u \cdot \chi_v = 2, \quad G = \chi_u \cdot \chi_v = -2$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{6}$$

$$m = \iint_S \delta(x, y, z) dA = \iint_S v(-2v+u+2) \|x_u \wedge x_v\| du dv =$$

$$\sqrt{6} \cdot \int_0^1 \int_{-v}^{v-2} (-2v^2 + uv + 2v) du dv = \sqrt{6} \cdot \int_0^1 [2(v-u^2)v + uv^2]_{-v}^{v-2} du =$$

$$\frac{\sqrt{6}}{2} \int_0^1 [5v^3 - 5v^2 + 5v] du = \frac{\sqrt{6}}{2} [5v^4 - 5v^3 + 5v^2]_0^1 = \frac{5\sqrt{6}}{24}$$

c) S: parte do paraboloide $x = 4 - y^2 - z^2$, $x \geq 0$

$$\delta(x, y, z) = x^2 + y^2$$

$$x(r, \theta) = (4 - r^2, r \cos \theta, r \sin \theta), \quad -2 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial x}{\partial r} = (-2r, \cos \theta, \sin \theta) = x_r$$

$$\frac{\partial x}{\partial \theta} = (0, -r \sin \theta, r \cos \theta) = x_\theta$$

$$E = x_r \cdot x_r = 4r^2 + \cos^2 \theta + \sin^2 \theta = 1 + 4r^2$$

$$F = x_\theta \cdot x_\theta = r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2$$

$$G = x_r \cdot x_\theta = -r \cancel{\sin 2\theta} + r \cancel{\sin 2\theta} = 0$$

$$\|x_r \wedge x_\theta\| = \sqrt{EF - G^2} = |r| \sqrt{1 + 4r^2}$$

$$m = \iint_S \delta(x, y, z) dA = \iint_S [(4 - r^2)^2 + r^2 \cos^2 \theta] \|x_r \wedge x_\theta\| dr d\theta =$$

$$-2 \int_0^2 \int_0^{2\pi} (4 - r^2)^2 |r| \sqrt{1 + 4r^2} + r^2 |r| \sqrt{1 + 4r^2} \cos^2 \theta dr d\theta =$$

$$2\pi [-2 \int_0^2 -r(4 - r^2)^2 \sqrt{1 + 4r^2} dr + \int_0^2 r(4 - r^2)^2 \sqrt{1 + 4r^2} dr] =$$

$$-\frac{\pi}{4} \int_0^2 8r(4 - r^2)^2 \sqrt{1 + 4r^2} dr + \frac{\pi}{4} \int_0^2 8r(4 - r^2)^2 \sqrt{1 + 4r^2} dr$$

$$\int 8r(4 - r^2)^2 \sqrt{1 + 4r^2} dr \stackrel{u=1+4r^2}{du} \int (4 - (u-1))^2 \sqrt{u} du =$$

$$\frac{1}{16} \int (17 - u)^2 \sqrt{u} du = \frac{1}{16} \int u^2 \sqrt{u} - 34u \sqrt{u} + 289 \sqrt{u} du =$$

$$\frac{1}{16} [2u^{7/2} - 34 \cdot 2u^{5/2} + 289 \cdot 2u^{3/2}] = \frac{u^{7/2}}{56} - \frac{17u^{5/2}}{20} + \frac{289u^{3/2}}{24} =$$

$$\frac{(1+4r^2)^3}{56} \sqrt{1+4r^2} - \frac{17}{20} (1+4r^2)^2 \sqrt{1+4r^2} + \frac{289}{24} (1+4r^2) \sqrt{1+4r^2} =$$

$$(1+4r^2)\sqrt{1+4r^2} \left[\frac{(1+4r^2)^2}{56} - \frac{17(1+4r^2)}{20} + \frac{289}{24} \right] + K, K \in \mathbb{R}$$

$$\therefore m = \frac{-\pi}{4} \left[\frac{(1+4r^2)\sqrt{1+4r^2}}{56} \left[\frac{(1+4r^2)^2}{56} - \frac{17(1+4r^2)}{20} + \frac{289}{24} \right] \right]_0^\infty$$

$$+ \frac{\pi}{4} \left[\frac{(1+4r^2)\sqrt{1+4r^2}}{56} \left[\frac{(1+4r^2)^2}{56} - \frac{17(1+4r^2)}{20} + \frac{289}{24} \right] \right]_0^\infty =$$

$$\frac{-\pi}{4} \left[\left(\frac{1}{56} - \frac{17}{20} + \frac{289}{24} \right) - \left(17\sqrt{17} \left[\frac{289}{56} - \frac{289}{20} + \frac{289}{24} \right] \right) \right]$$

$$+ \frac{\pi}{4} \left[\left(17\sqrt{17} \left[\frac{289}{56} - \frac{289}{20} + \frac{289}{24} \right] \right) - \left(\frac{1}{56} - \frac{17}{20} + \frac{289}{24} \right) \right] =$$

$$\frac{-\pi}{2} \cdot \left(\frac{15 - 714 + 10115}{840} \right) + \frac{\pi}{2} \cdot \frac{4913\sqrt{17}}{840} (15 - 42 + 35) =$$

$$\frac{\pi}{1640} (39304\sqrt{17} - 9416) = \frac{\pi}{205} (4913\sqrt{17} - 1177)$$

d) S: parte de $z = x^2 + y^2 + 2xy$ limitada por $x^2 + y^2 = 2$

$$\delta(x, y, z) = \frac{2x^2 + 3y^2}{\sqrt{1+8z}}$$

$$\chi(u, v) = (u, v, (u+v)^2), \quad 0 \leq u^2 + v^2 \leq 2$$

$$\frac{\partial \chi}{\partial u} = (1, 0, 2(u+v)) = \chi_u$$

$$\frac{\partial \chi}{\partial v} = (0, 1, 2(u+v)) = \chi_v$$

$$E = \chi_u \cdot \chi_u = 1 + 4(u+v)^2$$

$$F = \chi_v \cdot \chi_u = 1 + 4(u+v)^2$$

$$G = \chi_u \cdot \chi_v = 4(u+v)^2$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{1 + 8(u+v)^2 + 16(u+v)^4 - 16(u+v)^4} = \sqrt{1 + 8(u+v)^2}$$

$$m = \iint_S \delta(x, y, z) dA = \iint_S \frac{2u^2 + 3v^2}{\sqrt{1+8(u+v)^2}} \|\chi_u \wedge \chi_v\| du dv =$$

$$\iint_S 2U^2 + 3V^2 \, dudv$$

$$\begin{cases} U = r\cos\theta & 0 \leq \theta \leq 2\pi \\ V = r\sin\theta & 0 \leq r \leq \sqrt{2} \end{cases} \quad J\varphi = r$$

$$\therefore m = \int_0^{2\pi} \int_0^{\sqrt{2}} (2r^2 + r^2 \sin^2\theta) \cdot r \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (2r^3 + r^3 \sin^2\theta) \, dr \, d\theta = \int_0^{2\pi} (2 + \sin^2\theta) \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} \, d\theta =$$

$$\int_0^{2\pi} (2 + \sin^2\theta) \, d\theta = [2\theta + \theta - \sin\theta \cos\theta]_0^{2\pi} = 5\pi$$

e) S: parte de $z = \ln(x^2 + y^2)$ limitada pelos cilindros

$$x^2 + y^2 = 1 \text{ e } x^2 + y^2 = e^2$$

$$\delta(x, y, z) = x^2 + y^2$$

$$\chi(r, \theta) = (r\cos\theta, r\sin\theta, \ln r^2), 1 \leq r \leq e, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, \frac{2}{r}) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) = \chi_\theta$$

$$E = \chi_r \cdot \chi_r = \cos^2\theta + \sin^2\theta + \frac{4}{r^2} = 1 + \frac{4}{r^2} = \frac{r^2 + 4}{r^2}$$

$$F = \chi_r \cdot \chi_\theta = r^2 \sin^2\theta + r^2 \cos^2\theta = r^2$$

$$G = \chi_r \cdot \chi_\theta = -\frac{r\sin 2\theta}{2} + \frac{r\cos 2\theta}{2} = 0$$

$$\|\chi_r \wedge \chi_\theta\| = \sqrt{EF - G^2} = \sqrt{r^2 + 4}$$

$$m = \iint_S \delta(x, y, z) \, dA = \iint_S r^2 \|\chi_r \wedge \chi_\theta\| \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_1^e r^2 \sqrt{r^2 + 4} \, dr \, d\theta$$

$$\int r^2 \sqrt{r^2 + 4} \, dr = 2 \int r^2 \sqrt{(r)^2 + 1} \, dr \quad \text{Let } u = r^2 + 1, du = 2r \, dr$$

$$2 \int 4 \operatorname{tg}^2 u \cdot 2 \sec^3 u \, du = 16 \int \operatorname{tg}^2 u \cdot \sec^3 u \, du = 16 \int \sec^5 u - \sec^3 u \, du =$$

(I)

$$I: \int \sec^5 u \, du = \int \sec^2 u \cdot \sec^3 u \, du \rightarrow \begin{cases} f = \sec^2 u \rightarrow f' = \sec^2 u \\ g = \sec^3 u \rightarrow g' = 3\sec^2 u \tan u \end{cases}$$

$$\rightarrow \int \sec^5 u \, du = \sec^3 u \tan u - 3 \int \sec^3 u (\sec^2 u - 1) \, du \rightarrow \int \sec^5 u \, du = \sec^3 u \tan u - 3 \int \sec^5 u \, du + \int \sec^3 u \, du \rightarrow 4 \int \sec^5 u \, du = \sec^3 u \tan u + \frac{1}{4} \int \sec^3 u \, du \rightarrow \int \sec^5 u \, du = \frac{\sec^3 u \tan u}{4} + \frac{\sec u \tan u}{8} + \ln |\sec u + \tan u| + C, \quad C \in \mathbb{R}$$

$$\therefore \int r^2 \sqrt{r^2 + 4} \, dr = 16 \left(\frac{\sec^3 u \tan u}{4} + \frac{\sec u \tan u}{8} + \ln |\sec u + \tan u| \right)$$

$$- \left(\frac{\sec u \tan u}{2} + \ln |\sec u + \tan u| \right) = 4 \sec^3 u \tan u + \frac{3}{2} \left(\sec u \tan u + \ln |\sec u + \tan u| \right) =$$

$$4 \sec^3(\arctan(r/2)) \tan(\arctan(r/2)) + \frac{3}{2} [\sec(\arctan(r/2)) \tan(\arctan(r/2))]$$

$$+ \ln |\sec(\arctan(r/2)) + \tan(\arctan(r/2))| = \frac{4}{2} (r^2 + 4)^{3/2} + \frac{3}{2} [\sqrt{r^2 + 4} \cdot \frac{r}{2}]$$

$$+ \ln |\sqrt{r^2 + 4} + r| = \frac{(r^2 + 4)^{3/2}}{2} + \frac{3r\sqrt{r^2 + 4}}{4} + \frac{3(\ln|\sqrt{r^2 + 4} + r| - \ln 2)}{2}$$

+ C, C ∈ ℝ

$$\therefore \int_0^{2\pi} \int_0^e r^2 \sqrt{r^2 + 4} \, dr \, d\theta = \int_0^{2\pi} \left[\frac{(r^2 + 4)^{3/2}}{2} + \frac{3r\sqrt{r^2 + 4}}{4} \right]_1^e$$

$$+ \frac{3(\ln|\sqrt{r^2 + 4} + r| - \ln 2)}{2} \Big|_1^e = \int_0^{2\pi} \left[\frac{(e^2 + 4)^{3/2}}{2} + \frac{3e\sqrt{e^2 + 4}}{4} \right.$$

$$+ \frac{3(\ln|\sqrt{e^2 + 4} + e| - \ln 2)}{2} - \frac{3(\ln 2)}{2} - \left[\frac{5\sqrt{5}}{2} + \frac{3\sqrt{5}}{4} + \frac{3(\ln|\sqrt{5} + 1| - \ln 2)}{2} \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{(e^2 + 4)^{3/2}}{2} + \frac{3e\sqrt{e^2 + 4}}{4} + \frac{3(\ln(\sqrt{e^2 + 4} + e) - \ln 2)}{2} - \frac{13\sqrt{5}}{4} - \frac{3(\ln(1 + \sqrt{5}))}{2} \right] d\theta$$

$$= \frac{\pi}{2} \left[(e^2 + 4)^{3/2} + \frac{3e\sqrt{e^2 + 4}}{4} + \frac{3(\ln(\sqrt{e^2 + 4} + e) - \ln 2)}{2} - \frac{13\sqrt{5}}{4} \right]$$

$$5. a) \vec{F}(x, y, z) = (x^2y, -3xy^2, 4y^3)$$

Σ : parte da paraboloide $z = 9 - x^2 - y^2$, $z \geq 0$

$$\hat{n}(0,0,9) = \vec{k}$$

$$\chi(u, v) = (u, v, 9 - u^2 - v^2), \quad 0 \leq u^2 + v^2 \leq 9$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -2u) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -2v) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \\ -2u & -2v & \vec{k} \end{vmatrix} = -2u\vec{i} + 2v\vec{j} + \vec{k} = (2u, 2v, 1)$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(u, v)) \cdot (\chi_u \wedge \chi_v) du dv =$$

$$\iint_S (u^2v, -3uv^2, 4v^3) \cdot (2u, 2v, 1) du dv = \iint_S 2u^3v - 6uv^3 + 4v^3 du dv$$

$$= \iint_S 2uv(u^2 - 3v^2) + 4v^3 du dv$$

$$\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 3 \end{cases} \quad J_e = r$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = \int_0^{2\pi} \int_0^3 [r^2 \sin 2\theta (r^2 \cos^2 \theta - 3r^2 \sin^2 \theta) + 4r^3 \sin^3 \theta] r dr d\theta$$

$$\int r dr d\theta = \int_0^{2\pi} \int_0^3 r^5 \sin 2\theta \cos^2 \theta - 3r^5 \sin 2\theta \sin^2 \theta + 4r^4 \sin^3 \theta dr d\theta =$$

$$\int_0^{2\pi} [r^6 \sin \theta \cos^3 \theta - r^6 \sin^3 \theta \cos \theta + 4r^5 \sin^3 \theta] \Big|_0^3 d\theta =$$

$$6 \quad 2 \quad 5$$

$$\int_0^{2\pi} 243 \sin \theta \cos^3 \theta - 729 \sin^3 \theta \cos \theta + 972 \sin^3 \theta d\theta =$$

$$\frac{243}{2} \left[-\frac{\cos^4 \theta}{4} \right]_0^{2\pi} - \frac{729}{2} \left[\frac{\sin^4 \theta}{4} \right]_0^{2\pi} + 972 \int_0^{2\pi} \sin^3 \theta d\theta$$

$$\int \sin^3 \theta d\theta = \int \sin \theta (1 - \cos^2 \theta) d\theta \quad \begin{matrix} u = -\cos \theta \\ du = \sin \theta d\theta \end{matrix} \quad \int 1 - u^2 du =$$

$$\frac{u - u^3}{3} = -\cos \theta + \cos^3 \theta + K, \quad K \in \mathbb{R}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = \frac{972}{5} \left[-\cos \theta + \cos^3 \theta \right]_0^{2\pi} = 0$$

/ /

$$J) \vec{F}(x, y, z) = (x, xy, xz)$$

S : parte do plano $3x + 2y + z = 6$ interior ao cilindro

$$x^2 + y^2 = 1, \text{ com } \hat{n} = \frac{1}{\sqrt{14}} (3, 2, 1)$$

$$\chi(u, v) = (u, v, 6 - 3u - 2v), \quad 0 \leq u^2 + v^2 \leq 1$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -3) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -2) = \chi_v$$

$$E = \chi_u \cdot \chi_v = 10, \quad F = \chi_v \cdot \chi_v = 5, \quad G = \chi_u \cdot \chi_v = 6$$

$$\|\chi_u \wedge \chi_v\| = \sqrt{EF - G^2} = \sqrt{14}$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix} = 1\hat{k} + 3\hat{i} + 2\hat{j} = (3, 2, 1)$$

$$\therefore \hat{n} = \chi_u \wedge \chi_v = \frac{1}{\sqrt{14}} (3, 2, 1)$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(u, v)) \cdot (\chi_u \wedge \chi_v) du dv =$$

$$\iint_S (u, uv, 6u - 3u^2 - 2uv) \cdot (3, 2, 1) du dv = \iint_S 3u + 2uv + 6u - 3u^2 - 2uv du dv$$

$$= \iint_S 9u - 3u^2 du dv$$

$$\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{cases} \quad J_r = r$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = \int_0^{2\pi} \int_0^1 3(3r \cos \theta - r^2 \cos^2 \theta) r dr d\theta =$$

$$3 \int_0^{2\pi} \int_0^1 3r^2 \cos \theta - r^3 \cos^2 \theta dr d\theta = 3 \int_0^{2\pi} [r^3 \cos \theta - r^4 \cos^2 \theta]_0^1 d\theta =$$

4

$$3 \int_0^{2\pi} \cos \theta - \frac{\cos^2 \theta}{4} d\theta = 3 [\sin \theta - \frac{1}{8} (\theta + \sin \theta \cos \theta)]_0^{2\pi} = -\frac{3\pi}{4}$$

c) $\vec{F}(x, y, z) = (-x, -y, z^2)$

S : parte do cone $z = \sqrt{x^2 + y^2}$ entre os planos $z=1$ e $z=2$
 $\hat{n} \cdot \hat{k} < 0$

$$\chi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, 1) = \chi_r, \quad \frac{\partial \chi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos\theta & -r\sin\theta & \hat{i} \\ \sin\theta & r\cos\theta & \hat{j} \\ 0 & 0 & \hat{k} \end{vmatrix} = r\cos^2\theta \hat{i} - r\sin\theta \hat{j}$$

$$-r\cos\theta \hat{i} + r\sin\theta \hat{k} = (-r\cos\theta, -r\sin\theta, r) \rightarrow$$

$$\chi_\theta \wedge \chi_r = -(\chi_r \wedge \chi_\theta)$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \vec{F}(\chi(r, \theta)) \cdot (\chi_\theta \wedge \chi_r) \, dr \, d\theta =$$

$$\iint_S (-r\cos\theta, -r\sin\theta, r^2) \cdot (r\cos\theta, r\sin\theta, -r) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_1^2 -r^2 \cos^2\theta - r^2 \sin^2\theta - r^3 \, dr \, d\theta = \int_0^{2\pi} \int_1^2 -r^2 - r^3 \, dr \, d\theta =$$

$$-\int_0^{2\pi} \left[\frac{r^3}{3} + \frac{r^4}{4} \right]_1^2 \, d\theta = -\int_0^{2\pi} \frac{8}{3} + 4 - \left(\frac{1}{3} + \frac{1}{4} \right) \, d\theta =$$

$$-\frac{73}{12} \int_0^{2\pi} \, d\theta = -\frac{73\pi}{6}$$

d) $\vec{F}(x, y, z) = (x, y, z)$

S : sobre $x^2 + y^2 + z^2 = 9$, com \hat{n} exterior

$$\chi(\varphi, \theta) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi), \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial \varphi} = (3\cos\varphi\cos\theta, 3\cos\varphi\sin\theta, -3\sin\varphi) = \chi_\varphi$$

$$\frac{\partial \chi}{\partial \theta} = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 0) = \chi_\theta$$

$$\chi_\varphi \wedge \chi_\theta = 3 \begin{vmatrix} \cos\varphi\cos\theta & 3\cos\varphi\sin\theta & \hat{i} \\ \cos\varphi\sin\theta & -3\sin\varphi\sin\theta & \hat{j} \\ -\sin\varphi & 0 & \hat{k} \end{vmatrix} = 3[-3\sin 2\varphi \cos^2\theta \hat{k}]$$

$$-3\sin^2\varphi \cos\theta \hat{j} - 3\sin^2\varphi \sin\theta \hat{i} - 3\sin 2\varphi \cos^2\theta \hat{k} =$$

$(-9 \operatorname{sen}^2 \ell \operatorname{sen} \theta, -9 \operatorname{sen}^2 \ell \operatorname{cos} \theta, -9 \operatorname{sen} 2\ell) \rightarrow$ como \hat{n} é exterior,

2

devemos ter $\chi_0 \wedge \chi_\ell = -(\chi_\ell \wedge \chi_0)$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(\ell, \theta)) \cdot (\chi_0 \wedge \chi_\ell) d\ell d\theta =$$

$$0 \int_{2\pi}^{2\pi} \int_{\pi}^{\pi} (3 \operatorname{sen} \ell \operatorname{sen} \theta, 3 \operatorname{sen} \ell \operatorname{cos} \theta, 3 \operatorname{cos} \ell) \cdot (9 \operatorname{sen}^2 \ell \operatorname{sen} \theta, 9 \operatorname{sen}^2 \ell \operatorname{cos} \theta, 9 \operatorname{sen} 2\ell) d\ell d\theta$$

2

$$= 0 \int_{2\pi}^{2\pi} \int_{\pi}^{\pi} 27 \operatorname{sen}^3 \ell \operatorname{sen}^2 \theta + 27 \operatorname{sen}^3 \ell \operatorname{cos}^2 \theta + 27 \operatorname{sen} \ell \operatorname{cos}^2 \ell d\ell d\theta =$$

2

$$27 \int_{2\pi}^{2\pi} \int_{\pi}^{\pi} \operatorname{sen}^3 \ell + \operatorname{sen} \ell \operatorname{cos}^2 \ell d\ell d\theta = 27 \int_{2\pi}^{2\pi} \left[-\operatorname{cos}^3 \ell \right]_{\pi}^{\pi}$$

6

$$+ 0 \int_{2\pi}^{2\pi} \operatorname{sen} \ell (1 - \operatorname{cos}^2 \ell) d\ell d\theta = 27 \int_{2\pi}^{2\pi} \frac{1}{6} - \frac{(-1)}{6} - [\operatorname{cos} \ell + \operatorname{cos}^3 \ell]_{\pi}^{\pi} d\theta =$$

3

$$\frac{27}{3} \int_{2\pi}^{2\pi} \frac{1}{3} - \frac{-1}{3} - \frac{1}{3} - \frac{(1+1)}{3} d\theta = 54 \int_{2\pi}^{2\pi} d\theta = 108\pi$$

2) $\vec{F}(x, y, z) = (-y, x, 3z)$

S : hemisfério $z = \sqrt{16 - x^2 - y^2}$, com $\hat{n}(0,0,4) = \hat{k}$

$\chi(r, \theta) = (r \operatorname{cos} \theta, r \operatorname{sen} \theta, \sqrt{16 - r^2})$, $0 \leq r \leq 4$; $0 \leq \theta \leq 2\pi$

$$\frac{\partial \chi}{\partial r} = (\operatorname{cos} \theta, \operatorname{sen} \theta, \frac{-r}{\sqrt{16 - r^2}}) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \operatorname{sen} \theta, r \operatorname{cos} \theta, 0) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \operatorname{cos} \theta & -r \operatorname{sen} \theta & \hat{i} \\ \operatorname{sen} \theta & r \operatorname{cos} \theta & \hat{j} \\ \frac{-r}{\sqrt{16 - r^2}} & 0 & \hat{k} \end{vmatrix} = r \operatorname{cos}^2 \theta \hat{i} + r^2 \operatorname{sen} \theta \hat{j} - \frac{r^2 \operatorname{cos} \theta}{\sqrt{16 - r^2}} \hat{k}$$

$$+ r^2 \operatorname{cos} \theta \hat{i} + r \operatorname{sen}^2 \theta \hat{k} = \left(r^2 \operatorname{cos} \theta, \frac{r^2 \operatorname{sen} \theta}{\sqrt{16 - r^2}}, r \right)$$

$$E = \chi_r \cdot \chi_r = \operatorname{cos}^2 \theta + \operatorname{sen}^2 \theta + \frac{r^2}{16 - r^2} = \frac{16}{16 - r^2}$$

$$F = \chi_\theta \cdot \chi_\theta = r^2 \operatorname{sen}^2 \theta + r^2 \operatorname{cos}^2 \theta = r^2$$

$$G = \cancel{\chi_r} \cdot \cancel{\chi_\theta} = -\frac{r \sin \theta}{2} + \frac{r \sin \theta}{2} = 0$$

$$\|\chi_r \wedge \chi_\theta\| = \sqrt{F^2 - G^2} = 4r$$

$$\therefore \hat{n} = \chi_r \wedge \chi_\theta = \left(r \cos \theta, r \sin \theta, \frac{\sqrt{16-r^2}}{4} \right)$$

$$z = 4 \rightarrow r = 0 \rightarrow \hat{n}(0,0,4) = \hat{k}$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_D \vec{F}(\chi(r, \theta)) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

$$\int_0^{2\pi} \int_0^4 (-r \sin \theta, r \cos \theta, 3\sqrt{16-r^2}) \cdot (r^2 \cos \theta, r^2 \sin \theta, r) dr d\theta =$$

$$\frac{r^3 \sin 2\theta}{2\sqrt{16-r^2}} + \frac{r^3 \sin 2\theta}{2\sqrt{16-r^2}} + 3r \sqrt{16-r^2} dr d\theta =$$

$$\frac{-3}{2} \int_0^{2\pi} \int_0^4 -2r \sqrt{16-r^2} dr d\theta = \frac{-3}{2} \int_0^{2\pi} [2(16-r^2)^{3/2}]_0^4 d\theta =$$

$$64 \int_0^{2\pi} d\theta = 128\pi$$

$$f) \vec{F}(x, y, z) = (y, 0, -z)$$

S: paraboloide $y = x^2 + z^2$, $0 \leq y \leq 1$ unido ao disco $x^2 + z^2 \leq 1$, $y = 1$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = -1$$

onde S_1 o paraboloide e S_2 o disco:

$$\iint_{S_1 \cup S_2} \vec{F} \cdot \hat{n} dA = \iint_D \operatorname{div} \vec{F} dx dy dz = - \iiint_D dx dy dz$$

$$\begin{cases} x = r \cos \theta \\ 0 \leq \theta \leq 2\pi \end{cases} \quad \begin{cases} y = r \sin \theta \\ 0 \leq r \leq 1 \end{cases} \quad z = 0$$

$$\begin{cases} y = 1 \\ 0 \leq y \leq 1 \end{cases}$$

$$\iint_{S_1 \cup S_2} \vec{F} \cdot \hat{n} dA = - \int_0^{2\pi} \int_0^1 \int_0^1 r dy dr d\theta = - \int_0^{2\pi} [r^2]_0^1 d\theta =$$

$$\frac{-1}{2} \int_0^{2\pi} d\theta = -\pi$$

g) $\vec{F}(x, y, z) = (x, 2y, 3z)$

S : cubo de vértices $(\pm 1, \pm 1, \pm 1)$, \hat{n} exterior

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1+2+3 = 6$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 6 \cdot 1^M \cdot 1^M \cdot 1^M dx dy dz = 48$$

h) $\vec{F}(x, y, z) = (x+y, -2y-1, z)$

S : retângulo de vértices $(1, 0, 1), (1, 0, 0), (0, 1, 0)$ e $(0, 1, 1)$, com $\hat{n} \cdot \hat{j} > 0$

sejam $A = (1, 0, 0), B = (0, 1, 0), C = (1, 0, 1)$:

$$\vec{AB} = (-1, 1, 0)$$

$$\vec{AC} = (0, 0, 1) \rightarrow \text{determinar } S$$

$$P = (0, 1, 1)$$

$$S: \begin{vmatrix} -1 & 0 & x \\ 1 & 0 & y-1 \\ 0 & 1 & z-1 \end{vmatrix} = 0 \rightarrow x+y = 1$$

$$\chi(u, v) = (u, 1-u, v), 0 \leq u \leq 1, 0 \leq v \leq 1$$

$$\frac{\partial \chi}{\partial u} = (1, -1, 0) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 0, 1) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & \hat{i} \\ -1 & 0 & \hat{j} \\ 0 & 1 & \hat{k} \end{vmatrix} = -\hat{i} - \hat{j} = (-1, -1, 0) \rightarrow$$

deve-se usar $\chi_v \wedge \chi_u = -(\chi_u \wedge \chi_v)$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(u, v)) \cdot (\chi_v \wedge \chi_u) du dv =$$

$$\int_0^M \int_0^M (1, 2u-3, v) \cdot (1, 1, 0) du dv = \int_0^M \int_0^M 1+2u-3 du dv =$$

$$\int_0^M [u^2 - 2u]_0^1 dv = -1$$

$$G = \cancel{x_r} \cdot \cancel{x_\theta} = -\frac{r \sin 2\theta}{2} + \frac{r \sin 2\theta}{2} = 0$$

$$\|x_r \wedge x_\theta\| = \sqrt{F^2 - G^2} = \sqrt{16 - r^2}$$

$$\therefore \hat{n} = \cancel{x_r} \wedge \cancel{x_\theta} = \left(\frac{r \cos \theta}{4}, \frac{r \sin \theta}{4}, \frac{\sqrt{16 - r^2}}{4} \right)$$

$$z = 4 \rightarrow r = 0 \rightarrow \hat{n}(0,0,4) = \hat{k}$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_D \vec{F}(x(r,\theta)) \cdot (x_r \wedge x_\theta) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^4 (-r \sin \theta, r \cos \theta, 3\sqrt{16 - r^2}) \cdot (r^2 \cos \theta, r^2 \sin \theta, r) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^4 -r^3 \sin 2\theta + r^3 \sin 2\theta + 3r \sqrt{16 - r^2} \, dr \, d\theta =$$

$$\frac{-3}{2} \int_0^{2\pi} \int_0^4 -2r \sqrt{16 - r^2} \, dr \, d\theta = \frac{-3}{2} \int_0^{2\pi} [2(16 - r^2)^{3/2}]_0^4 \, d\theta =$$

$$64 \int_0^{2\pi} d\theta = 128\pi$$

$$f) \vec{F}(x,y,z) = (y, 0, -z)$$

S: parabolóide $y = x^2 + z^2$, $0 \leq y \leq 1$ unido ao disco $x^2 + z^2 \leq 1$, $y = 1$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = -1$$

onde S_1 o parabolóide e S_2 o disco:

$$\iint_{S_1 \cup S_2} \vec{F} \cdot \hat{n} \, dA = \iiint_V \operatorname{div} \vec{F} \, dx \, dy \, dz = - \iiint_V \, dx \, dy \, dz$$

$$\begin{cases} x = r \cos \theta \\ 0 \leq \theta \leq 2\pi \\ r^2 = y \end{cases} \quad \begin{cases} 0 \leq r \leq 1 \\ 0 \leq y \leq 1 \end{cases} \quad r \, dr \, d\theta \, dy$$

$$\begin{cases} z = r \sin \theta \\ 0 \leq y \leq 1 \end{cases}$$

$$\iint_{S_1 \cup S_2} \vec{F} \cdot \hat{n} \, dA = - \int_0^{2\pi} \int_0^1 \int_0^1 r \, dy \, dr \, d\theta = - \int_0^{2\pi} [r^2]_0^1 \, d\theta =$$

$$-\frac{1}{2} \int_0^{2\pi} d\theta = -\pi$$

$$i) \vec{F}(x, y, z) = (-yz, 0, 0)$$

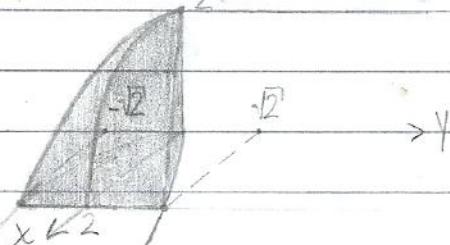
S: parte da esfera $x^2 + y^2 + z^2 = 4$ exterior ao cilindro
 $x^2 + y^2 \leq 1$, com $\hat{n}(2, 0, 0) = \hat{i}$ (exterior)

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 0$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 0$$

$$j) \vec{F}(x, y, z) = (y, z, x)$$

S: parte de $z = \sqrt{4-x^2}$ limitado por $x = y^2$, com $\hat{n}, \hat{i} > 0$



$$\chi(u, v) = (u, v, \sqrt{4-u^2}), 0 \leq u \leq v^2, -\sqrt{2} \leq v \leq \sqrt{2}$$

$$\frac{\partial \chi}{\partial u} = \left(1, 0, \frac{-1}{2\sqrt{4-u^2}} \right) = \chi_u$$

$$\frac{\partial \chi}{\partial v} = (0, 1, 0) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & \hat{i} \\ 0 & 1 & \hat{j} \\ -1/2\sqrt{4-u^2} & 0 & \hat{k} \end{vmatrix} = \hat{i} \wedge \frac{1}{2\sqrt{4-u^2}} \hat{j} = \left(\frac{1}{2\sqrt{4-u^2}}, 0, 1 \right)$$

$$\frac{1}{2\sqrt{4-u^2}} > 0 \rightarrow \hat{n}, \hat{i} > 0$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_D \vec{F}(\chi(u, v)) \cdot (\chi_u \wedge \chi_v) du dv =$$

$$-\sqrt{2} \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{v^2} (v, \sqrt{4-v^2}, u) \cdot \left(\frac{1}{2\sqrt{4-u^2}}, 0, 1 \right) du dv =$$

$$-\sqrt{2} \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{v^2} \frac{v}{2\sqrt{4-u^2}} + u \, du dv = -\sqrt{2} \int_{-\sqrt{2}}^{\sqrt{2}} \left[-v\sqrt{4-u^2} + \frac{u^2}{2} \right]_0^{v^2} dv =$$

$$-\sqrt{2} \int_{-\sqrt{2}}^{\sqrt{2}} -v\sqrt{4-v^2} + \frac{v^4}{2} + 2v \, dv = \frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} -2v\sqrt{4-v^2} \, dv + [v^5 + v^2]_{-\sqrt{2}}^{\sqrt{2}} =$$

$$\frac{1}{2} \left[\frac{2(4\sqrt{v^2})^{3/2}}{3} \right]_{-\sqrt{2}}^{\sqrt{2}} + 4\sqrt{2} + 2 - (-4\sqrt{2} + 2) = 4\sqrt{2}$$

K) $\vec{F}(x, y, z) = (x, y, -2z)$

S : parte do cone $z = \sqrt{x^2 + y^2}$ limitada pelo cilindro

$$x^2 + y^2 = 2x, \text{ com } \hat{n} \cdot \hat{k} < 0$$

$$\chi(u, v) = (u, v, \sqrt{u^2 + v^2}), \quad 0 \leq (x-1)^2 + y^2 \leq 1$$

$$\frac{\partial \chi}{\partial u} = (1, 0, \frac{u}{\sqrt{u^2 + v^2}}) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, \frac{v}{\sqrt{u^2 + v^2}}) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{u}{\sqrt{u^2 + v^2}} & \frac{v}{\sqrt{u^2 + v^2}} & k \end{vmatrix} = 1k - \frac{u}{\sqrt{u^2 + v^2}}$$

$$-\frac{v}{\sqrt{u^2 + v^2}} \hat{j} = \left(\frac{-u}{\sqrt{u^2 + v^2}}, \frac{-v}{\sqrt{u^2 + v^2}}, 1 \right) \rightarrow \text{direc da norma } \chi_v \wedge \chi_u =$$

$$-(\chi_u \wedge \chi_v)$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(u, v)) \cdot (\chi_v \wedge \chi_u) du dv =$$

$$\iint_S (u, v, -2\sqrt{u^2 + v^2}) \cdot \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}, -1 \right) du dv =$$

$$\iint_S \frac{u^2 + v^2}{\sqrt{u^2 + v^2}} + 2\sqrt{u^2 + v^2} du dv = \iint_S 3(u^2 + v^2) du dv = 3 \iint_S \sqrt{u^2 + v^2} du dv$$

$$\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases} \quad \begin{cases} -\pi/2 \leq \theta \leq \pi/2 \\ 0 \leq r \leq 2 \cos \theta \end{cases} \quad J\varphi = r$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \int_0^{\pi/2} \int_0^{2\cos \theta} r \cdot r dr d\theta = \int_0^{\pi/2} [r^3]_0^{2\cos \theta} d\theta =$$

$$\int_0^{\pi/2} \cos^3 \theta d\theta = \int_0^{\pi/2} \cos \theta (1 - \sin^2 \theta) d\theta = \int_0^{\pi/2} [\sin \theta - \frac{\sin^3 \theta}{3}] =$$

$$\frac{8}{3} \cdot \frac{4}{3} = \frac{32}{9}$$

$$6. a) \iint_S xz \, dy \, dz + yz \, dz \, dx + x^2 \, dx \, dy$$

S: semi-esfera $x^2 + y^2 + z^2 = a^2, z \geq 0$, no exterior

Dada $S_1: \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq a^2, z = 0 \}$, com $\hat{n} = -\hat{k}$,

tal que $S \cup S_1$ é fechada

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = z + z = 2z$$

$$\iint_{S \cup S_1} \vec{F} \cdot \hat{n} \, dA = \iint_{S_1} \operatorname{div} \vec{F} \, dx \, dy \, dz \rightarrow \iint_S \vec{F} \cdot \hat{n} \, dA =$$

$$2 \iint_V z \, dx \, dy \, dz - \iint_{S_1} \vec{F} \cdot \hat{n} \, dA$$

(I)

(II)

$$I: \begin{cases} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \\ 0 \leq r \leq a \end{cases} \quad J_r = r^2 \sin \varphi$$

$$\iint_V z \, dx \, dy \, dz = \int_0^{2\pi} \int_0^\pi \int_0^a r \cos \varphi \cdot r^2 \sin \varphi \, dr \, d\varphi \, d\theta =$$

$$\int_0^{2\pi} \int_0^\pi \int_0^a r^3 \sin 2\varphi \left[\frac{r^4}{4} \right]_0^a \, dr \, d\varphi \, d\theta = -a^4 \int_0^{2\pi} \int_0^\pi [\cos^2 \varphi]_0^\pi \, d\varphi \, d\theta = 0$$

$$II: \chi_1(r, \theta) = (r \cos \theta, r \sin \theta, 0), 0 \leq r \leq a, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi_1}{\partial r} = (\cos \theta, \sin \theta, 0) = \chi_r$$

Or

$$\frac{\partial \chi_1}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 1 \\ 0 & 0 & \hat{k} \end{vmatrix} = r \cos^2 \theta \hat{i} + r \sin^2 \theta \hat{j} = (0, 0, r) \quad (\text{divide-se usar } \chi_r \wedge \chi_r)$$

$$\iint_{S_1} \vec{F} \cdot \hat{n} \, dA = \iint_{S_1} \vec{F}(\chi_1(r, \theta)) \cdot (\chi_r \wedge \chi_\theta) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^a (0, 0, r^2 \cos^2 \theta) \cdot (0, 0, -r) \, dr \, d\theta = \int_0^{2\pi} \int_0^a -r^3 \cos^2 \theta \, dr \, d\theta =$$

$$\int_0^{2\pi} \cos^2 \theta \left[-r^4 \right]_0^a \, d\theta = -\frac{a^4}{4} [\theta + \sin \theta \cos \theta]_0^{2\pi} = -\frac{\pi a^4}{4}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, dA = -(-\frac{\pi a^4}{4}) = \frac{\pi a^4}{4}$$

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$$b) \iint_S x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$$

S : parte do plano $x+y+z=2$ no primeiro octante,
com $\hat{n}, \hat{k} > 0$

$$\chi(u, v) = (u, v, 2-u-v), 0 \leq u, v \leq 2$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -1) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -1) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & \hat{i} \\ 0 & 1 & \hat{j} \\ -1 & -1 & \hat{k} \end{vmatrix} = \hat{i} + \hat{j} + \hat{k} = (1, 1, 1)$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \vec{F}(\chi(u, v)) \cdot (\chi_u \wedge \chi_v) \, du \, dv =$$

$$0 \int_0^2 \int_0^2 (u, v, 2-u-v) \cdot (1, 1, 1) \, du \, dv = 0 \int_0^2 \int_0^2 2 \, du \, dv = 8$$

$$c) \iint_S x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$$

S : parte do paraboloide $z = 4 - x^2 - y^2$, $z \geq 2y+1$, com $\hat{n}, \hat{k} \geq 0$

$$4 - x^2 - y^2 = 2y + 1 \rightarrow x^2 + (y+1)^2 = 4$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta - 1, 3 + 2r \sin \theta - r^2), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, 2 \sin \theta - 2r) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 2r \cos \theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & \hat{i} \\ \sin \theta & r \cos \theta & \hat{j} \\ 2 \sin \theta - 2r & 2r \cos \theta & \hat{k} \end{vmatrix} = r \cos^2 \theta \hat{k} + r \sin 2\theta \hat{i}$$

$$+ (2r^2 \sin \theta - 2r \sin^2 \theta) \hat{j} + (2r^2 \cos \theta - r \sin 2\theta) \hat{i} + r \sin^2 \theta \hat{k} - 2r \cos^2 \theta \hat{j} =$$

$$(2r^2 \cos \theta, 2r^2 \sin \theta - 2r, r)$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \vec{F}(\chi(r, \theta)) \cdot (\chi_r \wedge \chi_\theta) \, dr \, d\theta =$$

$$0 \int^{2\pi} \int^2 (r \cos \theta, r \sin \theta - 1, 3 + 2r \sin \theta - r^2) \cdot (2r^2 \cos \theta, 2r^2 \sin \theta - 2r, r) \, dr \, d\theta$$

$$= 0 \int^{2\pi} \int^2 2r^3 \cos^2 \theta + 2r^3 \sin^2 \theta - 2r^2 \sin \theta - 2r^2 \sin \theta + 2r + 3r + 2r^2 \sin \theta$$

$$-r^3 \, dr \, d\theta = 0 \int^{2\pi} \int^2 r^3 + 5r - 2r^2 \sin \theta \, dr \, d\theta =$$

$$0 \int^{2\pi} [r^4 + 5r^2 - 2r^3 \sin \theta]^2 \, d\theta = 0 \int^{2\pi} 14 - 16 \sin \theta \, d\theta =$$

4 2 3

3

$$\frac{[140 + 1600\pi]}{3} \int_0^{\pi} = 28\pi$$

7. Seja o gráfico de $f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ de classe \mathcal{C}^1 , $n, k \geq 0$

$$\vec{F}(x, y, z) = (P, Q, R)$$

$$X(x, y) = (x, y, f(x, y)), \quad x, y \in D$$

$$\frac{\partial X}{\partial x} = (1, 0, \frac{\partial f}{\partial x})$$

$$\frac{\partial X}{\partial y} = (0, 1, \frac{\partial f}{\partial y})$$

$$X_x \wedge X_y = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cancel{\frac{\partial f}{\partial x}} & \cancel{\frac{\partial f}{\partial y}} & \cancel{k} \end{vmatrix} = 1k - \cancel{\frac{\partial f}{\partial x}} \hat{i} - \cancel{\frac{\partial f}{\partial y}} \hat{j} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$$

$$\iint_D \vec{F} \cdot \hat{n} \, dA = \iint_D \vec{F}(X(x, y)) \cdot (X_x \wedge X_y) \, dx \, dy =$$

$$\iint_D (P, Q, R) \cdot (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) \, dx \, dy = \iint_D -P \frac{\partial f}{\partial x} - Q \frac{\partial f}{\partial y} + R \, dx \, dy.$$

8. $\iint_S y^2 z^2 \, dy \wedge dz + x \, dz \wedge dx + y \, dx \wedge dy$

S: parte da $z^2 = x^2 + 2y^2$ entre os planos $z=1$ e $z=y+3$, $n, k \leq 0$
 $\text{div } \vec{F} = \nabla \cdot \vec{F} = 0$

$$z=1 \Rightarrow x^2 + 2y^2 = 1$$

$$z=y+3 \Rightarrow x^2 + 2y^2 = y^2 + 6y + 9 \Rightarrow x^2 + y^2 - 6y + 9 = 18 \Rightarrow$$

$$x^2 + (y-3)^2 = 18$$

$$\text{cônico } S_1: X_1(r, \theta) = (r \cos \theta, r \sin \theta, 1), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi,$$

$$\text{e } S_2: X_2(r, \theta) = (r \cos \theta, r \sin \theta + 3, r \sin \theta), \quad 0 \leq r \leq 3\sqrt{2},$$

$0 \leq \theta \leq 2\pi$, $n, k \geq 0$ tal que $S_1 \cup S_2$ é fechada e orientada segundo a normal interior

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$$\iint_S \vec{F} \cdot \hat{n} dA = - \iiint_V \operatorname{div} \vec{F} dx dy dz \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = - \iint_D \vec{F} \cdot \hat{n}_1 dA - \iint_D \vec{F} \cdot \hat{n}_2 dA$$

(I) (II)

I: $\chi_1(r, \theta) = (\frac{r \cos \theta}{\sqrt{2}}, \frac{r \sin \theta}{\sqrt{2}}, 1), 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$

$$\frac{\partial \chi_1}{\partial r} = (\cos \theta, \frac{\sin \theta}{\sqrt{2}}, 0) = \chi_{1r}, \quad \frac{\partial \chi_1}{\partial \theta} = (-r \sin \theta, \frac{r \cos \theta}{\sqrt{2}}, 0) = \chi_{1\theta}$$

$$\chi_{1r} \wedge \chi_{1\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & i \\ \frac{\sin \theta}{\sqrt{2}} & \frac{r \cos \theta}{\sqrt{2}} & j \\ 0 & 0 & k \end{vmatrix} = \left(\frac{r \cos^2 \theta + r \sin^2 \theta}{\sqrt{2}} \right) \hat{k} = (0, 0, \frac{r}{\sqrt{2}})$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_D \vec{F}(\chi_1(r, \theta)) \cdot (\chi_{1r} \wedge \chi_{1\theta}) dr d\theta =$$

$$0 \int_0^{2\pi} \int_0^1 (r^2 \sin^2 \theta, r \cos \theta, r \sin \theta) \cdot (0, 0, r) dr d\theta = 0 \int_0^{2\pi} \int_0^1 r^2 \sin \theta dr d\theta =$$

$$\frac{1}{2} \int_0^{2\pi} [r^3 \sin^3 \theta]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} r^3 \sin^3 \theta d\theta = 0$$

II: $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta + 3, r \sin \theta), 0 \leq r \leq 3\sqrt{2}, 0 \leq \theta \leq 2\pi$

$$\frac{\partial \chi_2}{\partial r} = (\cos \theta, \sin \theta, \sin \theta) = \chi_{2r}, \quad \frac{\partial \chi_2}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = \chi_{2\theta}$$

$$\chi_{2r} \wedge \chi_{2\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & i \\ \sin \theta & r \cos \theta & j \\ \sin \theta & r \cos \theta & k \end{vmatrix} = r \cos^2 \theta \hat{i} + r \sin 2\theta \hat{j}$$

$$-r \sin^2 \theta \hat{j} - r \sin 2\theta \hat{i} + r \sin^2 \theta \hat{k} - r \cos^2 \theta \hat{j} = (0, -r, r)$$

$$\iint_S \vec{F} \cdot \hat{n}_2 dA = \iint_D \vec{F}(\chi_2(r, \theta)) \cdot (\chi_{2r} \wedge \chi_{2\theta}) dr d\theta =$$

$$0 \int_0^{2\pi} \int_0^{3\sqrt{2}} ((r \sin \theta - 3)^2 r^2 \sin^2 \theta, r \cos \theta, r \sin \theta + 3) \cdot (0, -r, r) dr d\theta = 0 \int_0^{2\pi} \int_0^{3\sqrt{2}} -r^2 \cos^2 \theta - r^2 \sin^2 \theta - 3r dr d\theta = 0 \int_0^{2\pi} [(r \sin \theta - r \cos \theta)r^3 + 3r^2]_0^{3\sqrt{2}} dr d\theta =$$

$$0 \int_0^{2\pi} \frac{54\sqrt{2} \sin \theta}{3} - \frac{54\sqrt{2} \cos \theta}{3} + 54 \Big|_0^{2\pi} = 54\pi$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = -54\pi$$

9. $\iint_S e^{z^2} \ln(z+y) dy dz + (x^2+z^2) dz dx + z dx dy$

S : parte do paraboloide $z = 4 - x^2 - y^2$ limitada pelo plano $z = y - 4$, com $x, y \geq 0$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = 1$$

Seja S_1 a "tampa" do paraboloide, com $x, y \leq 0$, de modo que $S \cup S_1$ é fechada e orientada segundo a normal exterior

$$x_1(r, \theta) = (r \cos \theta, r \sin \theta, r \sin \theta - 4), \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial x_1}{\partial r} = (\cos \theta, \sin \theta, \sin \theta) = x_{1,r}, \quad \frac{\partial x_1}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = x_{1,\theta}$$

$$x_{1,r} \wedge x_{1,\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 0 \\ \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \cos^2 \theta \hat{k} + \frac{r \sin 2\theta}{2} \hat{i} - r \sin^2 \theta \hat{j}$$

$$-\cancel{r \sin 2\theta} \hat{i} + r \sin^2 \theta \hat{k} - r \cos^2 \theta \hat{j} = (0, -r, r) \rightarrow \text{dive-se usar } x_{1,\theta} \wedge x_{1,r}$$

$$\iint_{S \cup S_1} \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = \iiint_V dx dy dz \quad (\text{I})$$

(II)

$$\text{I: } \begin{cases} x = r \cos \theta & 0 \leq \theta \leq 2\pi \\ y = r \sin \theta & 0 \leq r \leq 2 \\ z = z & r \sin \theta - 4 \leq z \leq 4 - r^2 \end{cases} \quad \text{Je} = r$$

$$\begin{aligned} \iiint_V dx dy dz &= \int_0^{2\pi} \int_0^2 \int_{r \sin \theta - 4}^{4 - r^2} r dz dr d\theta = \\ &= \int_0^{2\pi} \int_0^2 r (4 - r^2 - r \sin \theta + 4) dr d\theta = \int_0^{2\pi} \int_0^2 (4r^2 - r^4 - r^3 \sin \theta) dr d\theta = \\ &= \int_0^{2\pi} \left[4r^3 - r^5 - r^4 \sin \theta \right]_0^2 d\theta = \int_0^{2\pi} (12 - 8 \sin \theta) d\theta = 24\pi \end{aligned}$$

$$\text{II: } \iint_{S_1} \vec{F} \cdot \hat{n} dA = \iint_{S_1} \vec{F}(x_1(r, \theta)) \cdot (x_{1,\theta} \wedge x_{1,r}) dr d\theta =$$

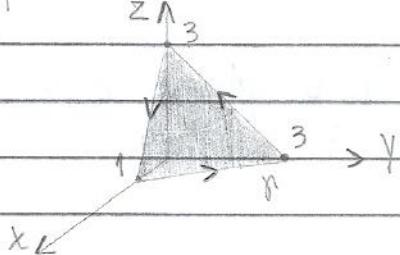
$$\begin{aligned} &\int_0^{2\pi} \int_0^2 (e^{(r \sin \theta - 4)^2} \ln(2r \sin \theta - 4), r^2 - 8r \sin \theta + 16, r \sin \theta - 4) \cdot (0, r, -r) dr d\theta = \\ &\int_0^{2\pi} \int_0^2 r^3 - 8r^2 \sin \theta + 16r - r^2 \sin \theta + 4r dr d\theta = \int_0^{2\pi} [r^4 - 3r^3 \sin \theta + 10r^2]_0^2 d\theta = 4 \end{aligned}$$

$$\int_0^{2\pi} 4 - 24 \sin \theta + 40 d\theta = 88\pi$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = 24\pi - 88\pi = -64\pi$$

10. a) $\vec{F}(x, y, z) = (xz, 2xy, 3xy)$

γ : fronteira da parte do plano $3x + y + z = 3$ contida no primeiro octante, orientada de modo que sua projeção em Oxy seja percorrida no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R - \partial Q}{\partial y - \partial z} \right) \hat{i} + \left(\frac{\partial P - \partial R}{\partial z - \partial x} \right) \hat{j} + \left(\frac{\partial Q - \partial P}{\partial x - \partial y} \right) \hat{k} =$$

$$3x\hat{i} + (x - 3y)\hat{j} + 2y\hat{k} = (3x, x - 3y, 2y)$$

$$\chi(u, v) = (u, v, 3 - 3u - v), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 3 - 3u$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -3) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -1) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -3 & -1 & 1 \end{vmatrix} = 1\hat{k} + 3\hat{i} + 1\hat{j} = (3, 1, 1)$$

$$\int_S \vec{F} \cdot d\vec{S} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} \, dA = \iint_S (\text{rot } \vec{F}(\chi(u, v))) \cdot (\chi_u \wedge \chi_v) \, du \, dv =$$

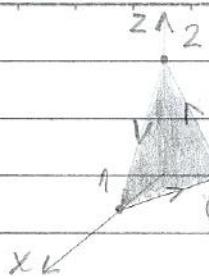
$$\int_0^1 \int_{3-3u}^{3-3u} (3u, u - 3v, 2v) \cdot (3, 1, 1) \, dv \, du = \int_0^1 \int_{3-3u}^{3-3u} 10u - v \, dv \, du =$$

$$\int_0^1 [10uv - \frac{v^2}{2}]_{3-3u}^{3-3u} \, du = \int_0^1 30u - 30u^2 - \frac{9u^2}{2} - 9u + \frac{9}{2} \, du =$$

$$\int_0^1 -\frac{69u^2}{2} + 21u + \frac{9}{2} \, du = [-23u^3 + 21u^2 + \frac{9u}{2}]_0^1 = 7$$

b) $\vec{F}(x, y, z) = (z^2 + e^{x^2}, y^2 + \ln(1+y^2), xy + \sin z^3)$

γ : fronteira do triângulo com vértices $(1, 0, 0)$, $(0, 1, 0)$ e $(0, 0, 2)$, orientada de modo que sua projeção em Oxy seja percorrida no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R - \partial Q}{\partial y - \partial z} \right) \hat{i} + \left(\frac{\partial P - \partial R}{\partial z - \partial x} \right) \hat{j} + \left(\frac{\partial Q - \partial P}{\partial x - \partial y} \right) \hat{k} =$$

$$x\hat{i} + (2z-y)\hat{j} = (x, 2z-y, 0)$$

$$\vec{B} = (0,0,2) - (1,0,0) = (-1,0,2)$$

$$\vec{C} = (0,1,0) - (1,0,0) = (-1,1,0) \rightarrow \text{determinant } S$$

$$P = (1,0,0)$$

$$\Sigma: \begin{vmatrix} -1 & -1 & x+1 \\ 0 & 1 & y \\ 2 & 0 & 2 \end{vmatrix} = 0 \rightarrow -2 - 2y - 2x + 2 = 0$$

$$\chi(u, v) = (u, v, -2u - 2v + 2), 0 \leq u \leq 1, 0 \leq v \leq 1-u$$

$$\frac{\partial \chi}{\partial u} = (1, 0, -2) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -2) = \chi_v$$

$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & \hat{i} \\ 0 & 1 & \hat{j} \\ -2 & -2 & \hat{k} \end{vmatrix} = \hat{i} + 2\hat{j} + 2\hat{k} = (2, 2, 1)$$

$$\int_S \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}(\chi(u, v))) \cdot (\chi_u \wedge \chi_v) du dv =$$

$$\int_0^1 \int_{4-u}^{1-u} (u, -4u - 4v + 4 - v, 0) \cdot (2, 2, 1) dv du =$$

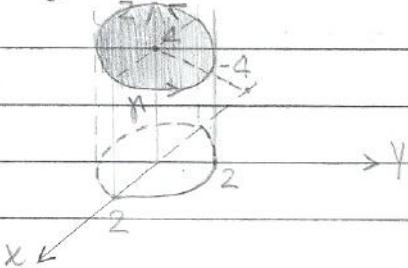
$$\int_0^1 \int_{4-u}^{1-u} (u, -4u - 5v + 4, 0) \cdot (2, 2, 1) dv du = \int_0^1 \int_{4-u}^{1-u} -6u - 10v + 8 dv du =$$

$$\int_0^1 [(8-6u)v - 5v^2]_{4-u}^{1-u} du = \int_0^1 8 - 8u - 6u + 6u^2 - 5u^2 + 10u - 5 du =$$

$$\int_0^1 u^2 - 4u + 3 du = [u^3 - 2u^2 + 3u]_0^1 = \frac{4}{3}$$

c) $\vec{F}(x, y, z) = (2z + \operatorname{sen}x^3, 4x, 5y + \operatorname{sen}(\operatorname{sen}z^2))$

$\begin{cases} x^2 + y^2 = 4 \\ z = x + 4 \end{cases} \rightarrow$ projeção em Oxy percorrida no sentido anti-horário



$$\operatorname{rot} \vec{F} = \left(\frac{\partial R - \partial Q}{\partial y - \partial z} \right) \hat{i} + \left(\frac{\partial P - \partial R}{\partial z - \partial x} \right) \hat{j} + \left(\frac{\partial Q - \partial P}{\partial x - \partial y} \right) \hat{k} =$$

$$5\hat{i} + 2\hat{j} + 4\hat{k} = (5, 2, 4)$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, r \cos \theta + 4), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, \cos \theta) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, -r \sin \theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & \hat{i} \\ \sin \theta & r \cos \theta & \hat{j} \\ \cos \theta & -r \sin \theta & \hat{k} \end{vmatrix} = r \cos^2 \theta \hat{k} - r \sin^2 \theta \hat{i} - r \sin 2\theta \hat{j}$$

$$-r \cos^2 \theta \hat{i} + r \sin^2 \theta \hat{k} + r \sin 2\theta \hat{j} = (-r, 0, r)$$

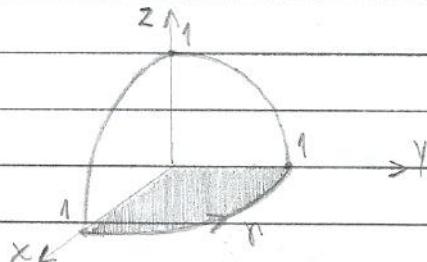
$$\int_S \vec{F} \cdot d\vec{s} = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 (5, 2, 4) \cdot (-r, 0, r) dr d\theta = \int_0^{2\pi} \int_0^2 -r dr d\theta = \int_0^{2\pi} [-r^2]_0^2 d\theta =$$

$$-2 \int_0^{2\pi} d\theta = -4\pi$$

d) $\vec{F}(x, y, z) = (x + \cos x^3, y, x^2 - y^2 + z^{100})$

Γ : frontaria da parte do paraboloide $z = 1 - x^2 - y^2$ contida no primeiro octante, com projeção im Oxy percorrida no sentido anti-horário.



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$-2y \hat{i} - 2x \hat{j} = (-2y, -2x, 0)$$

$$X(r, \theta) = (r \cos \theta, r \sin \theta, 0), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi/2$$

$$\frac{\partial X}{\partial r} = (\cos \theta, \sin \theta, 0) = X_r, \quad \frac{\partial X}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) = X_\theta$$

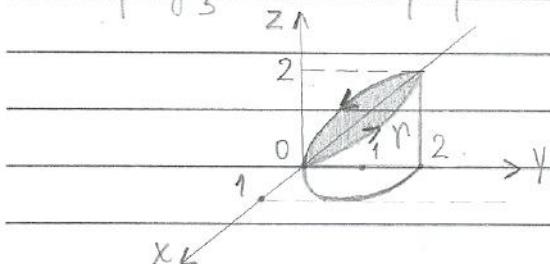
$$X_r \wedge X_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & \hat{i} \\ \sin \theta & r \cos \theta & \hat{j} \\ 0 & 0 & \hat{k} \end{vmatrix} = \hat{k} (r \cos^2 \theta + r \sin^2 \theta) = (0, 0, r)$$

$$\int_{\Gamma} \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}(X(r, \theta))) \cdot (X_r \wedge X_\theta) dr d\theta =$$

$$\int_0^{\pi/2} \int_0^1 (-2r \sin \theta, -2r \cos \theta, 0) \cdot (0, 0, r) dr d\theta = 0$$

e) $\vec{F}(x, y, z) = (y+z, 2x + (1+y^2)^{20}, x+y+z)$

Γ : intersecção do cilindro $x^2 + y^2 = 2y$ com o plano $z = y$, com projeção im Oxy percorrida em sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$\hat{i} + \hat{k} = (1, 0, 1)$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta + 1, r \sin \theta + 1), 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, \sin \theta) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & \hat{i} \\ \sin \theta & r \cos \theta & \hat{j} \\ \sin \theta & r \cos \theta & \hat{k} \end{vmatrix} = r \cos^2 \theta \hat{k} + \frac{r \sin 2\theta}{2} \hat{i}$$

$$-r \sin^2 \theta \hat{j} - \frac{r \sin 2\theta}{2} \hat{i} + r \sin^2 \theta \hat{k} - r \cos^2 \theta \hat{j} = (0, -r, r)$$

$$\int_S \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

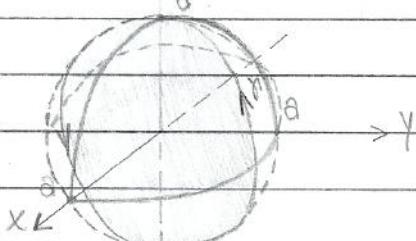
$$= \int_0^{2\pi} \int_0^1 (1, 0, 1) \cdot (0, -r, r) dr d\theta = \int_0^{2\pi} \int_0^1 r dr d\theta = \int_0^{2\pi} [r^2]_0^1 d\theta =$$

$$\frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

b) $\vec{F}(x, y, z) = (y, z, x)$

$\hat{n} : \int_0^2 x^2 + y^2 + z^2 = 8^2 \rightarrow$ projeção em Oxy percorrida

$$x + y + z = 0 \quad \text{no sentido anti-horário}$$



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = -1 \hat{i}$$

$$-1 \hat{j} - 1 \hat{k} = (-1, -1, -1)$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, -r \cos \theta - r \sin \theta), 0 \leq r \leq a, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, -\cos \theta - \sin \theta) = \chi_r$$

or

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \sin \theta - r \cos \theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 0 \\ -\cos \theta - \sin \theta & r \sin \theta - r \cos \theta & 0 \end{vmatrix} =$$

$$r \cos^2 \theta \hat{k} + (r \sin^2 \theta - r \sin 2\theta) \hat{i} + (r \sin^2 \theta + r \sin 2\theta) \hat{j} + (r \sin 2\theta + r \cos^2 \theta) \hat{i}$$

$$+ r \sin^2 \theta \hat{k} + (r \cos^2 \theta - r \sin 2\theta) \hat{j} = (r, r, r)$$

$$\int_S \vec{F} \cdot d\vec{S} = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

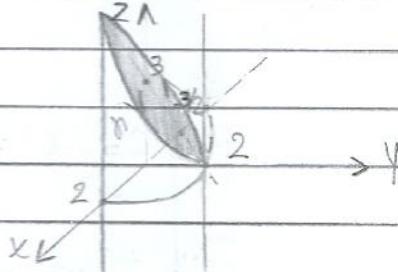
$$\int_0^{2\pi} \int_0^a (-1, -1, -1) \cdot (r, r, r) dr d\theta = -3 \int_0^{2\pi} \int_0^a r dr d\theta = -3a^2 \int_0^{2\pi} d\theta =$$

$$-3\pi a^2$$

by L. Leibniz

11. a) $\vec{F}(x, y, z) = (yz, xz + \ln(1+y^4), zy)$

$r: \begin{cases} x^2 + y^2 = 4 \\ z = 2x + 3 \end{cases} \rightarrow$ projeção em Oxy percorrida no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$(z-x)\hat{i} + y\hat{j} + (z-z)\hat{k} = (z-x, y, 0)$$

$$\chi(r, \theta) = (r\cos\theta, r\sin\theta, 2r\cos\theta + 3), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, 2\cos\theta) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r\sin\theta, r\cos\theta, -2r\sin\theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos\theta & -r\sin\theta & \hat{i} \\ \sin\theta & r\cos\theta & \hat{j} \\ 2\cos\theta & -2r\sin\theta & \hat{k} \end{vmatrix} = r\cos^2\theta \hat{k} - 2r\sin^2\theta \hat{i}$$

$$-r\sin 2\theta \hat{j} - 2r\cos^2\theta \hat{i} + r\sin^2\theta \hat{k} + r\sin 2\theta \hat{j} = (-2r, 0, r)$$

$$\int_{\Omega} \vec{F} \cdot d\vec{s} = \iint_{\Sigma} (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_{\Sigma} (\text{rot } \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

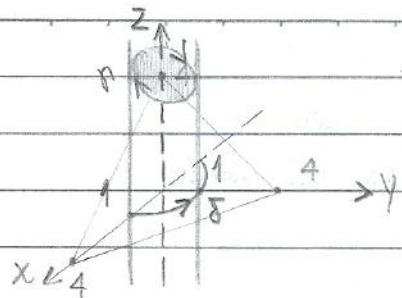
$$\int_0^{2\pi} \int_0^2 (r\cos\theta + 3, r\sin\theta, 0) \cdot (-2r, 0, r) dr d\theta = \int_0^{2\pi} \int_0^2 -2r^3 \cos\theta - 6r^2 dr d\theta =$$

$$= \int_0^{2\pi} [-2r^3 \cos\theta - 3r^2]_0^2 d\theta = \int_0^{2\pi} -16\cos\theta - 12 d\theta = -24\pi$$

b) $\vec{F}(x, y, z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, \frac{z^3}{1+z^2} \right)$

$r: \begin{cases} x^2 + y^2 = 1 \\ x + y + z = 4 \end{cases} \rightarrow$ projeção em Oxy percorrida no sentido horário

$$D\vec{F} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, z), z \in \mathbb{R}\}$$



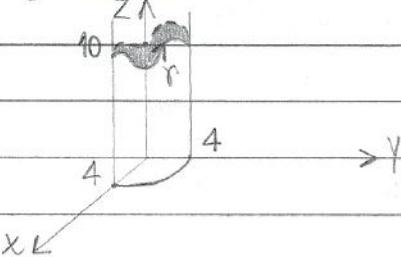
$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = (0, 0, 0)$$

Diga δ a circunferência $\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1, z = 0\}$ percorrida no sentido anti-horário e Σ a superfície tal que $\partial\Sigma = r \cup \delta$, com \hat{n} interior.

$$\int_{r \cup \delta} \vec{F} \cdot d\vec{s} = \iint_{\Sigma} (\text{rot } \vec{F}) \cdot \hat{n} dA \rightarrow \int_r \vec{F} \cdot d\vec{s} = - \int_{\delta} \vec{F} \cdot d\vec{s} = -2\pi$$

c) $\vec{F}(x, y, z) = (2xz^3 + y, x^2y^2, 3x^2z^2)$

$r: \begin{cases} z = 5\sin\theta + 10 \\ x^2 + y^2 = 16 \end{cases} \rightarrow$ projeção em Oxy percorrida no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$(6xz^2 - 6xz^2) \hat{j} + (2xy^2 - 1) \hat{k} = (0, 0, 2xy^2 - 1)$$

$$x(r, \theta) = (r\cos\theta, r\sin\theta, 5\sin(r\sin\theta) + 10), 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial x}{\partial r} = (\cos\theta, \sin\theta, 5\sin\theta \cdot \cos(r\sin\theta)) = x_r$$

ou

$$\frac{\partial x}{\partial \theta} = (-r\sin\theta, r\cos\theta, 5\cos\theta \cdot \cos(r\sin\theta)) = x_\theta$$

ou

$$X_{r\theta} = \begin{vmatrix} \cos\theta & -r\sin\theta & 1 \\ \sin\theta & r\cos\theta & 1 \\ 5\sin\theta\cos(r\sin\theta) & 5r\cos^2\theta\cos(r\sin\theta) & \vec{k} \end{vmatrix} =$$

$$\frac{r\cos^2\theta\vec{k} + 5r\sin^2\theta\cos(r\sin\theta)\vec{i} - 5r\sin^2\theta\cos(r\sin\theta)\vec{j}}{2}$$

$$-5r\sin^2\theta\cos(r\sin\theta)\vec{i} + r\sin^2\theta\vec{k} - 5r\cos^2\theta\cos(r\sin\theta)\vec{j} =$$

$$\frac{(0, -5r\cos(r\sin\theta), r)}{2}$$

$$\int_S \vec{F} \cdot d\vec{S} = \int_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \int_S (\text{rot } \vec{F}(X(r, \theta))) \cdot (X_{r\theta} X_{\theta}) dr d\theta =$$

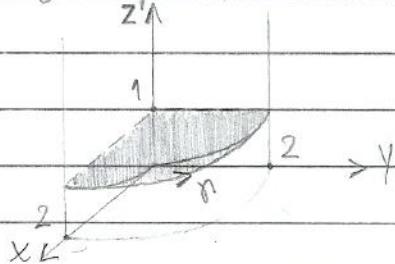
$$0 \int_0^{2\pi} \int_0^r (0, 0, 2r^3 \sin^2\theta \cos\theta - 1) \cdot (0, -5r\cos(r\sin\theta), r) dr d\theta =$$

$$0 \int_0^{2\pi} \int_0^r 2r^4 \sin^2\theta \cos\theta - r dr d\theta = 0 \int_0^{2\pi} [2r^5 \sin^2\theta \cos\theta - r^2]_0^r d\theta =$$

$$0 \int_0^{2\pi} 2048 \sin^2\theta \cos\theta - 8 d\theta = [2048 \frac{\sin^3\theta}{3} - 8\theta]_0^{2\pi} = -16\pi$$

d) $\vec{F}(x, y, z) = (x - y^2, x - z + \frac{y^2}{2 + \sin y}, y)$

$\gamma: \begin{cases} 4z = x^2 + y^2 \\ x^2 + y^2 = 4 \end{cases} \rightarrow \text{projecção em Oxy percorrida}$
 no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R - \partial Q}{\partial y} \right) \vec{i} + \left(\frac{\partial P - \partial R}{\partial z} \right) \vec{j} + \left(\frac{\partial Q - \partial P}{\partial x} \right) \vec{k} =$$

$$(1+1)\vec{i} + (1+2y)\vec{k} = (2, 0, 1+2y)$$

$$\chi(r, \theta) = \left(r \cos \theta, r \sin \theta, \frac{r^2}{4} \right), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = \left(\cos \theta, \sin \theta, \frac{r}{2} \right) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = \left(-r \sin \theta, r \cos \theta, 0 \right) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & i \\ \sin \theta & r \cos \theta & j \\ r/2 & 0 & k \end{vmatrix} = \frac{r \cos^2 \theta}{2} \hat{i} - \frac{r^2 \sin \theta}{2} \hat{j}$$

$$\frac{-r^2 \cos \theta}{2} \hat{i} + \frac{r^2 \sin \theta}{2} \hat{k} = \left(-\frac{r^2 \cos \theta}{2}, -\frac{r^2 \sin \theta}{2}, r \right)$$

$$\int_S \vec{F} \cdot d\vec{s} = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

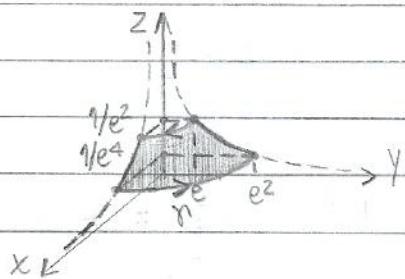
$$\int_0^{2\pi} \int_0^2 (2, 0, 1+2r \sin \theta) \cdot \left(-\frac{r^2 \cos \theta}{2}, -\frac{r^2 \sin \theta}{2}, r \right) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 -r^2 \cos \theta + r + 2r^2 \sin \theta dr d\theta = \int_0^{2\pi} [(2 \sin \theta - \cos \theta) r^3 + r^2]_0^2 d\theta =$$

$$\int_0^{2\pi} 16 \sin \theta - 8 \cos \theta + 2 d\theta = 4\pi$$

i) $\vec{F}(x, y, z) = (e^x \sin y, e^x \cos y - z, y)$

r : fronteira da superfície obtida pela rotação do gráfico de $z = \frac{1}{y^2}$ em torno do eixo Oz , $e \leq y \leq e^2$



$$\operatorname{rot} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$(1+1)\hat{i} + (e^x \cos y - e^x \cos y) \hat{k} = (2, 0, 0)$$

$$\chi(u, v) = (u, v, \frac{1}{v^2}), \quad e \leq u, v \leq e^2$$

$$\frac{\partial \chi}{\partial u} = (1, 0, 0) = \chi_u, \quad \frac{\partial \chi}{\partial v} = (0, 1, -\frac{2}{v^3}) = \chi_v$$

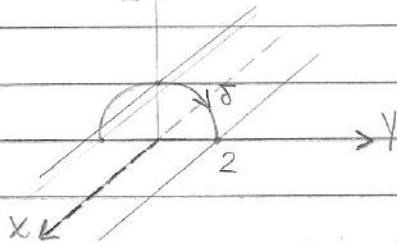
$$\chi_u \wedge \chi_v = \begin{vmatrix} 1 & 0 & \hat{i} \\ 0 & 1 & \hat{j} \\ 0 & -\frac{2}{v^3} & \hat{k} \end{vmatrix} = \frac{1}{v^3} \hat{i} + 2 \frac{1}{v^3} \hat{j} = \left(0, \frac{2}{v^3}, 1\right)$$

$$\int_S \vec{F} \cdot d\vec{s} = \iint_S \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(\chi(u, v))) \cdot (\chi_u \wedge \chi_v) du dv =$$

$$\int_0^{e^2} \int_0^{e^2} (2, 0, 0) \cdot \left(0, \frac{2}{v^3}, 1\right) du dv = 0$$

b) $\vec{F}(x, y, z) = (\cos(1+x^2), \frac{-z}{y^2+z^2} + e^{y^4}, \frac{y}{y^2+z^2})$

$\therefore \begin{cases} y^2+z^2=4 \\ x=y+z \end{cases} \rightarrow \text{projeção em Oyz percorrida no sentido anti-horário}$



$$D_F = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (x, 0, 0), \forall x \in \mathbb{R}\}$$

$$\operatorname{rot} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = (0, 0, 0)$$

deja δ a circunferência $\{(x, y, z) \in \mathbb{R}^3 : y^2+z^2=4, x=0\}$ percorrida no sentido horário e S a superfície tal que $dS = \vec{n} \wedge \vec{ds}$, com \vec{n}

$$\int_{\text{proj}} \vec{F} \cdot d\vec{s} = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = 0 \rightarrow \int_S \vec{F} \cdot d\vec{s} = - \int_{\delta} \vec{F} \cdot d\vec{s} = 2\pi$$

$$12. \text{ a) } \int_P (z+y^2) dx + (y^2+1) dy + (\ln(z^2+1) + y) dz$$

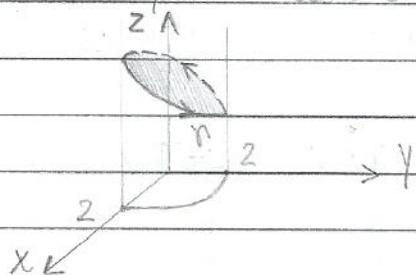
$$\vec{r}(t) = (2\cos t, 2\sin t, 10 - 2\sin t), \quad 0 \leq t \leq 2\pi$$

$$\vec{n}(t) = (-2\sin t, 2\cos t, -2\cos t)$$

$$\text{rot } \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y} \right) \hat{k} =$$

$$1\hat{i} + 1\hat{j} - 2y\hat{k} = (1, 1, -2y)$$

Se se analisar \vec{r} , vê-se que ela provém da intersecção do cilindro $x^2 + y^2 = 4$ com o plano $z = 10 - 2y$.



$$\chi(r, \theta) = (r\cos\theta, r\sin\theta, 10 - r\sin\theta), \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, -\sin\theta) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r\sin\theta, r\cos\theta, -r\cos\theta) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos\theta & -r\sin\theta & \hat{i} \\ \sin\theta & r\cos\theta & \hat{j} \\ -r\sin\theta & -r\cos\theta & \hat{k} \end{vmatrix} = r\cos^2\theta \hat{i} + r\sin^2\theta \hat{j}$$

$$-r\sin 2\theta \hat{i} + r\sin 2\theta \hat{i} + r\sin^2\theta \hat{k} + r\cos^2\theta \hat{j} = (0, r, r)$$

$$\int_P \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

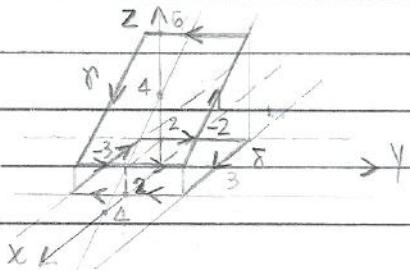
$$\int_0^{2\pi} \int_0^2 (1, 1, -2r\sin\theta) \cdot (0, r, r) dr d\theta = \int_0^{2\pi} \int_0^2 r - 2r^2 \sin\theta dr d\theta =$$

$$\int_0^{2\pi} [r^2 - 2r^3 \sin\theta]_0^2 d\theta = \int_0^{2\pi} [2 - 16\sin\theta] d\theta = [20 + 16\cos\theta]_0^{2\pi} = 4\pi$$

/ /

$$b) \int_P -\frac{(y-1)}{x^2+(y-1)^2} dx + \left(\frac{x}{x^2+(y-1)^2} + z \right) dy + \sin z dz$$

γ : intersecção do prisma de faces $x = -2, x = 2, y = -3, y = 3$ com o plano $z = 4 - x$, com projeção em Oxy percorrida no sentido anti-horário



$$D\vec{F} = \{(\vec{x}, \vec{y}, \vec{z}) \in \mathbb{R}^3 : (\vec{x}, \vec{y}, \vec{z}) \neq (0, 1, 2), \vec{z} \in \mathbb{R}\}$$

$$\text{rot } \vec{F} = \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y} \right) \hat{k} = (-1, 0, 0)$$

Seja δ o retângulo $\{(\vec{x}, \vec{y}, \vec{z}) \in \mathbb{R}^3 : -2 \leq \vec{x} \leq 2, -3 \leq \vec{y} \leq 3, \vec{z} = 0\}$ percorrido em sentido horário e S a superfície tal que $\partial S = \gamma \cup \delta$, com \vec{n} interior.

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \vec{n} dA \rightarrow \int_{\gamma} \vec{F} \cdot d\vec{s} = \iint_S (-1, 0, 0) \cdot \vec{n} dA - \int_{\delta} \vec{F} \cdot d\vec{s} \quad (I)$$

(II)

I: $S = S_1 \cup S_2 \cup S_3 \cup S_4$, onde S_1 é a face frontal, S_2 a face posterior, S_3 a face esquerda e S_4 a face direita

$$\begin{aligned} \iint_S (-1, 0, 0) \cdot \vec{n} dA &= \iint_{S_1} (-1, 0, 0) \cdot \vec{n}_1 dA + \iint_{S_2} (-1, 0, 0) \cdot \vec{n}_2 dA \\ &+ \iint_{S_3} (-1, 0, 0) \cdot \vec{n}_3 dA + \iint_{S_4} (-1, 0, 0) \cdot \vec{n}_4 dA = \iint_{S_1} (-1, 0, 0) \cdot (-1, 0, 0) dA \\ &+ \iint_{S_2} (-1, 0, 0) \cdot (1, 0, 0) dA + \iint_{S_3} (-1, 0, 0) \cdot (0, 1, 0) dA + \iint_{S_4} (-1, 0, 0) \cdot (0, -1, 0) dA \\ &= A_{S_1} - A_{S_2} = 6.2 - 6.6 = -24 \end{aligned}$$

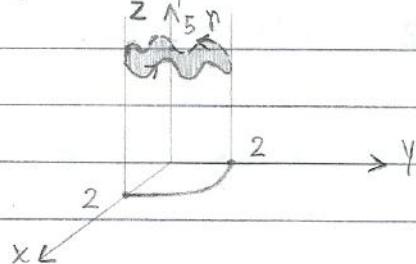
II: seja $\alpha(t) = (r \sin t, r \cos t, 1, 0)$, $0 \leq t \leq 2\pi$, $r \rightarrow 0$ e

D a superfície tal que $\partial D = \gamma \cup \alpha$

$$\begin{aligned} \int_{\partial D} \vec{F} \cdot d\vec{s} &= \iint_D (\text{rot } \vec{F}) \cdot \vec{n} dx dy \rightarrow \int_{\gamma} \vec{F} \cdot d\vec{s} = - \int_{\alpha} \vec{F} \cdot d\vec{s} = 2\pi \\ \therefore \int_{\gamma} \vec{F} \cdot d\vec{s} &= -24 + 2\pi \end{aligned}$$

c) $\int_P (xy + y) dx + 2yz dy + y^2 dz$

$r: \begin{cases} x^2 + y^2 = 4 \\ z = \cos(y^2) + 5 \end{cases} \rightarrow$ projeção em Oxy percorrida
no sentido anti-horário



$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$(2y - 2y) \hat{i} - (x+1) \hat{k} = (0, 0, -x-1)$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, \cos(r^2 \sin^2 \theta) + 5), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, 2r \sin^2 \theta \cdot \sin(r^2 \sin^2 \theta)) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, r^2 \sin 2\theta \cdot \sin(r^2 \sin^2 \theta)) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 2r \sin^2 \theta \cdot \sin(r^2 \sin^2 \theta) & r^2 \sin 2\theta \cdot \sin(r^2 \sin^2 \theta) & 0 \end{vmatrix} =$$

$$r \cos^2 \theta \hat{k} + r^2 \sin 2\theta \cdot \sin \theta \cdot \sin(r^2 \sin^2 \theta) \hat{i} - 2r^2 \sin^3 \theta \cdot \sin(r^2 \sin^2 \theta) \hat{j} - 2r^2 \sin^2 \theta \cos \theta \cdot \sin(r^2 \sin^2 \theta) \hat{i} + r \sin^2 \theta \hat{k} - r^2 \sin 2\theta \cos \theta \cdot \sin(r^2 \sin^2 \theta) \hat{j} =$$

$$(0, -2r^2 \sin \theta \cdot \sin(r^2 \sin^2 \theta) [\sin^2 \theta + \cos \theta], r)$$

$$\int_P \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 (0, 0, -r \cos \theta - 1) \cdot (0, -2r^2 \sin \theta \cdot \sin(r^2 \sin^2 \theta) [\sin^2 \theta + \cos \theta], r) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 -r^2 \cos \theta - r dr d\theta = \int_0^{2\pi} [-r^3 \cos \theta - r^2]_0^2 d\theta =$$

$$\int_0^{2\pi} -8 \cos \theta - 2 d\theta = -4\pi$$

13. γ : curva simples, fechada e plana

$\hat{n} = (a, b, c)$. vetor normal ao plano que contém γ

S : região plana limitada por γ ($\gamma = \partial S$)

$$A_S = \frac{1}{2} \int_{\gamma} (bz - cy) dx + (cx - az) dy + (ay - bx) dz$$

dizia $\vec{F}(x, y, z) = (bz - cy, cx - az, ay - bx)$, pelo teorema de Stokes tem-se:

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot (a, b, c) dA$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \cdot (a, b, c) dA = \iint_S (a - (-a), b - (-b), c - (-c)) \cdot (a, b, c) dA$$

$$\equiv 2 \iint_S (a^2 + b^2 + c^2) dA = 2A_S \quad \therefore A_S = \frac{1}{2} \int_{\gamma} \vec{F} \cdot d\vec{s}$$

14. a) $\int_P z^2 + \operatorname{sen}x^3 dx + y^2 dy + xy + e^{z^3} dz$

$\Gamma: \begin{cases} x^2 + y^2 + z^2 = 9 \\ z = 2x \end{cases}$, percorrida de $(0, 3, 0)$ a $(0, -3, 0)$, $z \geq 0$

$$x^2 + y^2 + 4x^2 = 9 \rightarrow 5x^2 + y^2 = 9$$

$$\chi(r, \theta) = \left(\frac{r \cos \theta}{\sqrt{5}}, \frac{r \sin \theta}{\sqrt{5}}, 2r \cos \theta \right), 0 \leq r \leq 3, -\pi \leq \theta \leq \pi$$

$$\frac{\partial \chi}{\partial r} = \left(\frac{\cos \theta}{\sqrt{5}}, \frac{\sin \theta}{\sqrt{5}}, 2\cos \theta \right) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = \left(-\frac{r \sin \theta}{\sqrt{5}}, \frac{r \cos \theta}{\sqrt{5}}, -2r \sin \theta \right) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta / \sqrt{5} & -\sin \theta / \sqrt{5} & \hat{i} \\ \sin \theta & \cos \theta & \hat{j} \\ 2\cos \theta / \sqrt{5} & -2r \sin \theta / \sqrt{5} & \hat{k} \end{vmatrix} = \frac{r \cos^2 \theta}{\sqrt{5}} \hat{i} - \frac{2r \sin^2 \theta}{\sqrt{5}} \hat{j} - \frac{r \sin 2\theta}{\sqrt{5}} \hat{k}$$

$$\frac{-r \sin 2\theta}{5} \hat{j} - \frac{2r \cos^2 \theta}{\sqrt{5}} \hat{i} + \frac{r \sin^2 \theta}{\sqrt{5}} \hat{k} + \frac{r \sin 2\theta}{5} \hat{j} = \left(-2r, 0, r \right).$$

$$\operatorname{rot} \vec{F} = \left(\frac{\partial R - \partial Q}{\partial y}, \frac{\partial P - \partial R}{\partial z}, \frac{\partial Q - \partial P}{\partial x} \right) =$$

$$x \hat{i} + (2z - y) \hat{j} = (x, 2z - y, 0)$$

$$\int_P \vec{F} \cdot d\vec{s} = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(\chi(r, \theta))) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

$$\int_0^{2\pi} \int_0^3 (r \cos \theta, 4r \cos \theta - r \sin \theta, 0) \cdot (-2r, 0, r) dr d\theta =$$

$$\int_0^{2\pi} \int_0^3 -2r^2 \cos \theta dr d\theta = -2 \int_0^{2\pi} [r^3 \cos \theta]_0^3 d\theta = -2 \int_0^{2\pi} \cos^4 \theta d\theta = 0$$

b) $\vec{F}(x, y, z) = \vec{F}_1(x, y, z) + \vec{F}_2(x, y, z)$

$$\left(\frac{-y}{(x^2 + y^2)^2}, \frac{x}{(x^2 + y^2)^2}, 0 \right) + (e^{x^4}, \operatorname{sen}(\operatorname{sen} y^5), \ln(1+z^4))$$

$$\gamma: \begin{cases} x^2 + y^2 = 4 \\ z = 5 + \ln(1+y^2) \end{cases} \rightarrow \begin{array}{l} \text{projeção em } Oxy \text{ percorrida} \\ \text{no sentido horário} \end{array}$$

$D\vec{F} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, z), \forall z \in \mathbb{R}\}$

$D\vec{F} = \mathbb{R}^3$

$\text{rot } \vec{F}_1 = \vec{0}$

$$\text{rot } \vec{F}_2 = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = \vec{0}$$

diga δ a circunferência $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4, z = 0\}$ percorrida em sentido anti-horário e S a superfície tal que $ds = r d\alpha$, com \hat{n} exterior

$$\int_{\text{proj}_{xy}} \vec{F} \cdot d\vec{s} = \iint_S (\text{rot } \vec{F}) \cdot \hat{n} dA = \iint_S (\text{rot } \vec{F}_1 + \text{rot } \vec{F}_2) \cdot \hat{n} dA = 0$$

$$\Rightarrow \int_S \vec{F} \cdot d\vec{s} = - \int_{\text{proj}_{xy}} \vec{F} \cdot d\vec{s}$$

diga $\alpha(t) = (r \sin t, r \cos t, 0)$, $r \rightarrow 0$, $0 \leq t \leq 2\pi$, de modo que a região D seja tal que $\partial D = \delta \cup \alpha$

$$\int_{\text{proj}_{xy}} \vec{F} \cdot d\vec{s} = \iint_D (\text{rot } \vec{F}) \cdot \hat{k} dx dy \rightarrow \int_D \vec{F} \cdot d\vec{s} = - \int_D \vec{F} \cdot d\vec{s} = 2\pi$$

$$\therefore \int_S \vec{F} \cdot d\vec{s} = -2\pi$$

c) $\int_S -2x + y \, dx + z^2 \cos y - y \, dy + x^2 + z \, dz$

$r : \begin{cases} z = \sqrt{x^2 + 2y^2} \\ z = y+1 \end{cases} \rightarrow \text{projeto em Oxy percorrida}$

$\begin{cases} z = y+1 \\ z = \sqrt{x^2 + 2y^2} \end{cases} \text{em sentido horário}$

$$x^2 + 2y^2 = y^2 + 2y + 1 \rightarrow x^2 + y^2 - 2y + 1 = 2 \rightarrow$$

$$x^2 + (y-1)^2 = 2$$

$$\text{rot } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} =$$

$$-2z \cos y \hat{i} - 2x \hat{j} - 1 \hat{k} = (-2z \cos y, -2x, -1)$$

$$X(r, \theta) = (r \cos \theta, r \sin \theta + 1, r \sin \theta + 2), 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial X}{\partial r} = (\cos \theta, \sin \theta, \sin \theta) = \hat{x}, \frac{\partial X}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = \hat{y}$$

$$X_r \wedge X_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & \hat{x} \\ \sin \theta & r \cos \theta & \hat{y} \\ \sin \theta & r \cos \theta & \hat{z} \end{vmatrix} = r \cos^2 \theta \hat{z} - r \sin^2 \theta \hat{i} + r \sin 2\theta \hat{j}$$

$$-r \sin 2\theta \hat{i} + r \sin^2 \theta \hat{z} - r \cos^2 \theta \hat{j} = (0, -r, r)$$

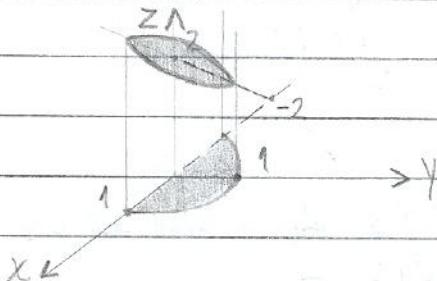
$$\begin{aligned} \int_S \vec{F} \cdot d\vec{s} &= \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}(X(r, \theta))) \cdot (X_r \hat{i} + X_\theta \hat{j}) dr d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} (-2(r \sin \theta + 1) \cos(r \sin \theta + 1), -2r \cos \theta, -1) \cdot (0, -r, r) dr d\theta = \\ &\quad \int_0^{2\pi} \int_0^{\sqrt{2}} 2r^2 \cos \theta - r dr d\theta = \int_0^{2\pi} [2r^3 \cos \theta - r^2]_0^{\sqrt{2}} d\theta = \end{aligned}$$

$$\int_0^{2\pi} 4\sqrt{2} \cos^2 \theta - 1 d\theta = -2\pi$$

15. $\vec{F}(x, y, z) = (x^2 + ye^z, y^2 + ze^x, z^2 + xe^y)$

S : fronteira da região limitada pelo cilindro $x^2 + y^2 = 1$ entre os planos $z = 0$ e $z = x + 2$, com n exterior

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$



$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 2 \iiint_V x + y + z dx dy dz$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ 0 \leq z \leq r \cos \theta + 2 \end{cases} \quad Je = r$$

$$\begin{aligned} \iiint_V x + y + z dx dy dz &= \int_0^{2\pi} \int_0^1 \int_0^{r \cos \theta + 2} (r \cos \theta + r \sin \theta + z) r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \int_0^{r \cos \theta + 2} r^2 \cos \theta + r^2 \sin \theta + rz dz dr d\theta = \int_0^{2\pi} \int_0^1 [(r^2(\cos \theta + \sin \theta))z \\ &\quad + \frac{rz^2}{2}]_0^{r \cos \theta + 2} dr d\theta = \int_0^{2\pi} \int_0^1 (r^2 \cos \theta + r^2 \sin \theta)(r \cos \theta + 2) + r(r^2 \cos^2 \theta + 4r \cos \theta + 4) \end{aligned}$$

$$dr d\theta = \int_0^{2\pi} \int_0^1 r^3 \cos^2 \theta + 2r^2 \cos \theta + r^3 \sin 2\theta + 2r^2 \sin \theta + r^3 \cos^2 \theta + 2r^2 \cos \theta + 2r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 3r^3 \cos^2 \theta + 4r^2 \cos \theta + 2r^2 \sin \theta + r^3 \sin 2\theta + 2r dr d\theta =$$

$$\int_0^{2\pi} [3r^4 \cos^2 \theta + r^4 \sin 2\theta + (4 \cos \theta + 2 \sin \theta)r^3 + r^2]_0^1 d\theta =$$

$$\int_0^{2\pi} \frac{3\cos^2\theta}{8} + \frac{\sin 2\theta}{8} + \frac{4\cos\theta}{3} + \frac{2\sin\theta}{3} + 1 \, d\theta =$$

$$\int_0^{2\pi} \frac{19}{16} + \frac{3\cos 2\theta}{16} + \frac{\sin 2\theta}{8} \, d\theta = \left[\frac{19\theta}{16} + \frac{3\sin 2\theta}{32} - \frac{\cos 2\theta}{16} \right]_0^{2\pi} =$$

$$\frac{19\pi}{8} - 1 - \left(-\frac{1}{16} \right) = \frac{19\pi}{8}$$

$$16. \int_S dy \wedge dz + y^3 dz \wedge dx + z^2 dx \wedge dy$$

a) S: esfera $x^2 + y^2 + z^2 = r^2$, com \hat{n} exterior.

$$\operatorname{div} \vec{F} = 3y^2 + 2z$$

$$\int_S \vec{F} \cdot \hat{n} \, dA = \iiint_V \operatorname{div} \vec{F} \, dx \, dy \, dz = \iiint_V 3y^2 + 2z \, dx \, dy \, dz$$

$$\begin{cases} x = r \sin \varphi \cos \theta \\ 0 \leq \theta \leq 2\pi \end{cases} \quad \begin{cases} y = r \sin \varphi \sin \theta \\ 0 \leq \varphi \leq \pi \end{cases} \quad J = r^2 \sin \varphi$$

$$\begin{cases} z = r \cos \varphi \\ 0 \leq r \leq n \end{cases}$$

$$\int_S \vec{F} \cdot \hat{n} \, dA = \int_0^{2\pi} \int_0^\pi \int_0^n (3r^2 \sin^2 \varphi \cos^2 \theta + 2r \cos \varphi) \cdot r^2 \sin \varphi \, dr \, d\varphi \, d\theta =$$

$$\int_0^{2\pi} \int_0^\pi \int_0^n 3r^4 \sin^3 \varphi \cos^2 \theta + r^3 \sin 2\varphi \, dr \, d\varphi \, d\theta =$$

$$\int_0^{2\pi} \int_0^\pi \left[3r^5 \sin^3 \varphi \cos^2 \theta + r^4 \sin 2\varphi \right]_0^n \, d\varphi \, d\theta =$$

$$\int_0^{2\pi} \int_0^\pi 3r^5 \sin^3 \varphi \cos^2 \theta + r^4 \sin 2\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^\pi 3r^5 \cos^2 \theta \sin \varphi$$

$$-3r^5 \cos^2 \theta \cdot \sin \varphi \cos^2 \varphi + r^4 \sin 2\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \left[-3r^5 \cos^2 \theta \cos \varphi \right]$$

$$+ r^5 \cos^2 \theta \cos^3 \varphi - r^4 \cos 2\varphi \right]_0^\pi \, d\theta = \int_0^{2\pi} 3r^5 \cos^2 \theta - r^5 \cos^2 \theta$$

$$-r^4 - \left(-\frac{3r^5 \cos^2 \theta}{5} + \frac{r^5 \cos^2 \theta}{5} - \frac{r^4}{8} \right) \, d\theta = \int_0^{2\pi} 4r^5 \cos^2 \theta \, d\theta =$$

$$\frac{2r^5}{5} [\theta + \sin \theta \cos \theta]_0^{2\pi} = 4\pi r^5$$

b) S : fronteira da região limitada por $z = 4$ e $z = x^2 + y^2$, com interior

$$\operatorname{div} \vec{F} = 3y^2 + 2z$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_D \operatorname{div} \vec{F} dx dy dz = \iiint_D 3y^2 + 2z dx dy dz$$

$$\begin{cases} x = r \cos \theta \\ 0 \leq \theta \leq 2\pi \\ y = r \sin \theta \\ 0 \leq r \leq 2 \\ z = z \end{cases}$$

$$r^2 \leq z \leq 4$$

$$\iiint_D 3y^2 + 2z dx dy dz = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (3r^2 \sin^2 \theta + 2z) \cdot r dz dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 r^2 \int_{r^2}^4 (3r^3 \sin^2 \theta + 2rz) dz dr d\theta = \int_0^{2\pi} \int_0^2 [3r^3 \sin^2 \theta z + rz^2]_{r^2}^4 dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 [12r^3 \sin^2 \theta + 16r - (3r^5 \sin^2 \theta + r^5)] dr d\theta =$$

$$\int_0^{2\pi} [3r^4 \sin^2 \theta + 8r^2 - r^6 \sin^2 \theta - r^6] dr =$$

$$2 \quad 6$$

$$\int_0^{2\pi} [48 \sin^2 \theta + 32 - 32 \sin^2 \theta - \frac{32}{3}] dr = \int_0^{2\pi} [16 \sin^2 \theta + \frac{64}{3}] dr =$$

$$[\frac{8(\theta + \sin \theta \cos \theta)}{3} + 64\theta]_{0}^{2\pi} = \frac{16\pi}{3} + \frac{128\pi}{3} = \frac{176\pi}{3}$$

17. $\vec{F}(x, y, z) = \vec{r} = \left(\frac{x}{|\vec{r}|^3}, \frac{y}{(x^2+y^2+z^2)^{2/3}}, \frac{z}{(x^2+y^2+z^2)^{2/3}} \right)$

a) S : esfera de raio a centrada na origem, com \hat{n} exterior.

Modo 1: $\iint_S \vec{F} \cdot \hat{n} dA = \iint_{S_1} \vec{r} \cdot \hat{r} \cdot r^2 dA = \frac{1}{a^2} \iint_{S_1} dA = 4\pi$

Modo 2: $D_F = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, $\operatorname{div} \vec{F} = 0$.

Deja S_1 a esfera $x^2 + y^2 + z^2 = b^2$, $b \rightarrow 0$, com $\hat{n} = -\hat{r}$, tal que $S \cup S_1$ é fechada e orientada pela normal exterior.

$$\begin{aligned} \iint_{S \cup S_1} \vec{F} \cdot \hat{n} dA &= \iiint_V \operatorname{div} \vec{F} dx dy dz = 0 \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = \\ - \iint_{S_1} \vec{F} \cdot \hat{n} dA &= - \iint_{S_1} b \hat{r} \cdot (-\hat{r}) dA = \frac{1}{b^3} \iint_{S_1} dA = 4\pi \end{aligned}$$

b) S : superfície fechada lisa por partes tal que $(0, 0, 0) \notin S$ e $(0, 0, 0) \in \overset{\circ}{S}$, com \hat{n} exterior.

$D_F = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, $\operatorname{div} \vec{F} = 0$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 0$$

c) S : superfície fechada lisa por partes, $(0, 0, 0) \in \overset{\circ}{S}$, com \hat{n} exterior

$D_F = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, $\operatorname{div} \vec{F} = 0$

Analogamente ao item a), $\iint_S \vec{F} \cdot \hat{n} dA = - \iint_{S_1} \vec{F} \cdot \hat{n} dA = 4\pi$

18. a) $\vec{F}(x, y, z) = (y, z, x)$

S : parte do paraboloide $z = 9 - x^2 - y^2$ acima de $z = 5$, $\hat{n} = \hat{k}$

$$9 - x^2 - y^2 = 5 \rightarrow x^2 + y^2 = 4$$

$$\vec{r}(t) = (2\cos t, 2\sin t, 5), \quad 0 \leq t \leq 2\pi \quad (\theta = 0)$$

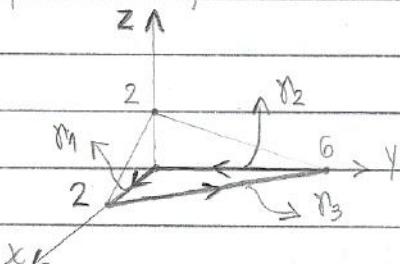
$$\vec{r}'(t) = (-2\sin t, 2\cos t, 0)$$

$$\iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \int_P \vec{F} \cdot \vec{ds} = \int_P \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt =$$

$$\begin{aligned} \int_0^{2\pi} (2\sin t, 2\cos t, 5) \cdot (-2\sin t, 2\cos t, 0) dt &= \int_0^{2\pi} -4\sin^2 t + 10\cos^2 t dt = \\ -2 [2\theta - \sin \theta \cos \theta]_0^{2\pi} &= -4\pi \end{aligned}$$

b) $\vec{F}(x, y, z) = (xz, x-y, x^2y)$

S: superfície formada pelas 3 faces que não estão em Oxy do tetraedro formado pelas planas coordenadas e o plano $3x+y+3z=6$, com \hat{n} exterior



$$\begin{cases} \gamma_1(t) = (t, 0, 0), 0 \leq t \leq 2 \\ \gamma_1'(t) = (1, 0, 0) \end{cases}$$

$$\begin{cases} \gamma_2(t) = (0, t, 0), 0 \leq t \leq 6 \\ \gamma_2'(t) = (0, 1, 0) \end{cases}$$

$$\begin{cases} \gamma_3(t) = (t, 6-3t, 0), 0 \leq t \leq 2 \\ \gamma_3'(t) = (1, -3, 0) \end{cases}$$

$$\partial S = \gamma_1 \cup (-\gamma_2) \cup (-\gamma_3)$$

$$\iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \int_{\gamma_1 \cup (-\gamma_2) \cup (-\gamma_3)} \vec{F} \cdot d\vec{s} = \int_{\gamma_1} \vec{F}(\gamma_1(t)) \cdot \gamma_1'(t) dt$$

$$- \int_{\gamma_2} \vec{F}(\gamma_2(t)) \cdot \gamma_2'(t) dt - \int_{\gamma_3} \vec{F}(\gamma_3(t)) \cdot \gamma_3'(t) dt = 0 \int^2_0 (0, t, 0) \cdot (1, 0, 0) dt$$

$$-0 \int^6_0 (0, -t, 0) \cdot (0, 1, 0) dt - 0 \int^2_0 (0, 4t-6, -3t^2+6t^2) \cdot (1, -3, 0) dt =$$

$$0 \int^6_0 t dt - 0 \int^2_0 -12t + 18 dt = 18 - [-6t^2 + 18t]_0^2 = 18 - 12 = 6$$

c) $\vec{F}(x, y, z) = (x^2, z, yz)$

S: parte do hipoboloide $x^2 + y^2 - z^2 = 1$ limitada pelo cilindro $x^2 + y^2 = 4$, com \hat{n} apontando para Oz

$$4 - z^2 = 1 \rightarrow z^2 = 3$$

$$\begin{cases} \gamma_1(t) = (2 \cos t, 2 \sin t, \sqrt{3}), 0 \leq t \leq 2\pi \\ \gamma_1'(t) = (-2 \sin t, 2 \cos t, 0) \end{cases}$$

$$\begin{cases} \gamma_2(t) = (2 \sin t, 2 \cos t, -\sqrt{3}), 0 \leq t \leq 2\pi \\ \gamma_2'(t) = (2 \cos t, -2 \sin t, 0) \end{cases}$$

$$\partial S = \gamma_1 \cup \gamma_2$$

$$\iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \int_{\gamma_1 \cup \gamma_2} \vec{F} \cdot d\vec{s} = \int_{\gamma_1} \vec{F}(\gamma_1(t)) \cdot \gamma_1'(t) dt$$

$$+ \int_{\gamma_2} \vec{F}(\gamma_2(t)) \cdot \gamma_2'(t) dt = 0 \int^{2\pi} (4 \cos^2 t, \sqrt{3}, 2\sqrt{3} \sin t) \cdot (-2 \sin t, 2 \cos t, 0) dt$$

$$+ 0 \int^{2\pi} (4 \sin^2 t, -\sqrt{3}, -2\sqrt{3} \cos t) \cdot (2 \cos t, -2 \sin t, 0) dt =$$

$$0 \int^{2\pi} -8 \cos^2 t \sin t + 2\sqrt{3} \cos^2 t + 8 \sin^2 t \cos t + 2\sqrt{3} \sin^2 t dt =$$

$$[8 \cos^3 t + 8 \sin^3 t]_0^{2\pi} = 0$$

19. a) $\vec{F}(x, y, z) = (-xz, y^3 - yz, z^2)$

S : elipsóide $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, com \hat{n} exterior

$$\operatorname{div} \vec{F} = -z + 3y^2 - z + 2z = 3y^2$$

$$\begin{cases} x = ap \sin \vartheta \sin \varphi \\ y = bp \sin \vartheta \cos \varphi \\ z = cp \cos \vartheta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \\ 0 \leq p \leq 1 \end{cases} \quad J = abc p^2 \sin \vartheta$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz =$$

$$3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 b^2 p^2 \sin^2 \vartheta \cos^2 \theta \cdot abc p^2 \sin \vartheta dp d\vartheta d\theta =$$

$$3ab^3 c \int_0^{2\pi} \int_0^{\pi} \int_0^1 p^4 \sin^3 \vartheta \cos^2 \theta dp d\vartheta d\theta = 3ab^3 c \int_0^{2\pi} \int_0^{\pi} [p^5 \sin^3 \vartheta \cos^2 \theta]_0^1 d\vartheta d\theta =$$

$$= 3ab^3 c \int_0^{2\pi} \cos^2 \theta \int_0^{\pi} \sin \vartheta - \sin \vartheta \cos^2 \vartheta d\vartheta d\theta =$$

5

$$\frac{3ab^3 c}{5} \int_0^{2\pi} \cos^2 \theta [-\cos \vartheta + \cos^3 \vartheta]_0^{\pi} d\theta = 4ab^3 c \int_0^{2\pi} \cos^2 \theta d\theta =$$

$$\frac{2ab^3 c}{5} [\theta + \sin \theta \cos \theta]_0^{2\pi} = 4\pi ab^3 c$$

b) $\vec{F}(x, y, z) = (x^3 + y \sin z, y^3 + z \sin x, 3z)$

S : superfície do sólido limitado pelos hemisférios

$$z = \sqrt{4-x^2-y^2}, z = \sqrt{1-x^2-y^2} \text{ e por } z=0, \text{ com } \hat{n} \text{ exterior}$$

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3 = 3(x^2 + y^2 + 1)$$

$$\begin{cases} x = p \sin \vartheta \sin \varphi \\ y = p \sin \vartheta \cos \varphi \\ z = p \cos \vartheta \end{cases} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/2 \\ 1 \leq p \leq 2 \end{cases} \quad J = p^2 \sin \vartheta$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 (p^2 \sin^2 \vartheta + 1) \cdot p^2 \sin \vartheta$$

$$dp d\vartheta d\theta = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 p^4 \sin^3 \vartheta + p^2 \sin \vartheta dp d\vartheta d\theta =$$

$$3 \int_0^{2\pi} \int_0^{\pi/2} [\frac{p^5}{5} \sin^3 \vartheta + \frac{p^3}{3} \sin \vartheta]_1^2 d\vartheta d\theta = 3 \int_0^{2\pi} \int_0^{\pi/2} \frac{31}{5} \sin^3 \vartheta + 7 \sin \vartheta d\vartheta d\theta =$$

$$3 \int_0^{2\pi} [\frac{31}{5} (-\cos \vartheta + \cos^3 \vartheta) - 7 \cos \vartheta]_0^{\pi/2} d\theta = 3 \int_0^{2\pi} \frac{31}{5} - 31 + 7 d\theta =$$

$$\frac{97}{5} \cdot \int_0^{2\pi} d\theta = 194\pi$$

c) $\vec{F}(x, y, z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} + z \right)$

$S: \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = (z-2)^2, 0 \leq z \leq 2 \}$, com $\hat{n}, \hat{k} > 0$

$D\vec{e} = \{ (x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0) \}$

$$\operatorname{div} \vec{F} = 1$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, -r+2), 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \partial \chi &= (\cos \theta, \sin \theta, -1) = \chi_r, \quad \partial \chi = (-r \sin \theta, r \cos \theta, 0) = \chi_\theta \\ \text{on} \end{aligned}$$

$$\chi_r \cdot \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 0 \\ -1 & 0 & \hat{k} \end{vmatrix} = r \cos^2 \theta \hat{i} + r \sin \theta \hat{j} + r \cos \theta \hat{k}$$

$$+ r \sin^2 \theta \hat{k} = (r \cos \theta, r \sin \theta, r)$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(r, \theta)) \cdot (\chi_r \cdot \chi_\theta) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 \left(\frac{r \cos \theta}{[2(r^2-2r+2)]^{3/2}} \frac{r \sin \theta}{[2(r^2-2r+2)]^{3/2}} \frac{(-r+2)}{[2(r^2-2r+2)]^{3/2}} + (-r+2) \right) \cdot (r \cos \theta, r \sin \theta, r) dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r}{\sqrt{2(r^2-2r+2)^{3/2}}} \left[-r^2 + 2r \right] dr d\theta = \int_0^{2\pi} \left[\frac{(r-2)}{\sqrt{2(r^2-2r+2)}} - \frac{r^3 - r^2}{3} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{8}{3} + 4 - (-1) \right] d\theta = \int_0^{2\pi} \left[\frac{7}{3} \right] d\theta = \frac{14\pi}{3}$$

/ /

d) $\vec{F}(x, y, z) = (2y, x-y, 4x)$

S : porção do paraboloide $z = x^2 + y^2 - 8$ acima do plano $z = 2x + 4y + 3$, $\hat{n} \cdot \hat{k} < 0$

$$x^2 + y^2 - 8 = 2x + 4y + 3 \Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 16 \Rightarrow (x-1)^2 + (y-2)^2 = 16$$

$$\operatorname{div} \vec{F} = -1$$

Seja S_1 a "tampa" do paraboloide, com $\hat{n} \cdot \hat{k} \geq 0$, tal que $S \cup S_1$ é fechada e orientada pela normal exterior

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_V \operatorname{div} \vec{F} dx dy dz \rightarrow \iint_{S_1} \vec{F} \cdot \hat{n} dA = - \iint_V dx dy dz - \iint_{S_1} \vec{F} \cdot \hat{m} dA$$

(I)

(II)

$$\text{I: } \begin{cases} x = r \cos \theta & 0 \leq \theta \leq 2\pi \\ y = r \sin \theta & 0 \leq r \leq 4 \\ z = z & r^2 - 8 \leq z \leq 2r \cos \theta + 4r \sin \theta + 3 \end{cases} \quad J_r = r$$

$$\iint_V dx dy dz = \int_0^{2\pi} \int_0^4 \int_{r^2 - 8}^{2r \cos \theta + 4r \sin \theta + 3} r dz dr d\theta =$$

$$\int_0^{2\pi} \int_0^4 r (2r \cos \theta + 4r \sin \theta + 3 - r^2 + 8) dr d\theta = \int_0^{2\pi} \int_0^4 2r^2 \cos \theta +$$

$$+ 4r^2 \sin \theta + 11r - r^3 dr d\theta = \int_0^{2\pi} \left[2r^3 \cos \theta + 4r^3 \sin \theta + \frac{11}{2}r^2 - \frac{1}{4}r^4 \right]_0^4 d\theta =$$

$$\int_0^{2\pi} \left[128 \cos^3 \theta + 256 \sin^3 \theta + 88 - 64 \right] d\theta = 48\pi$$

II: $\chi_1(r, \theta) = (r \cos \theta + 1, r \sin \theta + 2, 2r \cos \theta + 4r \sin \theta + 13)$

$$0 \leq r \leq 4, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi_1}{\partial r} = (\cos \theta, \sin \theta, 2\cos \theta + 4\sin \theta) = \chi_{1r}$$

$$\frac{\partial \chi_1}{\partial \theta} = (-r \sin \theta, r \cos \theta, -2r \sin \theta + 4r \cos \theta) = \chi_{1\theta}$$

$$\chi_{1r} \wedge \chi_{1\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 1 \\ 2\cos \theta + 4\sin \theta & -2r \sin \theta + 4r \cos \theta & 1 \end{vmatrix} =$$

$$\begin{aligned} & r\cos^2\theta \hat{k} + (-2r\sin^2\theta + 2r\sin 2\theta) \hat{i} + (-r\sin 2\theta - 4r\sin^2\theta) \hat{j} + r\sin^2\theta \hat{k} \\ & + (-2r\cos^2\theta - 2r\sin 2\theta) \hat{i} + (r\sin 2\theta - 4r\cos^2\theta) \hat{j} = (-2r, -4r, r) \\ & \iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(x_1(r, \theta)) \cdot (\hat{x}_{1r} \hat{x}_{1\theta}) dr d\theta = \\ & \int_0^{2\pi} \int_0^4 (2r\sin 2\theta + 4, r\cos\theta - r\sin\theta - 1, 4r\cos\theta + 4) \cdot (-2r, -4r, r) dr d\theta = \\ & \int_0^{2\pi} \int_0^4 -4r^2 \sin\theta - 8r - 4r^2 \cos\theta + 4r^2 \sin\theta + 4r + 4r^2 \cos\theta + 4r dr d\theta = 0 \\ & \therefore \iint_S \vec{F} \cdot \hat{n} dA = -48\pi \end{aligned}$$

20. S : superfície fechada lisa por partes, com \hat{n} exterior,
 $R = \oint_S$

a) $V_R = \frac{1}{3} \iint_S x dy dz + y dz dx + z dx dy$

seja $\vec{F}(x, y, z) = (x, y, z)$; como S satisfaz as hipóteses do Teorema de Gauss, tem-se:

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n} dA &= \iiint_V \operatorname{div} \vec{F} dx dy dz = \iiint_V 3 dx dy dz = 3V_R \\ \therefore V_R &= \frac{1}{3} \iint_S \vec{F} \cdot \hat{n} dA = \frac{1}{3} \iint_S x dy dz + y dz dx + z dx dy \end{aligned}$$

b) $\iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = 0$, se $\vec{F}(x, y, z)$ de classe C^1 numa região que contém S

seja $\vec{G}(x, y, z) = \operatorname{rot} \vec{F}(x, y, z)$; como S satisfaz as hipóteses do Teorema de Gauss, tem-se:

$$\iint_S \vec{G} \cdot \hat{n} dA = \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA = \iiint_V \operatorname{div} (\operatorname{rot} \vec{F}) \cdot \hat{n} dA$$

Pelo Teorema de Schubert, as derivadas parciais mistas de uma função de classe C^1 são iguais; sendo assim, tem-se:

$$\begin{aligned} \iint_S (\operatorname{rot} \vec{F}) \cdot \hat{n} dA &= \iiint_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial R - \partial Q}{\partial y}, \frac{\partial P - \partial R}{\partial z}, \frac{\partial Q - \partial P}{\partial x} \right) dx dy dz \\ &= \iint_S \frac{\partial^2 R}{\partial x \partial z} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial z} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial x} = 0 \end{aligned}$$

21. a) $\iint_S x \, dy \wedge dz + yze^{z^2} \, dz \wedge dx - e^{z^2} \, dx \, dy$

S : parte de $z = x^2 + y^2$ limitada por $x^2 + y^2 = 1$, com $\hat{n} \cdot \hat{k} > 0$
 $\text{div } \vec{F} = 1 + z e^{z^2} - z e^{z^2} = 1$

Diga S_1 : $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 1\}$, com $\hat{m} = -\hat{k}$,
tal que $S \cup S_1$ é fechada e orientada pela normal interior

$$\iint_{S \cup S_1} \vec{F} \cdot \hat{n} \, dA = - \iint_S \text{div } \vec{F} \, dx \, dy \, dz \rightarrow \iint_S \vec{F} \cdot \hat{n} \, dA =$$

$$- \iiint_V dx \, dy \, dz - \iint_{S_1} \vec{F} \cdot \hat{m} \, dA$$

(I) (II)

$$\begin{aligned} I: & \begin{cases} x = r \cos \theta & 0 \leq \theta \leq 2\pi \\ y = r \sin \theta & 0 \leq r \leq 1 \\ z = z & r^2 \leq z \leq 1 \end{cases} \quad r\hat{e} = \hat{n} \end{aligned}$$

$$\iint_V dx \, dy \, dz = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r(1-r^2) \, dr \, d\theta =$$

$$\int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}$$

$$II: \chi_{(r, \theta)} = (r \cos \theta, r \sin \theta, 1), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\chi_{r\hat{e}} \wedge \chi_{\theta\hat{e}} = (0, 0, r) \rightarrow \text{deve-se usar } \chi_{\theta\hat{e}} \wedge \chi_{r\hat{e}}$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \vec{F}(\chi_{(r, \theta)}) \cdot (\chi_{\theta\hat{e}} \wedge \chi_{r\hat{e}}) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^1 (r \cos \theta, e, r \sin \theta, -e) \cdot (0, 0, -r) \, dr \, d\theta = \int_0^{2\pi} \int_0^1 er \, dr \, d\theta =$$

$$\frac{e}{4} \int_0^{2\pi} d\theta = \frac{\pi e}{2}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, dA = -\frac{\pi}{2} - \frac{\pi e}{2} = -\frac{\pi}{2}(1+e)$$

$$b) \iint_S (x^2 + z^3) dy dz + z^5 dz dx + (e^{x^2+y^2} + z^2) dx dy$$

S : parte da $x^2 + y^2 + (z-1)^2 = 1$ interior ao cone $z^2 = x^2 + y^2$, com \hat{n} exterior

$$\operatorname{div} \vec{F} = 2x + 2z = 2(x+z)$$

$$z^2 + (z-1)^2 = 1 \rightarrow z^2 + z^2 - 2z + 1 = 1 \rightarrow 2z(z-1) = 0 \rightarrow$$

$z=0$ ou $z=1$

diga Si a superfície do cone $\{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2, 0 \leq z \leq 1\}$ com $\hat{n}, \hat{k} \leq 0$, tal que $S \cup S_1$ é fechada e orientada segundo a normal exterior

$$\iint_{S \cup S_1} \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = 2 \iiint_V x + z dx dy dz - \iint_{S_1} \vec{F} \cdot \hat{n} dA \quad (I)$$

(II)

$$I: \begin{cases} x = r \sin \varphi \cos \theta & 0 \leq \theta \leq 2\pi \\ y = r \sin \varphi \sin \theta & 0 \leq \varphi \leq \pi/4 \\ z = r \cos \varphi & 0 \leq r \leq 2 \cos \varphi \end{cases} \quad J_r = r^2 \sin \varphi$$

$$y = r \sin \varphi \cos \theta \quad 0 \leq \varphi \leq \pi/4$$

$$z = r \cos \varphi \quad 0 \leq r \leq 2 \cos \varphi$$

$$\iiint_V x + z dx dy dz = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2 \cos \varphi} r (\sin \varphi \cos \theta + \cos \varphi) dr d\varphi d\theta$$

$$r^2 \sin \varphi \cos \theta dx dy dz = \int_0^{2\pi} \int_0^{\pi/4} \left[(\sin^2 \varphi \cos^2 \theta + \sin 2\varphi) r^4 \right]_0^{2 \cos \varphi} =$$

$$\int_0^{2\pi} \int_0^{\pi/4} 4 \cos^4 \varphi \sin^2 \varphi \cos^2 \theta + 2 \cos^5 \varphi \sin \varphi \cos \theta d\theta d\varphi = \int_0^{\pi/4} \int_0^{2\pi} 4 \cos^4 \varphi \sin^2 \varphi \cos^2 \theta d\theta d\varphi$$

$$+ 2 \cos^5 \varphi \sin \varphi \cos \theta d\theta d\varphi = 4\pi \int_0^{\pi/4} \cos^5 \varphi \sin \varphi d\varphi = -\frac{\pi}{6} [\cos^6 \varphi]_0^{\pi/4} = -\frac{\pi}{6}$$

$$II: \chi_r(r, \theta) = (r \cos \theta, r \sin \theta, r), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi_r}{\partial r} = (\cos \theta, \sin \theta, 1) = \chi_{r_r}, \quad \frac{\partial \chi_r}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) = \chi_{r_\theta}$$

$$\chi_{r_r} \wedge \chi_{r_\theta} = (-r \cos \theta, -r \sin \theta, r) \rightarrow \text{dive -1e usar } \chi_{r_\theta} \wedge \chi_{r_r}$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \int_0^1 \int_0^{\pi/4} (r^2 \cos^2 \theta + r^3, r^5, e^{r^2} + r^2) \cdot (r \cos \theta, r \sin \theta, -r) dr d\theta$$

$$dr d\theta = \int_0^1 \int_0^{2\pi} r^3 \cos^3 \theta + r^4 \cos^2 \theta + r^6 \sin^2 \theta - r \cdot e^{r^2} - r^3 d\theta dr =$$

$$\int_0^1 [r^3 (\sin \theta - \sin^3 \theta)/3 - (r, e^{r^2}) \theta - r^3 \theta]_0^{2\pi} dr = -\pi \int_0^1 2r \cdot e^{r^2} dr - 2\pi \int_0^1 r^3 dr$$

$$= -\pi [e^{r^2}]_0^1 - \pi/2 = -\pi(e-1) - \pi/2 = -\pi(e-1/2)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = -\pi + \pi e + \pi = \pi(e+1)$$

c) $\iint_F e^{z^2} \cos(zy^2) dy dz + x dz \wedge dx + y dx \wedge dy$

S : parte de $x^2 + y^2 = 1$ limitada por $z = 0$ e $z = y+3$, com normal exterior

$$\operatorname{div} \vec{F} = 0$$

dijam $\chi_1(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, com $\hat{n}_1 = -\hat{k}$ e $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, r \sin \theta + 3)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, com \hat{n}_2 . $R \geq 0$ tal que $S \cup S_1 \cup S_2$ é fechada e orientada pela normal exterior

$$\iint_{S \cup S_1 \cup S_2} \vec{F} \cdot \hat{n} dA = \iiint_V \operatorname{div} \vec{F} dx dy dz = 0 \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = - \iint_{S_1} \vec{F} \cdot \hat{n}_1 dA - \iint_{S_2} \vec{F} \cdot \hat{n}_2 dA$$

(I)

(II)

I: $\chi_1(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$\chi_{1r} \wedge \chi_{1\theta} = (0, 0, r) \rightarrow \text{dive-se usar } \chi_{1\theta} \wedge \chi_{1r}$$

$$\iint_{S_1} \vec{F} \cdot \hat{n}_1 dA = \iint_{S_1} \vec{F}(\chi_1(r, \theta)) \cdot (\chi_{1\theta} \wedge \chi_{1r}) dr d\theta =$$

$$\int_0^{2\pi} \int_0^1 (1, r \cos \theta, r \sin \theta) \cdot (0, 0, -r) dr d\theta = \int_0^{2\pi} \int_0^1 -r^2 \sin \theta dr d\theta = -\frac{1}{3} \int_0^{2\pi} \sin^3 \theta d\theta = 0$$

II: $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, r \sin \theta + 3)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$\partial \chi_2 = (\cos \theta, \sin \theta, \sin \theta) = \chi_{2r}$$

or

$$\frac{\partial \chi_2}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = \chi_{2\theta}$$

$$\chi_{2r} \wedge \chi_{2\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & i \\ \sin \theta & r \cos \theta & j \\ \sin \theta & r \cos \theta & k \end{vmatrix} = r \cos^2 \theta \hat{i} + r \sin 2\theta \hat{j}$$

$$-r \sin^2 \theta \hat{j} - r \sin 2\theta \hat{i} + r \sin^2 \theta \hat{k} - r \cos^2 \theta \hat{j} = (0, -r, r)$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}_2 dA = \iint_{S_2} \vec{F}(\chi_2(r, \theta)) \cdot (\chi_{2r} \wedge \chi_{2\theta}) dr d\theta =$$

$$\int_0^{2\pi} \int_0^1 (e^{(r \sin \theta + 3)^2} \cos((r \sin \theta + 3) r^2 \sin^2 \theta), r \cos \theta, r \sin \theta) \cdot (0, -r, r) dr d\theta =$$

$$\int_0^{2\pi} \int_0^1 -r^2 \cos \theta + r^2 \sin \theta \, dr d\theta = \int_0^{2\pi} -\frac{\cos \theta}{3} + \frac{\sin \theta}{3} \, d\theta = 0$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, dA = 0$$

$$d) \iint_S z^2 dz \wedge dx + x^2 \ln(x^2+y^2) dx \wedge dy$$

S: parte de $z^2 = x^2 + y^2 - 9$ com $0 \leq z \leq 4$, $\hat{n} \cdot \vec{k} \geq 0$
 $\operatorname{div} \vec{F} = 0$

$$D\vec{F} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, z), z \in \mathbb{R}\}$$

$$\chi(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{r^2 - 9}), 3 \leq r \leq 5, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos \theta, \sin \theta, -\frac{1}{\sqrt{r^2 - 9}}) = \chi_r$$

$$\frac{\partial \chi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos \theta & -r \sin \theta & i \\ \sin \theta & r \cos \theta & j \\ \frac{1}{\sqrt{r^2 - 9}} & 0 & k \end{vmatrix} = r \cos^2 \theta \hat{i} - \frac{r^2 \sin \theta}{\sqrt{r^2 - 9}} \hat{j}$$

$$\frac{-r^2 \cos \theta}{\sqrt{r^2 - 9}} \hat{i} + \frac{r^2 \sin \theta}{\sqrt{r^2 - 9}} \hat{j} = \left(-\frac{r^2 \cos \theta}{\sqrt{r^2 - 9}}, -\frac{r^2 \sin \theta}{\sqrt{r^2 - 9}}, r \right)$$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \vec{F}(\chi(r, \theta)) \cdot (\chi_r \wedge \chi_\theta) \, dr d\theta = \int_0^{2\pi} \int_3^5 (0, r^2 - 9, r^2 \cos^2 \theta \ln r^2) \, dr d\theta.$$

$$\left(-\frac{r^2 \cos \theta}{\sqrt{r^2 - 9}}, -\frac{r^2 \sin \theta}{\sqrt{r^2 - 9}}, r \right) \, dr d\theta = \int_0^{2\pi} \int_3^5 -r^2 \sqrt{r^2 - 9} \sin \theta + r^3 \ln r^2 \cos^2 \theta \, dr d\theta =$$

$$3 \int_3^5 \int_0^{2\pi} -r^2 \sqrt{r^2 - 9} \sin \theta + r^3 \ln r^2 \cos^2 \theta \, d\theta dr = 3 \int_3^5 r^3 \ln r^2 [\theta + \sin \theta \cos \theta]_0^{2\pi} \, dr =$$

$$\pi 3 \int_3^5 r^3 \ln r^2 \, dr$$

$$\int r^3 \ln r^2 \, dr \rightarrow \begin{cases} f = r^3 \rightarrow f = \frac{r^4}{4} \\ g = \ln r^2 \rightarrow g = \frac{2}{r} \end{cases} \rightarrow \int r^3 \ln r^2 \, dr =$$

$$\int r^3 \ln r^2 \, dr = \frac{r^4}{4} \ln r^2 - \frac{1}{2} \int r^3 \, dr = \frac{r^4}{4} \ln r^2 - \frac{1}{2} r^4 + K, K \in \mathbb{R}$$

$$\frac{r^4 \ln r^2}{4} - \frac{1}{2} \int r^3 \, dr = \frac{r^4}{4} \left(\ln r^2 - \frac{1}{2} \right) + K, K \in \mathbb{R}$$

/ /

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = \frac{\pi}{4} [r^4 (\ln r^2 - 1)]_3^5 = \frac{\pi}{4} \left[\frac{625(2\ln 5 - 1)}{2} - \frac{81(2\ln 3 - 1)}{2} \right] = \frac{\pi}{4} [2.625\ln 5 - 2.81\ln 3 - 272] = \frac{\pi(625\ln 5 - 81\ln 3 - 68)}{2}$$

1) $\iint_S \frac{x}{(x^2+y^2+z^2)^{3/2}} dy dz + \frac{y}{(x^2+y^2+z^2)^{3/2}} dz dx + \frac{z}{(x^2+y^2+z^2)^{3/2}} + z dx dy$

S : parte de $(z-3)^2 = x^2+y^2$, $0 \leq z \leq 3$, com $\hat{n} \cdot \hat{k} \geq 0$
 $\text{div } \vec{F} = 1$

$$D\vec{F} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$$

$$\chi(r, \theta) = (r\cos\theta, r\sin\theta, r+3), 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi}{\partial r} = (\cos\theta, \sin\theta, 1) = \chi_r, \frac{\partial \chi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) = \chi_\theta$$

$$\chi_r \wedge \chi_\theta = \begin{vmatrix} \cos\theta & -r\sin\theta & i \\ \sin\theta & r\cos\theta & j \\ 0 & 0 & k \end{vmatrix} = r\cos^2\theta \hat{i} + r\sin\theta \hat{j} + r\cos\theta \hat{k}$$

$$+ r\sin^2\theta \hat{k} = (r\cos\theta, r\sin\theta, r)$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}(\chi(r, \theta)) \cdot (\chi_r \wedge \chi_\theta) dr d\theta =$$

$$= \int_0^{2\pi} \int_0^3 \left(\frac{r\cos\theta}{(2r^2-6r+9)^{3/2}}, \frac{r\sin\theta}{(2r^2-6r+9)^{3/2}}, \frac{(-r+3)}{(2r^2-6r+9)^{3/2}} + (-r+3) \right) \cdot (r\cos\theta, r\sin\theta, r) dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 \frac{3r}{(2r^2-6r+9)^{3/2}} - r^2 + 3r dr d\theta = \int_0^{2\pi} \left[\frac{(r-3)}{\sqrt{2r^2-6r+9}}, \frac{-r^3 + 3r^2}{3}, \frac{2}{2} \right]_0^3 d\theta =$$

$$\int_0^{2\pi} \frac{-9 + 27}{2} - (-1) d\theta = \frac{11}{2} \int_0^{2\pi} d\theta = 11\pi$$

$$f) \iint_S \frac{x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz + \frac{y}{(x^2+y^2+z^2)^{3/2}} dz \wedge dx + \frac{z}{(x^2+y^2+z^2)^{3/2}} + z dx \wedge dy$$

S : parte de $x^2+y^2-z^2=1$, $0 \leq z \leq 1$, com $\hat{n} \cdot \hat{k} \leq 0$

$$\vec{F}(x, y, z) = \vec{F}_1(x, y, z) + \vec{F}_2(x, y, z), \quad \vec{F}_2(x, y, z) = z\hat{k}$$

$$D\vec{F}_1 = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}, \quad D\vec{F}_2 = \mathbb{R}^3$$

$$\operatorname{div} \vec{F}_1 = 0, \quad \operatorname{div} \vec{F}_2 = 1$$

\vec{F}_1 : seja S_1 a semi-esfera $\chi_1(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$

$0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq 2\pi$, com $\hat{n}_1 \cdot \hat{k} \leq 0$, e S_2 : $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, 1)$,

$0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$, com $\hat{n}_2 = \hat{k}$, tal que $S \cup S_1 \cup S_2$ é fechada e orientada pela normal exterior

$$\begin{aligned} \iint_{S \cup S_1 \cup S_2} \vec{F}_1 \cdot \hat{n} dA &= \iint_S \operatorname{div} \vec{F}_1 = 0 \rightarrow \iint_S \vec{F}_1 \cdot \hat{n} dA = \\ &= \iint_{S_1} \vec{F}_1 \cdot \hat{m} dA - \iint_{S_2} \vec{F}_1 \cdot \hat{n}_2 dA \end{aligned}$$

(I) (II)

I: $\chi_1(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$, $0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq 2\pi$

$$\chi_{1r} \wedge \chi_{1\theta} = (-\sin^2 \varphi \cos \theta, -\sin^2 \varphi \sin \theta, -\sin(2\varphi)/2)$$

$$\iint_{S_1} \vec{F}_1 \cdot \hat{m} dA = \int_0^{\pi/2} \int_0^{2\pi} (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi).$$

$$(-\sin^2 \varphi \cos \theta, -\sin^2 \varphi \sin \theta, -\sin(2\varphi)/2) d\varphi d\theta = \int_0^{\pi/2} \int_0^{2\pi} -\sin^3 \varphi - \sin \varphi \cos^2 \varphi d\varphi d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\pi} -\sin \varphi d\varphi d\theta = \int_0^{\pi/2} [\cos \varphi]_0^{2\pi} d\theta = -\int_0^{\pi/2} d\theta = -2\pi$$

II: $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, 1)$, $0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$

$$\chi_{2r} \wedge \chi_{2\theta} = (0, 0, r)$$

$$\iint_{S_2} \vec{F}_1 \cdot \hat{n}_2 dA = \int_0^{\pi/2} \int_0^{\sqrt{2}} \left(\frac{r \cos \theta}{(r^2+1)^{3/2}}, \frac{r \sin \theta}{(r^2+1)^{3/2}}, \frac{1}{(r^2+1)^{3/2}} \right) \cdot (0, 0, r) dr d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^{\sqrt{2}} \frac{2r}{(r^2+1)^{3/2}} dr d\theta = \frac{1}{2} \int_0^{\pi/2} \left[-\frac{2}{\sqrt{r^2+1}} \right]_0^{\sqrt{2}} d\theta = -\frac{1}{\sqrt{3}} \int_0^{\pi/2} d\theta = 2\pi/(\sqrt{3})$$

\vec{F}_2 : $\chi(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{r^2-1})$, $1 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$

$$\chi_{r\theta} = \left(-\frac{r^2 \cos \theta}{\sqrt{r^2-1}}, -\frac{r^2 \sin \theta}{\sqrt{r^2-1}}, 1 \right) \rightarrow \text{dificil de usar } \chi_r \wedge \chi_\theta$$

$$\begin{aligned} \iint_S \vec{F}_2 \cdot \hat{n} dA &= \int_0^{\pi/2} \int_0^{\sqrt{2}} (0, 0, \sqrt{r^2-1}) \cdot \left(\frac{r^2 \cos \theta}{\sqrt{r^2-1}}, \frac{r^2 \sin \theta}{\sqrt{r^2-1}}, -1 \right) dr d\theta \\ &= \int_0^{\pi/2} \int_0^{\sqrt{2}} -r \sqrt{r^2-1} dr d\theta = -2\pi/3 \end{aligned}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} dA = \iint_S \vec{F}_1 \cdot \hat{n} dA + \iint_S \vec{F}_2 \cdot \hat{n} dA = 2\pi(\sqrt{3}-1)$$

$$g) \iint_S \frac{x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz + \frac{y}{(x^2+y^2+z^2)^{3/2}} dz \wedge dx + \frac{z}{(x^2+y^2+z^2)^{3/2}} dx \wedge dy$$

Σ : elipsóide $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$, com \hat{n} exterior

$$D_E = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$$

$$\operatorname{div} \vec{F} = 0$$

diga se a esfera $x^2+y^2+z^2=a^2$, $a \rightarrow 0$, com \hat{n} interior,
tal que $\Sigma \cup \Sigma_1$ é fechada e orientada pela normal exterior

$$\begin{aligned} \iint_{\Sigma \cup \Sigma_1} \vec{F} \cdot \hat{n} dA &= \iiint_V \operatorname{div} \vec{F} dx dy dz = 0 \rightarrow \iint_{\Sigma} \vec{F} \cdot \hat{n} dA = \\ - \iint_{\Sigma_1} \vec{F} \cdot \hat{n} dA &= - \iint_{\Sigma_1} r \hat{r} \wedge (-\hat{r}) dA = 1 \iint_{\Sigma_1} dA = 4\pi \end{aligned}$$

$$22. a) \iint_S \frac{x}{(4x^2+9y^2+25z^2)^{3/2}} dy \wedge dz + \frac{y}{(4x^2+9y^2+25z^2)^{3/2}} dz \wedge dx + \frac{z}{(4x^2+9y^2+25z^2)^{3/2}} dx \wedge dy$$

Σ : semi-esfera $x^2+y^2+z^2=1$, $z \geq 0$, com $\hat{n} \cdot \hat{k} \geq 0$

$$D_E = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \in \mathbb{R}^3\}$$

$$\operatorname{div} \vec{F} = 0$$

diga se o elipsóide $\frac{4x^2}{a^2} + \frac{9y^2}{a^2} + \frac{25z^2}{a^2} = 1$, $a \rightarrow 0$, com

\hat{n} interior, e Σ_2 : $\chi_2(r, \theta) = (\operatorname{sen} \varphi \operatorname{sen} \theta, \operatorname{sen} \varphi \operatorname{cos} \theta, \operatorname{cos} \varphi)$, $\pi/2 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$,
com $\hat{n}_2 \cdot \hat{k} \leq 0$, tal que $\Sigma \cup \Sigma_2$ é fechada e orientada pela normal ext.

$$\begin{aligned} \iint_{\Sigma \cup \Sigma_2} \vec{F} \cdot \hat{n} dA &= \iiint_V \operatorname{div} \vec{F} dx dy dz = 0 \rightarrow \iint_{\Sigma} \vec{F} \cdot \hat{n} dA = \\ - \iint_{\Sigma_2} \vec{F} \cdot \hat{n}_2 dA &= - \iint_{\Sigma_2} \vec{F} \cdot \hat{n}_2 dA \end{aligned}$$

(I)

(II)

$$I: \chi_1(\varphi, \theta) = \left(\frac{a \operatorname{sen} \varphi \operatorname{sen} \theta}{2}, \frac{a \operatorname{sen} \varphi \operatorname{cos} \theta}{3}, \frac{a \operatorname{cos} \varphi}{5} \right), 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi$$

$$\frac{\partial \chi_1}{\partial \varphi} = \left(\frac{a \operatorname{cos} \varphi \operatorname{sen} \theta}{2}, \frac{a \operatorname{cos} \varphi \operatorname{cos} \theta}{3}, -\frac{a \operatorname{sen} \varphi}{5} \right) = \chi_{1\varphi}$$

$$\frac{\partial \chi_1}{\partial \theta} = \left(\frac{a \operatorname{sen} \varphi \operatorname{cos} \theta}{2}, -\frac{a \operatorname{sen} \varphi \operatorname{sen} \theta}{3}, 0 \right) = \chi_{1\theta}$$

$$\chi_{1\varphi \wedge 1\theta} = \begin{vmatrix} \text{acose sen}\theta & \text{sen}\varphi \text{cos}\theta & i \\ 2 & 2 & i \\ \text{acose cos}\theta & -\text{sen}\varphi \text{sen}\theta & i \\ 3 & 3 & i \\ -\text{sen}\varphi & 0 & k \\ 5 & 0 & k \end{vmatrix} = -a^2 \text{sen}2\varphi \text{sen}^2\theta \hat{k}$$

$$-\frac{a^2 \text{sen}^2\varphi \text{cos}\theta}{10} \hat{j} - \frac{a^2 \text{sen}^2\varphi \text{sen}\theta}{15} \hat{i} - \frac{a^2 \text{sen}2\varphi \text{cos}^2\theta}{12} \hat{k} =$$

$$(-\frac{a^2 \text{sen}^2\varphi \text{sen}\theta}{15}, -\frac{a^2 \text{sen}^2\varphi \text{cos}\theta}{10}, -\frac{a^2 \text{sen}2\varphi}{12})$$

$$\iint_{S_1} \vec{F} \cdot \hat{n} dA = \iint_{S_1} \vec{F}(\chi_1(\varphi, \theta)) \cdot (\chi_{1\varphi} \wedge \chi_{1\theta}) d\varphi d\theta =$$

$$0 \int_{2\pi}^{2\pi} \int_{0}^{\pi} (-\frac{\text{sen}\varphi \text{sen}\theta}{2a^2}, \frac{\text{sen}\varphi \text{cos}\theta}{3a^2}, \frac{\text{cos}\varphi}{5a^2}) \cdot (-\frac{a^2 \text{sen}^2\varphi \text{sen}\theta}{15}, -\frac{a^2 \text{sen}^2\varphi \text{cos}\theta}{10},$$

$$-\frac{a^2 \text{sen}2\varphi}{12}) d\varphi d\theta = 0 \int_{30}^{30} \int_{60}^{30} -\text{sen}^3\varphi - \text{cos}^2\varphi \text{sen}\varphi d\varphi d\theta =$$

$$0 \int_{30}^{2\pi} \left[-1 \left(-\text{cos}\theta + \text{cos}^3\theta \right) + \text{cos}^3\varphi \right]_{180}^{\pi} d\theta = -\frac{5}{3} \int_{90}^{2\pi} d\theta = -\frac{10\pi}{9}$$

II: Pela simetria de \vec{F} , tem-se que $\iint_S \vec{F} \cdot \hat{n} dA = \iint_{S_2} \vec{F} \cdot \hat{n} dA$

$$\therefore 2 \iint_S \vec{F} \cdot \hat{n} dA = -(-\pi) \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = \frac{\pi}{18}$$

/ /

$$b) \iint_S \ln(2 + \cos(y+z)) dy dz - yz dz dx + z^2 dx dy$$

S : parte de $x^2 + y^2 = 4$, $0 \leq z \leq y+7$, com \hat{n} exterior
 $\text{dir } \vec{F} = -\hat{z} + \hat{z} = 0$

dijam $\chi_1(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, com
 $\hat{n}_1 = -\hat{r}$, e $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, r \sin \theta + 7)$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$,
com $\hat{n}_2 \cdot \hat{k} \geq 0$, tal que $S \cup S_1 \cup S_2$ é fechada e orientada pela normal
exterior

$$\iint_{S \cup S_1 \cup S_2} \vec{F} \cdot \hat{n} dA = \iiint_V \text{div } \vec{F} dx dy dz = 0 \rightarrow \iint_S \vec{F} \cdot \hat{n} dA = -\iint_{S_1} \vec{F} \cdot \hat{m} dA - \iint_{S_2} \vec{F} \cdot \hat{n}_2 dA$$

(I)

(II)

I: $\chi_1(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

$\chi_{1r} \wedge \chi_{1\theta} = (0, 0, r) \rightarrow \text{div } -\text{se usar } \chi_{1\theta} \wedge \chi_{1r}$

$$\iint_S \vec{F} \cdot \hat{m} dA = \iint_S \vec{F}(\chi_1(r, \theta)) \cdot (\chi_{1\theta} \wedge \chi_{1r}) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 (\ln(2 + \cos(r \sin \theta)), 0, 0) \cdot (0, 0, r) dr d\theta = 0$$

II: $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, r \sin \theta + 7)$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

$$\partial \chi_2 = (\cos \theta, \sin \theta, \sin \theta) = \chi_{2r}$$

or

$$\frac{\partial \chi_2}{\partial \theta} = (-r \sin \theta, r \cos \theta, r \cos \theta) = \chi_{2\theta}$$

$$\chi_{2r} \wedge \chi_{2\theta} = \begin{vmatrix} \cos \theta & -r \sin \theta & 1 \\ \sin \theta & r \cos \theta & 1 \\ \sin \theta & r \cos \theta & 1 \end{vmatrix} = r \cos^2 \theta \hat{k} + r \sin 2\theta \hat{i}$$

$$-r \sin^2 \theta \hat{j} - r \sin 2\theta \hat{i} - r \cos^2 \theta \hat{j} + r \sin^2 \theta \hat{k} = (0, -r, r)$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}_2 dA = \iint_{S_2} \vec{F}(\chi_2(r, \theta)) \cdot (\chi_{2r} \wedge \chi_{2\theta}) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 (\ln(2 + \cos(2r \sin \theta + 7)), -r^2 \sin^2 \theta - 7r \sin \theta, (r \sin \theta + 7)^2) \cdot (0, -r, r) dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 \sin^2 \theta + 7r^2 \sin \theta + r^3 \sin^2 \theta + 7r^2 \sin \theta + 49r dr d\theta =$$

$$\frac{1}{2} \int_0^2 \int_{0}^{2\pi} 3r^3 \sin^2 \theta + 14r^2 \sin \theta + 49r \, d\theta dr = \frac{1}{2} \int_0^2 \left[\frac{3r^3}{2} (\theta - \sin \theta \cos \theta) \right]_0^{2\pi}$$

$$+ 49r\theta \right]_0^{2\pi} dr = \frac{1}{2} \int_0^2 3\pi r^3 + 98\pi r \, dr = \frac{1}{2} \left[\frac{3\pi r^4}{4} + 49\pi r^2 \right]_0^2 =$$

$$6\pi + 98\pi = 104\pi$$

$$c) \iint_S \frac{x}{(x^2+y^2+z^2)^{3/2}} dy dz + \frac{y}{(x^2+y^2+z^2)^{3/2}} dz dx + \frac{z}{(x^2+y^2+z^2)^{3/2}} dx dy$$

\therefore parte do cilindro $x^2+y^2=2$, $-\sqrt{2} \leq z \leq \sqrt{2}$, com \hat{n} tal que $\hat{n}(\sqrt{2}, 0, 0) = (-1, 0, 0)$ (interior)

$$DF = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$$

$$\operatorname{div} \vec{F} = 0$$

dijam $\chi_1(r, \theta) = (r \cos \theta, r \sin \theta, -\sqrt{2})$, $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$, com $\hat{n}_1 = -\hat{r}$, e $\chi_2(r, \theta) = (r \cos \theta, r \sin \theta, \sqrt{2})$, $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$, com $\hat{n}_2 = \hat{r}$, tal que $(S) \cup S_1 \cup S_2$ é fechada e orientada pela normal exterior:

$$\iint_{(S) \cup S_1 \cup S_2} \vec{F} \cdot \hat{n} \, dA = \iiint_V \operatorname{div} \vec{F} \, dx dy dz = 0 \rightarrow \iint_S \vec{F} \cdot \hat{n} \, dA = \iint_{S_1} \vec{F} \cdot \hat{n}_1 \, dA + \iint_{S_2} \vec{F} \cdot \hat{n}_2 \, dA$$

(I) (II)

$$I: \chi_1(r, \theta) = (r \cos \theta, r \sin \theta, -\sqrt{2}), 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$$

$$\chi_{1r} \wedge \chi_{1\theta} = (0, 0, r) \rightarrow \text{dive-ai usar } \chi_{1\theta} \wedge \chi_{1r}$$

$$\iint_{S_1} \vec{F} \cdot \hat{n}_1 \, dA = \iint_{S_1} \vec{F}(\chi_1(r, \theta)) \cdot (\chi_{1\theta} \wedge \chi_{1r}) \, dr d\theta =$$

$$\frac{1}{2} \int_0^{2\pi} \int_0^{\sqrt{2}} \left(\frac{r \cos \theta}{(r^2+2)^{3/2}}, \frac{r \sin \theta}{(r^2+2)^{3/2}}, -\frac{\sqrt{2}}{(r^2+2)^{3/2}} \right) \cdot (0, 0, -r) \, dr d\theta =$$

$$\frac{1}{\sqrt{2}} \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{2r}{(r^2+2)^{3/2}} \, dr d\theta = \frac{1}{\sqrt{2}} \int_0^{2\pi} \left[-2 \frac{1}{\sqrt{r^2+2}} \right]_0^{\sqrt{2}} d\theta = -\sqrt{2} \left(\frac{1}{2} - 1 \right) \int_0^{2\pi} d\theta =$$

$$\frac{2\pi(1-1)}{\sqrt{2}}$$

/ /

$$\text{II: } \chi_2(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{2}), 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi$$

$$\chi_{2r} \wedge \chi_{2\theta} = (0, 0, r)$$

$$\iint_S \vec{F} \cdot \hat{n}_2 \, dA = \iint_{S_2} \vec{F}(\chi_2(r, \theta)) \cdot (\chi_{2r} \wedge \chi_{2\theta}) \, dr \, d\theta =$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\sqrt{2}} \left(\frac{r\cos\theta}{(r^2+2)^{3/2}}, \frac{r\sin\theta}{(r^2+2)^{3/2}}, \frac{\sqrt{2}}{(r^2+2)^{3/2}} \right) \cdot (0, 0, r) \, dr \, d\theta = 2\pi \left(1 - \frac{1}{\sqrt{2}} \right) \quad (\text{analog-}$$

opamente a I)

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, dA = 4\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$