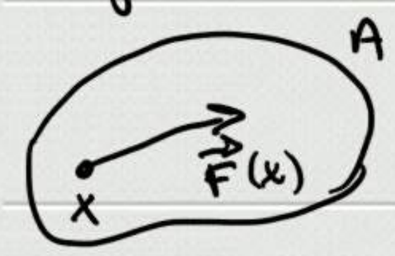
Mini Resumo feito por Caio Alcântara

Rotacional de uma campo função vetorial

Seja $A \subset \mathbb{R}^n$ $\quad F: A \to \mathbb{R}^n$ $\quad F(x) = \vec{F} \mid x \in A$, x é um vetor



$$\boxed{\operatorname{rot}(\vec{F}) = \nabla \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

tal que $\vec{F} = P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{k}$ $\quad$ Obs: em $\mathbb{R}^2$ $R = 0$

- Condição para $\vec{F}$ ser irrotacional

$$\boxed{\vec{F} \text{ é irrotacional} \Leftrightarrow \operatorname{rot}(\vec{F}) = \vec{0}}$$

Divergente de um campo vetorial (um campo escalar)

$\vec{F} = (F_1, F_2, \dots, F_n)$ $\quad \vec{F}: \Omega \to \mathbb{R}^n \mid \Omega \subset \mathbb{R}^n$

$$\operatorname{div} \vec{F}: \Omega \to \mathbb{R}$$

$$\operatorname{div} \vec{F} = \sum \frac{\partial F_i}{\partial x_i} \quad \therefore \quad \boxed{\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}}$$

Os 3 "produtos": gradiente, divergente e rotacional

$$\nabla \varphi, \quad \nabla \cdot \vec{F} \quad e \quad \nabla \wedge \vec{F}$$

Integrais de linha

$\vec{F}: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ e $\gamma: [a,b] \to \Omega$, $\gamma \in C^1 \mid a, b \in \mathbb{R}$ e $a < b$

$$\boxed{\int_\gamma \vec{F} \cdot d\gamma = \int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t) \cdot dt} \mid \vec{r}(t) = \gamma(t)$$

↳ proviniente da regra da cadeia

ou $\displaystyle\int_\gamma \vec{F}\, d\vec{r} = \int_a^b \left(P(x(t), y(t)),\ Q(x(t), y(t)) \right) \cdot \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right) \cdot dt$

$$= \int_a^b \left(P(x(t), y(t)) \frac{dx(t)}{dt} + Q(x(t), y(t)) \cdot \frac{dy(t)}{dt} \right) \cdot dt = \int_\gamma \left(P(x,y) \cdot dx + Q(x,y)\, dy \right)$$

↳ Esta sempre será a forma direta de integração

$$\therefore \boxed{\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma P(x,y)\, dx + Q(x,y)\, dy = \text{Eq. Amarela}}$$

Identificando Campos Conservativos

O primeiro passo para que um campo vetorial $\vec{F}$ seja um forte candidato a ser um campo conservative é ter $rot(\vec{F}) = \vec{0}$

Um campo vetorial $\vec{F}: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ $(n = 2$ ou $3)$ é denominado conservativo se existe campo escalar $\varphi: \Omega \subset \mathbb{R}^n \to \mathbb{R}$, com $\vec{F} \in \mathcal{C}^1$ e com $\varphi \in \mathcal{C}^2$, tal que:

$$\boxed{\vec{\nabla}\varphi(x,y,z) = \vec{F}(x,y,z)} \quad \text{ou} \quad \boxed{\vec{\nabla}\varphi(x,y) = \vec{F}(x,y)}$$

Ou seja, $\vec{F}(x,y,z)$ será da forma:

$$\vec{F}(x,y,z) = \left(\frac{\partial}{\partial x}\varphi(x,y,z), \frac{\partial}{\partial y}\varphi(x,y,z), \frac{\partial}{\partial z}\varphi(x,y,z)\right)$$

Obs: Haverá casos que só isto não definirá que $\vec{F}(x,y,z)$ é conservativo, para estes casos, deverá se trabalhar com o domínio Ω de $\vec{F}$, utilizando-se uma topologia de <u>Domínios Simplesmente Conexos</u>

Note que se $\vec{F}(x,y,z) = \left(\frac{\partial}{\partial x}\varphi(x,y,z), \frac{\partial}{\partial y}\varphi(x,y,z), \frac{\partial}{\partial z}\varphi(x,y,z)\right)$

$rot(\vec{F})$ é necessariamente $\vec{0}$:

Obs: $\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial z}\varphi = \frac{\partial^2}{\partial y \partial z}\varphi$

$$\nabla \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x}\varphi & \frac{\partial}{\partial y}\varphi & \frac{\partial}{\partial z}\varphi \end{vmatrix} = \underbrace{\left(\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial z}\varphi - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\varphi\right)}_{0}\hat{i} + \underbrace{\left(\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial x}\varphi - \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial z}\varphi\right)}_{0}\hat{j} + \underbrace{\left(\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial y}\varphi - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\varphi\right)}_{0}\hat{k}$$

$$\nabla \wedge \vec{F} = \vec{0}$$

pelo Teorema de Schwarz, se $\varphi \in \mathcal{C}^2$, então $\frac{\partial^2}{\partial x \partial y}\varphi = \frac{\partial^2}{\partial y \partial x}\varphi$

Uma percepção importante de se ter é o reconhecimento da forma diferencial exata de uma integral de linha "$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy$", sendo que $\underline{P(x,y)\,dx + Q(x,y)\,dy}$ (Eq.0) só será a <u>forma diferencial exata</u> quando $\underline{\vec{F} = (P(x,y), Q(x,y))}$ <u>for conservative</u>. (Por que isto?)

O porquê:

$\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ é conservativo, logo, existe o campo escalar $\varphi: \Omega \to \mathbb{R} \mid \nabla\varphi(x,y) = \vec{F}(x,y)$ $\therefore$ $\vec{F}(x,y) = \left(\frac{\partial}{\partial x}\varphi, \frac{\partial}{\partial y}\varphi\right)$

Sendo $\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma \left(\frac{\partial}{\partial x}\varphi, \frac{\partial}{\partial y}\varphi\right)\cdot(dx, dy) = \int_\gamma \frac{\partial}{\partial x}\varphi \cdot dx + \frac{\partial}{\partial y}\varphi\, dy$ (Eq. 1)

Lembrando, do cálculo II, que a derivada de um campo escalar $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (com $n = 2$ ou 3 no nosso caso) é:

$$d\varphi = \frac{\partial}{\partial x}\varphi \cdot dx + \frac{\partial}{\partial y}\varphi\, dy \quad \text{(Eq. 2)}$$

Mais conhecido como o diferencial de φ, lembrando que φ é diferenciável

De eq. 0, eq. 1 e de eq. 2:

$$P(x,y)\,dx + Q(x,y)\cdot dy = \frac{\partial}{\partial x}\varphi(x,y)\,dx + \frac{\partial}{\partial y}\varphi(x,y)\cdot dy = d\varphi(x,y)$$

Por isso é a forma diferencial exata

Além do mais, isso demonstra a independência do caminho na integral de linha quando o campo é conservativo:

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma d\varphi \quad , \quad \gamma : [a,b] \to \Omega \text{ com } \gamma(a) = A \text{ e } \gamma(b) = B$$

então: $\int_\gamma \vec{F}\cdot d\vec{r} = \int_A^B d\varphi = \varphi(B) - \varphi(A)$ → só basta saber o ponto final e inicial

Resumindo: para saber se é conservativo da maneira convencional, basta verificar se $\text{rot}(\vec{F}) = \vec{0}$. Se for, sendo $\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$, basta integrar $P(x,y,z)$ em relação a x: $\varphi(x,y,z) = \int P(x,y,z)\cdot dx$ e comparar a constante $K(y,z)$ que resultará desta integral, derivando-a parcialmente e igualando com $Q(x,y,z) \Rightarrow \frac{\partial}{\partial y}K(y,z) = Q$ ou com $R(x,y,z) \Rightarrow \frac{\partial}{\partial z}K(y,z) = R$. O exemplo desta operação encontra-se na página 21 ex. 10 b

Verificando se um campo é conservativo por conjunto simplesmente conexo

Se você só quer decorar, avance uma página e leia as caixas azuis

A grande obstrução para que um campo vetorial irrotacional seja conservativo é o fato de existirem pontos P internos ao domínio Ω de $\vec{F}$ | $P \notin \Omega$, possibilitando a existência de curvas $\gamma_i : [a_i, b_i] \to \Omega$, tal que $\gamma_i \in \Omega$, mas cuja área interna, K_i, não esteja contida no domínio Ω de $\vec{F}$:

Para o caso em que $\Omega \subset \mathbb{R}^2$

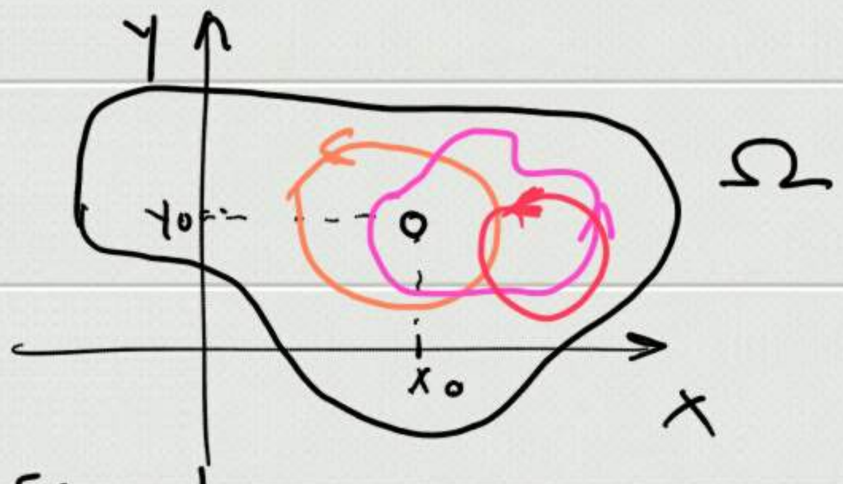


Figura 1

$P = (x_0, y_0) \notin \Omega$

$\gamma_1 \subset \Omega$ e $K_1' \not\subset \Omega$

$\gamma_2 \subset \Omega$ e $K_2' \not\subset \Omega$

Se o domínio Ω de $\vec{F}$ fosse mais restrito, isto seria impossível de acontecer:

$\gamma_3 \subset \Omega$ e $K_3' \subset \Omega$

Conclui-se que, se existir alguma curva simples e fechada em Ω que envolva algum ponto fora de Ω, mesmo que o campo $\vec{F}$ seja irrotacional, ele pode não ser conservativo (no caso em que a circulação do campo nessa curva não seja nulo $\oint_{\gamma_i} \vec{F} \cdot d\gamma_i \neq 0$)

Consequentemente, percebe-se que a propriedade crucial do domínio Ω que implica que todo campo $\vec{F}$ irrotacional seja conservativo é que toda curva simples e fechada γ_i em Ω envolva uma região $K_i' \subset \Omega$ e que não de volta em nenhum ponto fora de Ω. Tais regiões chamam-se simplesmente conexas

Portanto, pelo Teorema de Green, defini-se o critério:

O campo $\vec{F}$, definido num domínio simplesmente conexo Ω, é conservativo se, e somente se, o $\text{rot}(\vec{F}) = \vec{0}$

- Se Ω não for simplesmente conexo

O campo $\vec{F}$ é conservativo em Ω se, e somente se, $\text{rot}(\vec{F}) = \vec{0}$ e $\oint_{\gamma_i} \vec{F} \cdot d\gamma_i = 0$, $\forall \gamma_i \subset \Omega$

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Integral de linha sobre campo conservativo

Se $\vec{F}: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ for um campo vetorial contínuo e conservativo, se $\varphi: \Omega \to \mathbb{R}$ for uma função potencial para $\vec{F}$ ($\vec{\nabla}\varphi = \vec{F}$) e se $\gamma: [a,b] \to \Omega \mid \gamma \in \mathcal{C}^1$ com $A = \gamma(a)$ e $B = \gamma(b)$:

$$\boxed{\int_\gamma \vec{F} \cdot d\gamma = \int_\gamma \vec{\nabla}\varphi \cdot d\gamma = \varphi(B) - \varphi(A)}$$

Condição necessária para $\vec{F}$ ser conservativo

$$\boxed{\text{rot}(\vec{F}) = \vec{0}}$$

Teorema de Green

Seja $K \subset \mathbb{R}^2$ um compacto com interior não-vazio cuja fronteira é imagem de uma curva $\gamma: [a,b] \to \mathbb{R}^2$ fechada, simples, $\mathcal{C}^1$ por partes e orientada no sentido anti-horário. E sejam P e $Q \in \mathcal{C}^1$ num aberto contendo K. Nestas condições, tem-se que:

$$\boxed{\oint_\gamma P\,dx + Q\,dy = \iint_K \underbrace{\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]}_{\text{Rotacional de } \vec{F} = (P,Q)} \cdot dx\,dy}$$

Teorema de Stokes plano ≡ Teorema de Green

$\vec{F}$ um campo vetorial de classe C^1 no aberto $\Omega \subset \mathbb{R}^2$ e com γ e K satisfazendo o Teorema de Green

$$\boxed{\oint_\gamma \vec{F} \cdot d\gamma = \iint_K \text{rot}(\vec{F}) \cdot \vec{K}\,dx\,dy}$$

Ou seja é a mesma coisa, só que ele pode tratar o caso no espaço quando se usa a forma mais geral que não cairá no nosso curso.

⇒ Falta fazer o resumo de Integral de Superfície

Obs: Creio que não irá cair na P2

Obs: onde estiver com marca texto amarelo, não bateu o gabarito, verifique se não errei conta.

Lista 2 - Cálculo III

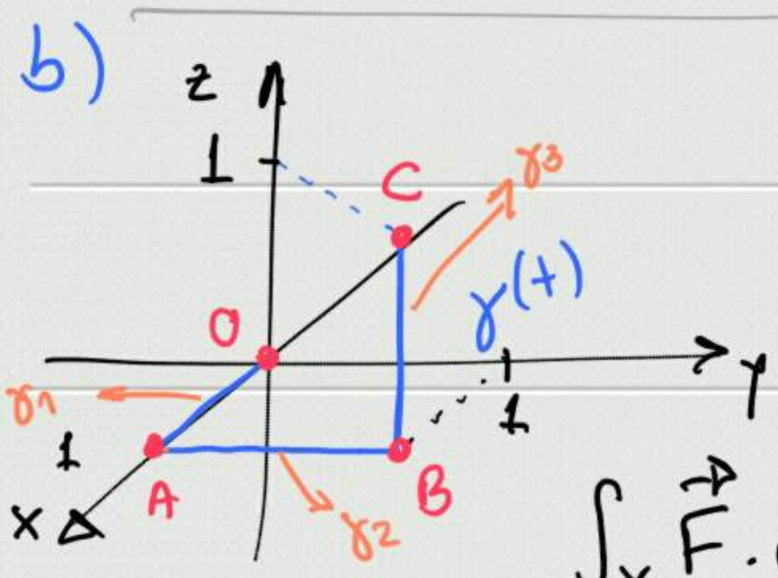
1) $\int_\gamma \vec{F}\cdot d\vec{r}$, $\vec{F}(x,y,z) = (x^2+y, -7yz, 2xz^2)$, $A=(0,0,0)$, $B=(1,1,1)$

a) $\gamma(t) = (t, t^2, t^3) \Leftrightarrow \frac{d}{dt}\gamma(t) = (1, 2t, 3t^2) \Leftrightarrow d\gamma(t) = (1, 2t, 3t^2).dt$

$$\int_\gamma \vec{F}(x,y,z).d\vec{r} = \int_A^B \vec{F}(x(t),y(t),z(t)).d\gamma(t) = \int_0^1 (t^2+t^2, -7t^2.t^3, 2t(t^3)^2)\cdot (1,2t,3t^2).dt = \int_0^1 (2t^2 - 14t^6 + 6t^9).dt = \frac{2}{3} - \frac{14}{7} + \frac{6}{10} = \frac{20-60+18}{30}$$

$$\frac{-22}{30} = \frac{-11}{15}$$

b)



$$\gamma(t) = \begin{cases} \gamma_1(t) = (t,0,0) \\ \gamma_2(t) = (1,t,0) \\ \gamma_3(t) = (1,1,t) \end{cases}, \quad t\in[0,1]$$

$$\int_\gamma \vec{F}.d\vec{r} = \int_{\gamma_1} \vec{F}.d\vec{r} + \int_{\gamma_2} \vec{F}.d\vec{r} + \int_{\gamma_3} \vec{F}.d\vec{r}$$

$$\int_{\gamma_1} \vec{F}.d\vec{r} = \int_0^A \vec{F}(x(t),y(t),z(t))\cdot d\gamma(t) = \int_0^1 (t^2, 0, 0)\cdot(1,0,0).dt = \int_0^1 t^2.dt = 1/3$$

$$\int_{\gamma_2} \vec{F}.d\vec{r} = \int_A^B \vec{F}(x(t),y(t),z(t))\cdot d\gamma(t) = \int_0^1 (1+t, 0, 0)\cdot(0,1,0).dt = 0$$

$$\int_{\gamma_3} \vec{F}.d\vec{r} = \int_B^C \vec{F}(x(t),y(t),z(t)).d\gamma(t) = \int_0^1 (1+1, -7.1.t, 2.1.t^2)\cdot(0,0,1).dt = \int_0^1 2t^2.dt = 2/3$$

$$\therefore \int_\gamma \vec{F}.d\vec{r} = \frac{1}{3} + 0 + \frac{2}{3} = 1$$

2) a) $\gamma(x) = (x, \sqrt{4-x^2})$; $y = \sqrt{4-x^2} \Rightarrow y \geq 0$ → semicircunferência

parametrização

$y^2 = 4-x^2 \Leftrightarrow x^2+y^2 = 4$; começa no $(-2,0)$ e acaba no $(2,0)$

$$\therefore \begin{cases} x = -2\cos t \\ y = 2\,\text{sen}\,t \end{cases}, \quad t\in[0,\pi]$$

$$\int_\gamma \vec{F}.d\vec{r} = \int_{(-2,0)}^{(2,0)} F(x(t),y(t))\cdot d\gamma(t) = \int_0^\pi (2\,\text{sen}\,t, 4)\cdot(2\,\text{sen}\,t, 2\cos t).dt =$$

$$\int_0^\pi (4\,\text{sen}^2 t + 8\cos t).dt = \int_0^\pi 4\,\text{sen}^2 t.dt + 8\int_0^\pi \cos t.dt = 4\left(\left[-\frac{\cos t.\text{sen}\,t}{2}\right]_0^\pi + \frac{1}{2}\int_0^\pi .dt\right) = 4.\frac{\pi}{2} = 2\pi$$

b) $\vec{F}(x,y,z) = (2x+2y,\ x-y,\ 0)$

parametrização

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, sentido anti-horário : $\begin{cases} x = a.\cos t \\ y = b.\operatorname{sen} t \end{cases}$, $t \in [0, 2\pi]$

$\gamma(t) = (a\cos t,\ b\operatorname{sen} t,\ 0)$

$$\oint_\gamma \vec{F}.d\vec{r} = \oint_\gamma \vec{F}(x(t),y(t),z(t)).\ d\gamma(t) = \int_0^{2\pi} (2a\cos t + 2b\operatorname{sen} t,\ a\cos t - b\operatorname{sen} t, 0)\cdot$$

$$(a(-\operatorname{sen} t),\ b\cos t,\ 0).dt =$$

$$\int_0^{2\pi}\left[-2(a^2\cos t\operatorname{sen} t + ba\operatorname{sen}^2 t) + ba\cos^2 t - b^2\cos t\operatorname{sen} t\right].dt =$$

$$\int_0^{2\pi} -2a^2\cos t\operatorname{sen} t.dt - \int_0^{2\pi} 2ba.\operatorname{sen}^2 t.dt + \int_0^{2\pi} ba\cos^2 t.dt - b^2\int_0^{2\pi}\cos t\operatorname{sen} t.dt =$$

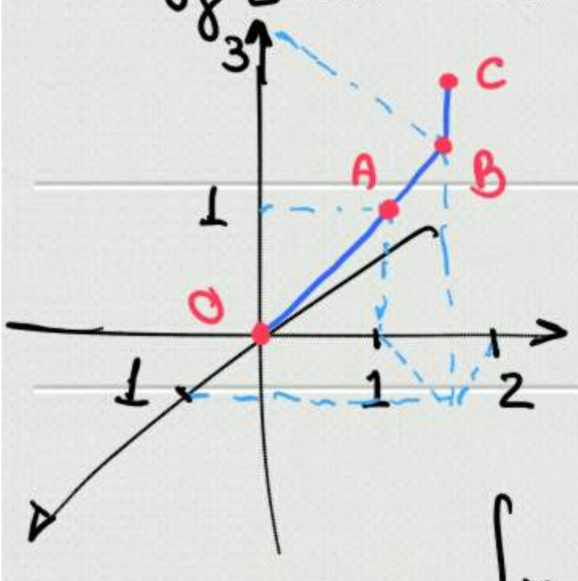
$$-a^2\int_0^{2\pi}\operatorname{sen} 2t.dt - 2ba\left(\left[\frac{-\cos t.\operatorname{sen} t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right) + ba\left(\left[\frac{\operatorname{sen} t.\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right)$$

$$-\frac{b^2}{2}\int_0^{2\pi}\operatorname{sen} 2t.dt = -a^2.0 - 2ba.\frac{1}{2}.2\pi + ba\frac{1}{2}2\pi - \frac{b^2}{2}.0 = -\pi ab$$

c) $\int_\gamma x^3y^2z\,dz = \int_\gamma (0, 0, x^3y^2z).\ d\gamma$; $\gamma(t) = (2t, t^2, t^2)$, $t \in [0,1]$

$$\int_\gamma x^3y^2z\,dz = \int_0^1 (0, 0, (2t)^3.(t^2)^2.t^2).(2, 2t, 2t).dt = \int_0^1 8.t^9.2t.dt = \frac{16}{11}$$

d) $\int_\gamma z^2dx - z\,dy + 2y\,dz = \int_\gamma (z^2, -z, 2y).d\gamma = \int_\gamma \underbrace{(z^2(t), -z(t), 2y(t))}_{\vec{F}(t)}.d\gamma(t)$



$$\gamma(t) = \begin{cases} \overline{OA},\ \gamma_1(t) = (0, t, t),\ t \in [0,1] \\ \overline{AB},\ \gamma_2(t) = (t, 1+t, 1+2t),\ t \in [0,1] \\ \overline{BC},\ \gamma_3(t) = (1, 2, 3+t),\ t \in [0,1] \end{cases}$$

$$\int_\gamma \vec{F}(t).d\gamma = \int_{\gamma_1}\vec{F}(t).d\gamma_1 + \int_{\gamma_2}\vec{F}(t).d\gamma_2 + \int_{\gamma_3}\vec{F}(t).d\gamma_3$$

$$\int_{\gamma_1}\vec{F}(t).d\gamma_1(t) = \int_0^1 (t^2, -t, 2t).(0,1,1).dt = \int_0^1(-t+2t).dt = \frac{1}{2}$$

$$\int_{\gamma_2}\vec{F}(t).d\gamma_2(t) = \int_0^1 ((1+2t)^2,\ 1+2t,\ 2(1+t))\cdot(1,1,2).dt = 8+18+24$$

$$\int_0^1(1+4t+4t^2-(1+2t)+4+4t).dt = \int_0^1(4t^2+6t+4).dt = \frac{4}{3}+\frac{6}{2}+4 = \frac{50}{6}$$

$$\int_{\gamma_3}\vec{F}(t).d\gamma_3(t) = \int_0^1((3+t)^2, -(3+t), 2.2).(0,0,1).dt = 4$$

$$\int_\gamma \vec{F}.d\gamma = \frac{1}{2}+\frac{50}{6}+4 = \frac{3+50+24}{6} = \frac{77}{6}$$

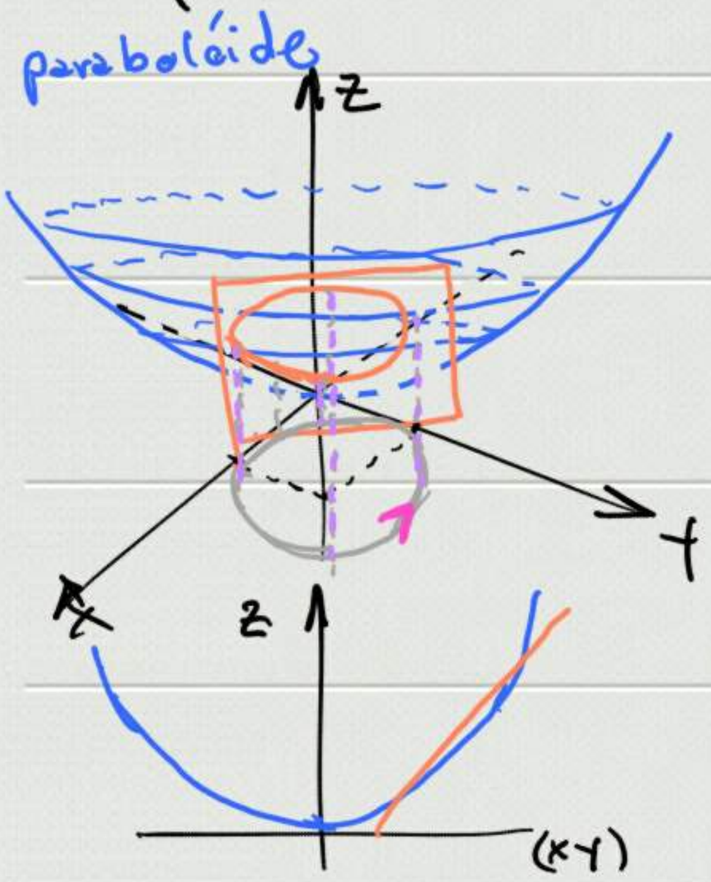
Uma reflexão:

3) O mais interessante deste exercício é a descrição de $\gamma(t)$, pois o cálculo da integral segue o mesmo padrão das anteriores: Compor $\vec{F}$ e $d\gamma(t)$ e fazer o produto escalar entre eles antes de integrar.

a) $\int_\gamma x\,dx + (x+y)\,dy + z\,dz = \int_\gamma (x, x+y, z)\cdot d\gamma$

$$\gamma: \begin{cases} z = x^2+y^2 \\ z = 2x+2y-1 \end{cases} \iff x^2+y^2 = 2x+2y-1 \iff x^2-2x+1+y^2-2y+1 = 1$$

$$(x-1)^2+(y-1)^2 = 1$$ circunferência projetada no plano $z=0$



no sentido horário é igual de 0 à -2π

Parametrizando-se

$$\begin{cases} x = \cos(-t)+1 \\ y = \operatorname{sen}(-t)+1 \\ z = 2(\cos(-t)+1)+2(\operatorname{sen}(-t)+1)-1 \end{cases}, \quad t\in[0,2\pi]$$

$$\gamma(t) = (\cos t+1, -\operatorname{sen} t+1, 2(\cos t-\operatorname{sen} t)+3)$$

$$\int_\gamma (x(t), x(t)+y(t), z(t))\cdot d\gamma(t) =$$

$$\int_0^{2\pi} (\cos t+1, 1-\operatorname{sen} t+\cos t+1, 2(\cos t-\operatorname{sen} t)+3)\cdot(-\operatorname{sen} t, -\cos t, 2(-\operatorname{sen} t-\cos t))\,dt =$$

$$\int_0^{2\pi} (\cancel{-\operatorname{sen} t\cos t}-\operatorname{sen} t+\cancel{\operatorname{sen} t\cos t}-\cos^2 t-2\cos t-4(\cos^2 t-\operatorname{sen}^2 t)-6(\operatorname{sen} t+\cos t))\cdot dt$$

$$= -\int_0^{2\pi}\operatorname{sen} t\,dt - \int_0^{2\pi}\cos^2 t\,dt - 2\int_0^{2\pi}\cos t\,dt - 4\int_0^{2\pi}\cos 2t\,dt - 6\int_0^{2\pi}\operatorname{sen} t\,dt - 6\int_0^{2\pi}\cos t\,dt =$$

$$= 0 - \left(\left[\frac{\operatorname{sen} t\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1\,dt\right) + 0 + 0 + 0 + 0 = -\pi$$

b) $\int_\gamma (2y+1)\,dx + z\,dy + x\,dz = \int_\gamma (2y+1, z, x)\,d\gamma$

$$\gamma: \begin{cases} x^2+4y^2 = 1 \\ x^2+z^2 = 1 \\ y\geq 0 \\ z\geq 0 \end{cases} \iff \begin{cases} x^2+4y^2=1 \\ z^2 = 4y^2 \\ y\geq 0 \\ z\geq 0 \end{cases} \iff \begin{cases} x^2+4y^2=1 \\ z=2y \\ y\geq 0 \\ z\geq 0 \end{cases} \iff \begin{cases} x^2+4y^2=1 \\ z=2y \\ y\geq 0 \end{cases}$$

parametrizando:

$$\begin{cases} x = \cos t \\ y = \frac{\operatorname{sen} t}{2} \\ z = \operatorname{sen} t \end{cases} \qquad \gamma(t) = \left(\cos t, \frac{\operatorname{sen} t}{2}, \operatorname{sen} t\right), \quad t\in[0,\pi]$$

$$\int_\gamma (2y+1,\ z,\ x)\cdot d\gamma = \int_0^{\pi} (2y(t)+1,\ z(t),\ x(t))\cdot\left(-\operatorname{sen}t,\ \frac{\cos t}{2},\ \cos t\right)\cdot dt$$

$$= \int_0^{\pi} \left(2\frac{\operatorname{sen}t}{2}+1,\ \operatorname{sen}t,\ \cos t\right)\left(-\operatorname{sen}t,\ \frac{\cos t}{2},\ \cos t\right)\cdot dt = \int_0^{\pi}\left(-\operatorname{sen}^2 t - \operatorname{sen}t + \frac{\operatorname{sen}t\cos t}{2} + \cos^2 t\right)dt$$

$$= \int_0^{\pi} \cos 2t\cdot dt - \int_0^{\pi} \operatorname{sen}t\cdot dt + \frac{1}{4}\int_0^{\pi} \operatorname{sen}2t\cdot dt = 0 - \left[-\cos t\right]_0^{\pi} + \frac{1}{4}\left[\frac{1}{2}(-\cos 2t)\right]_0^{\pi} =$$

$$-2 - \frac{1}{8}(1-1) = -2$$

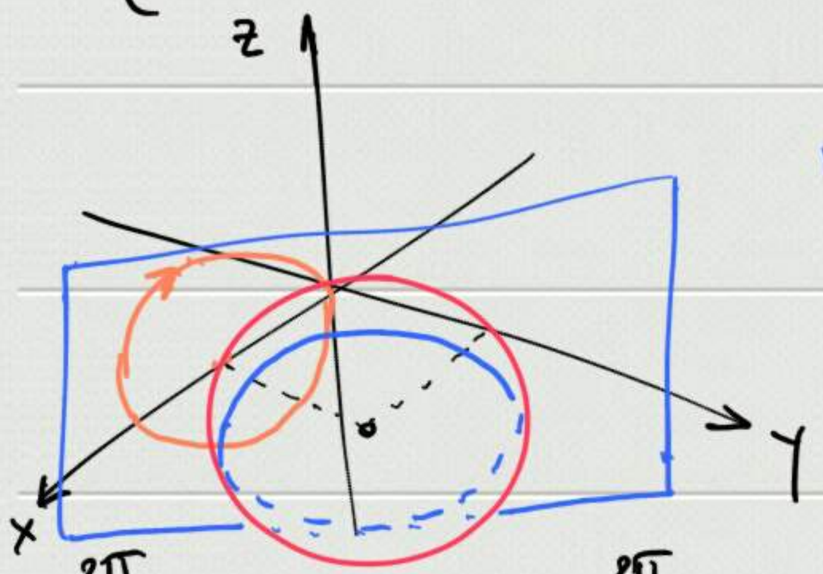
c) $\int_\gamma y\,dx + z\,dy + x\,dz = \int_\gamma (y, z, x)\,d\gamma = \int_\gamma (y(t), z(t), x(t))\cdot d\gamma(t)$

$\gamma: \begin{cases} x+y = 2 & \to \text{plano} \\ x^2+y^2+z^2 = 2x+2y & \to \text{esfera} \end{cases}$, orientada tal que a $\text{proj}_{xz}\gamma$ é percorrida no horário

$$\begin{cases} x+y=2 \\ x^2+y^2+z^2 = 2x+2y \end{cases} \Leftrightarrow \begin{cases} x+y=2 \\ x^2-2x+1+y^2-2x+1+z^2 = 2 \end{cases} \Leftrightarrow \begin{cases} x+y=2 \\ (x-1)^2+(y-1)^2+z^2=2 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 2-x \\ (x-1)^2+(y-1)^2+z^2 = 2 \end{cases} \Leftrightarrow (x-1)^2+(1-x)^2+z^2 = 2 \ ;\ (1-x)^2 = (x-1)^2$$

$$2(x-1)^2+z^2 = 2 \Leftrightarrow (x-1)^2 + \frac{z^2}{2} = 1$$



parametrizando no sentido horário

$$\gamma(t) = \begin{cases} x = \cos(-t)+1 \\ y = 1-\cos(-t) \\ z = \sqrt{2}\operatorname{sen}(-t) \end{cases} \therefore \gamma(t) = (1+\cos t,\ 1-\cos t,\ -\sqrt{2}\operatorname{sen}t) \quad | \ t\in[0,2\pi]$$

$$\int_0^{2\pi} \vec{F}(t)\cdot d\gamma(t) = \int_0^{2\pi} (1-\cos t,\ -\sqrt{2}\operatorname{sen}t,\ 1+\cos t)\cdot(-\operatorname{sen}t,\ \operatorname{sen}t,\ -\sqrt{2}\cos t)\cdot dt$$

$$= \int_0^{2\pi}\left(-\operatorname{sen}t + \operatorname{sen}t\cos t - \sqrt{2}\operatorname{sen}^2 t - \sqrt{2}\cos t - \sqrt{2}\cos^2 t\right)dt = -\int_0^{2\pi}\operatorname{sen}t\,dt + \frac{1}{2}\int_0^{2\pi}\operatorname{sen}2t\cdot dt$$

$$-\sqrt{2}\int_0^{2\pi}\cos t\,dt - \sqrt{2}\int_0^{2\pi}(\operatorname{sen}^2 t+\cos^2 t)\,dt = 0+0+0-\sqrt{2}\cdot 2\pi = -2\sqrt{2}\pi$$

d) $\int_\gamma x\,dx + (y+x)\,dy + z\,dz = \int_\gamma (x,\ y+x,\ z)\cdot d\gamma = \int_\gamma (x(t),\ x(t)+y(t),\ z(t))\cdot d\gamma(t)$

Obs: para se obter a projeção de γ no plano xy, basta obter uma equação com apenas as variáveis x e y.

$\gamma: \begin{cases} z = xy \\ x^2+y^2 = 1 \end{cases}$ parametrizando $\gamma(t): \begin{cases} x = \cos(-t) \\ y = \operatorname{sen}(-t) \\ z = \cos(-t)\cdot\operatorname{sen}(-t) \end{cases}$, $t\in[0,2\pi]$ $\to z = -\operatorname{sen}t\cos t$

$$\int_0^{2\pi}\vec{F}(t)\cdot d\gamma(t) = \int_0^{2\pi}(\cos t,\ \cos t-\operatorname{sen}t,\ -\operatorname{sen}t\cos t)\cdot(-\operatorname{sen}t,\ -\cos t,\ -(\cos t\cdot\cos t - \operatorname{sen}t\cdot\operatorname{sen}t))\cdot dt =$$

$$\int_0^{2\pi}\left(-\operatorname{sen}t\cos t - \cos^2 t + \operatorname{sen}t\cos t + \operatorname{sen}t\cdot\cos t\cdot\cos 2t\right)dt = -\int_0^{2\pi}\cos^2 t\cdot dt + \frac{1}{2}\int_0^{2\pi}\operatorname{sen}2t\cdot\cos 2t\cdot dt$$

$$= -\left(\left[\frac{\operatorname{sen}t\cdot\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}1\,dt\right) + \frac{1}{4}\int_0^{2\pi}\operatorname{sen}4t\cdot dt = 0 - \pi + 0 = -\pi$$

e) $\int_\gamma x^2dx + xdy + zdz = \int_\gamma (x^2, x, z).d\gamma$

$\gamma: \begin{cases} z = \frac{x^2}{9} \\ z = 1 - \frac{y^2}{4} \end{cases} \Leftrightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

parametrização

$\begin{cases} x = 3\cos t \\ y = 2\,\text{sen}\,t \\ z = \cos^2 t \end{cases}, \; t \in [0, 2\pi]$

$\gamma(t) = (3\cos t, 2\,\text{sen}\,t, \cos^2 t)$

$\int_\gamma \vec{F}.d\gamma = \int_0^{2\pi} (9\cos^2 t, 3\cos t, \cos^2 t) \cdot (-3\,\text{sen}\,t, 2\cos t, 2\cos t(-\text{sen}\,t)).dt$

$= \int_0^{2\pi} (-27.\cos^3 t + 6\cos^2 t - 2\cos^3 t.\text{sen}\,t).dt = -27\int_0^{2\pi} \cos^3 t.dt + 6\int_0^{2\pi} \cos^2 t.dt$

$- 2\int_0^{2\pi} \cos^3 t.\text{sen}\,t\,dt = -27\left(\left[\frac{\text{sen}\,t.\cos^2 t}{3}\right]_0^{2\pi} + \frac{2}{3}\int_0^{2\pi} \cos t\,dt\right) + 6\left(\left[\frac{\text{sen}\,t.\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right)$

$- 2\int_1^1 u^3.\text{sen}\,t.\frac{-du}{\text{sen}\,t} = 0 + 6\pi + 2.0 = 6\pi$

---

f) $\int_\gamma y^2dx + 3zdy = \int_\gamma (y^2, 3z, 0).d\gamma$

$\gamma: \begin{cases} z = x^2 + y^2 \\ z = 2x + 4y \end{cases} \Leftrightarrow x^2 + y^2 - 2x - 4y = 0 \Leftrightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 5$

$(x-1)^2 + (y-2)^2 = 5 \Leftrightarrow \left(\frac{x-1}{\sqrt{5}}\right)^2 + \left(\frac{y-2}{\sqrt{5}}\right)^2 = 1$

parametrização

$\begin{cases} x = \sqrt{5}\cos t + 1 \\ y = \sqrt{5}\,\text{sen}\,t + 2 \\ z = 2.\sqrt{5}\cos t + 2 + 4\sqrt{5}\,\text{sen}\,t + 8 \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{5}\cos t + 1 \\ y = \sqrt{5}\,\text{sen}\,t + 2 \\ z = 4\sqrt{5}.\text{sen}\,t + 2\sqrt{5}\cos t + 10 \end{cases}, \; t \in [0, 2\pi]$

$\gamma(t) = (\sqrt{5}\cos t + 1, \sqrt{5}\,\text{sen}\,t + 2, 4\sqrt{5}\,\text{sen}\,t + 2\sqrt{5}\cos t + 10)$

$\int_\gamma \vec{F}.d\gamma = \int_\gamma y^2dx + 3zdy = \int_0^{2\pi} \left[(\sqrt{5}\,\text{sen}\,t + 2)^2(-\sqrt{5}\,\text{sen}\,t.dt) + 3(4\sqrt{5}\,\text{sen}\,t + 2\sqrt{5}\cos t + 10)\sqrt{5}\cos t\,dt\right]$

$= \int_0^{2\pi} (-5\sqrt{5}\,\text{sen}^3 t - 20\,\text{sen}^2 t - 4\sqrt{5}\,\text{sen}\,t + 3.20\,\text{sen}\,t.\cos t + 3.10\cos^2 t + 30\sqrt{5}\cos t)\,dt$

$= \int_0^{2\pi} (-20\,\text{sen}^2 t + 20\cos^2 t + 10\cos^2 t)\,dt = \int_0^{2\pi} (20\cos 2t + 10\cos^2 t)\,dt =$

$\int_0^{2\pi} 10\cos^2 t\,dt = 10\left(\left[\frac{\text{sen}\,t\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right) = 10 \cdot \frac{2\pi}{2} = 10\pi$

g) $\int_\gamma z\,dy - x\,dz = \int_\gamma (0, z, -x)\cdot d\gamma$

$$\gamma: \begin{cases} \frac{x^2}{6} + \frac{y^2}{4} + \frac{z^2}{6} = \frac{4}{3} \\ x + z = 2 \end{cases} \Leftrightarrow \begin{cases} 4x^2 + 6y^2 + 4z^2 = 32 \\ z = 2 - x \end{cases} \Leftrightarrow$$

$$4x^2 + 6y^2 + 4(2-x)^2 = 32 \Leftrightarrow 4x^2 + 6y^2 + 4(4 - 4x + x^2) = 32 \Leftrightarrow$$

$$8x^2 - 16x + 6y^2 + 16 = 32 \Leftrightarrow 8(x-1)^2 + 6y^2 = 24 \Leftrightarrow$$

$$\left(\frac{x-1}{\sqrt{3}}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

parametrizando

$$\gamma(t) = (\sqrt{3}\cos t + 1, 2\,\text{sen}\,t, 1 - \sqrt{3}\cos t) \quad \gamma: \begin{cases} x = \sqrt{3}\cos t + 1 \\ y = 2\,\text{sen}\,t \\ z = 2 - \sqrt{3}\cos t - 1 \end{cases}, \; t \in [0, 2\pi]$$

$$\int_\gamma \vec{F}\,d\gamma = \int_0^{2\pi} (0, 1 - \sqrt{3}\cos t, -\sqrt{3}\cos t - 1)\cdot(-\sqrt{3}\,\text{sen}\,t, 2\cos t, \sqrt{3}\,\text{sen}\,t)\,dt$$

$$\int_0^{2\pi} (2\cos t - 2\sqrt{3}\cos^2 t - 3\cos t\,\text{sen}\,t - \sqrt{3}\,\text{sen}\,t)\,dt = -\int_0^{2\pi} 2\sqrt{3}\cos^2 t\,dt =$$

$$-2\sqrt{3}\left(\left[\frac{\text{sen}\,t\cdot\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1\,dt\right) = -2\sqrt{3}\pi$$

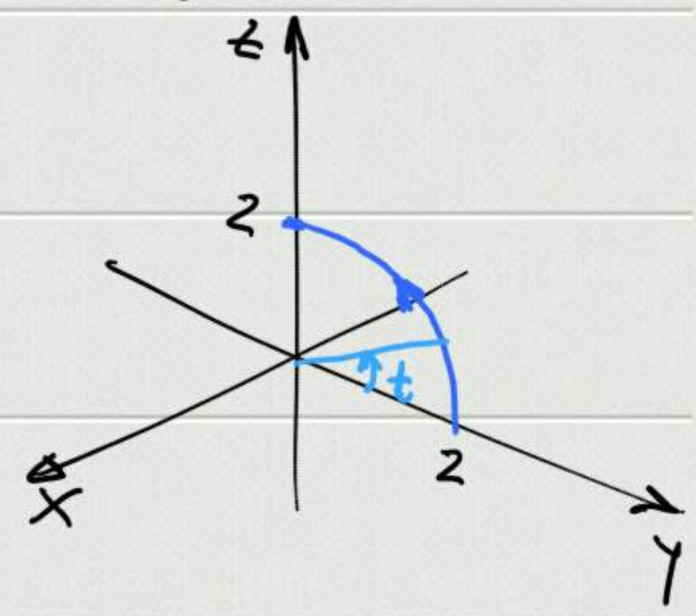
4) a) $\int_\gamma 2x\,dx + \left(z^2 - \frac{y^2}{2}\right)dz = \int_\gamma \left(2x, 0, z^2 - \frac{y^2}{2}\right)\cdot d\gamma$

$$\gamma: \begin{cases} x = 0 \\ y^2 + z^2 = 4 \end{cases};$$

parametrizando

$$\begin{cases} y = 2\cos t \\ z = 2\,\text{sen}\,t \end{cases}, \; t \in \left[0, \frac{\pi}{2}\right]$$

$$\gamma(t) = (0, 2\cos t, 2\,\text{sen}\,t)$$



$$\int_\gamma \vec{F}\,d\gamma = \int_0^{\pi/2} \left(2x, 0, z^2 - \frac{y^2}{2}\right)\cdot(0, -2\,\text{sen}\,t, 2\cos t)\,dt$$

$$= \int_0^{\pi/2} (2\cdot 0, 0, 4\,\text{sen}^2 t - 2\cos^2 t)\cdot(0, -2\,\text{sen}\,t, 2\cos t)\,dt$$

$$= \int_0^{\pi/2} (8\,\text{sen}^2 t\cdot\cos t - 4\cos^3 t)\,dt = \int_0^1 8u^2\cdot\cos t\cdot\frac{du}{\cos t} - 4\left(\left[\frac{\text{sen}\,t\cos^2 t}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\cos t\,dt\right)$$

$$\frac{8}{3} - \frac{8}{3}(1 - 0) = 0$$

b) $\displaystyle\int_\gamma \frac{(x+y)\,dx - (x-y)\,dy}{x^2+y^2} = \int_\gamma \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)\cdot d\gamma$

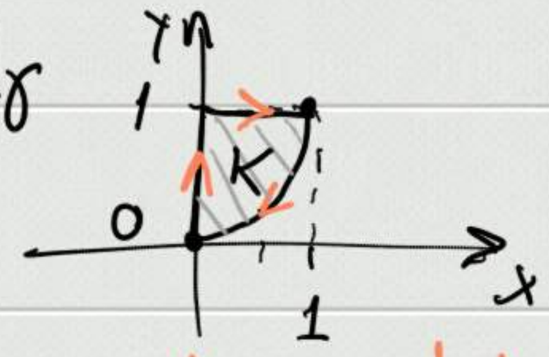
$\gamma: x^2+y^2 = a^2 \qquad \gamma: \begin{cases} x = a\cos(-t) \\ y = a\,\text{sen}(-t) \end{cases}, \quad t \in [0, 2\pi]$ → sentido horário

$\gamma(t) = a(\cos t, -\text{sen}\, t)$

$$\int_\gamma \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)\cdot d\gamma = \int_0^{2\pi}\left(\frac{a\cos t - a\,\text{sen}\, t}{a^2}, -\frac{a\,\text{sen}\, t - a\cos t}{a^2}\right)\cdot a\cdot(-\text{sen}\, t, -\cos t)\,dt$$

$$= \int_0^{2\pi}\left(-\text{sen}\, t\cdot\cos t + \text{sen}^2 t + \text{sen}\, t\cdot\cos t + \cos^2 t\right)\cdot dt = 2\pi$$

c) $\displaystyle\oint_\gamma \sqrt{y}\,dx + \sqrt{x}\,dy = \oint_\gamma (\sqrt{y}, \sqrt{x})\cdot d\gamma = \oint_\gamma \vec{F}\cdot d\gamma$



$\gamma: \begin{cases} x = 0 \\ y = 1 \\ y = x^2 \end{cases}$, Dado o fato de que γ é fechada e de que a região K está contida no domínio Ω de $\vec{F}$ ($\Omega = \mathbb{R}^2 - \{(x,y) \mid x < 0 \text{ e } y < 0\}$), pode-se utilizar o Teorema de Green. Logo:

$$\oint_\gamma \vec{F}\cdot d\gamma = -\oint \vec{F}\cdot d\gamma = -\iint_K \left(\frac{\partial}{\partial x}\sqrt{x} - \frac{\partial \sqrt{y}}{\partial y}\right)\cdot dx\,dy = \iint_K \left(\frac{\partial}{\partial y}\sqrt{y} - \frac{\partial}{\partial x}\sqrt{x}\right)dx\,dy$$

$$\therefore \oint_\gamma (\sqrt{y}, \sqrt{x})\cdot d\gamma = \iint_K \left(\frac{\partial}{\partial y}\sqrt{y} - \frac{\partial}{\partial x}\sqrt{x}\right)\cdot dx\,dy$$

$K = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \text{ e } x^2 \le y \le 1\}$

Portanto: $\displaystyle\iint_K \left(\frac{\partial}{\partial y}\sqrt{y} - \frac{\partial}{\partial x}\sqrt{x}\right)dx\,dy = \int_0^1\int_{x^2}^1 \left(\frac{1}{2\sqrt{y}} - \frac{1}{2\sqrt{x}}\right)dy\,dx = \int_0^1 \left[\sqrt{y} - \frac{y}{2\sqrt{x}}\right]_{x^2}^1 dx$

$$= \int_0^1 \left(1 - \sqrt{x^2} - \frac{1-x^2}{2\sqrt{x}}\right)dx = \int_0^1 \left(1 - x - \frac{x^{-1/2}}{2} + \frac{x^{3/2}}{2}\right)dx$$

$$= \left(\cancel{1} - \frac{1}{2} - \cancel{\frac{1}{2}\cdot 2} + \frac{\cancel{2}}{5} - \frac{1}{\cancel{2}}\right) = \frac{-5+2}{10} = -\frac{3}{10}$$

$$\therefore \boxed{\oint \sqrt{y}\,dx + \sqrt{x}\,dy = -\frac{3}{10}}$$

5) $\vec{F}(x,y) = (x, y+2) \qquad \vec{r}(t) = (t - \text{sen}\, t, 1 - \cos t), \quad t \in [0, 2\pi]$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(x(t), y(t))\cdot d\vec{r}(t) = \int_0^{2\pi} (t - \text{sen}\, t, 1 - \cos t + 2)\cdot(1 - \cos t, \text{sen}\, t)\,dt$$

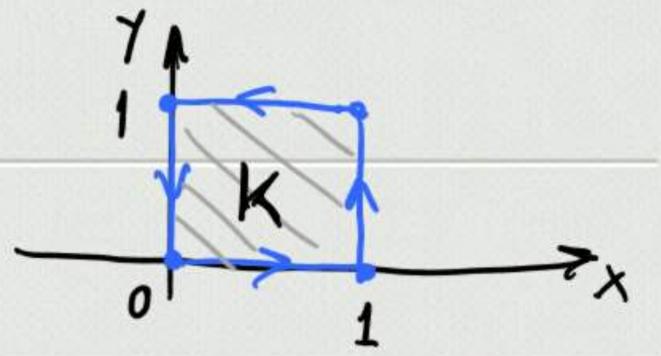
$$\int_0^{2\pi} \left(t - \text{sen}\, t - t\cdot\cos t + \cancel{\text{sen}\, t\cos t} + 3\,\text{sen}\, t - \cancel{\text{sen}\, t\cos t}\right)dt$$

$$\int_0^{2\pi} t\cdot dt - \int_0^{2\pi} t\cdot\cos t\cdot dt + 2\int_0^{2\pi} \text{sen}\, t\cdot dt = \frac{4\pi^2}{2} - \left(\left[t\cdot\text{sen}\, t\right]_0^{2\pi} - \int_0^{2\pi} 1\cdot\text{sen}\, t\cdot dt\right) + 0$$

$$= 2\pi^2$$

6) Este exercício simplesmente se resume a aplicar o teorema de Green

a) $\oint x^2y\,dx + xy^3\,dy = \oint (x^2y, xy^3)\,d\gamma$ $\qquad \gamma:$



$\vec{F}: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2$, logo, $K \subset \Omega$, portanto, o Teorema de Green é válido

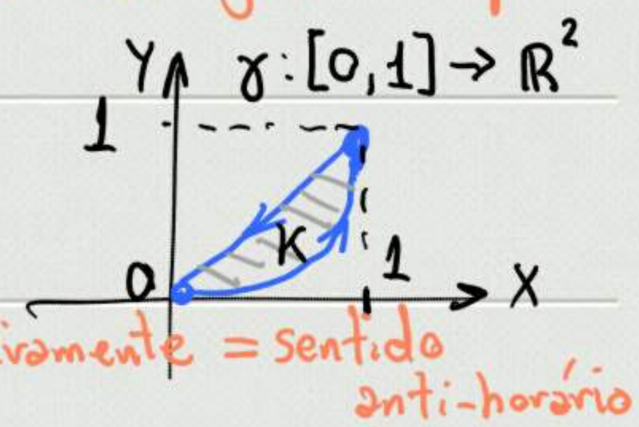
$$\text{rot}(\vec{F}) = \vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xy^3 & 0 \end{vmatrix} = \cancelto{0}{0 + \frac{\partial}{\partial z} x^2y\,\hat{j}} + \frac{\partial}{\partial x} xy^3\,\hat{k} - \frac{\partial}{\partial y} x^2y\,\hat{k} \cancelto{0}{- \frac{\partial}{\partial z} xy^3\,\hat{i}} - 0 = (y^3 - x^2)\hat{k}$$

$$\therefore \boxed{\oint (x^2y, xy^3) \cdot d\gamma = \iint_K (y^3 - x^2)\,dx\,dy}$$

$$\iint_K (y^3 - x^2)\,dx\,dy = \int_0^1\int_0^1 (y^3 - x^2)\,dx\,dy = \int_0^1 \left(y^3 - \frac{1}{3}\right)dy = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

Perceba a utilidade do Teorema de Green: Pode se reduzir várias integrais de linha (Neste caso 4) a apenas uma integral dupla.

b) $\oint (x+2y)\,dx + (x-2y)\,dy = \oint \underbrace{(x+2y, x-2y)}_{\vec{F}} \cdot d\gamma$



$\vec{F}: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2$, logo: $K \subset \Omega$, portanto, o Teorema de Green é válido

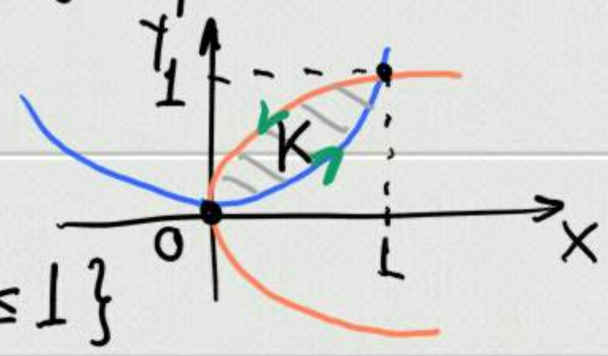
$$\text{rot}(\vec{F}) = \vec{\nabla} \wedge \vec{F} = \left(\frac{\partial}{\partial x}(x-2y) - \frac{\partial}{\partial y}(x+2y)\right)\hat{k} = (1-2)\hat{k} = -1\,\hat{k}$$

$$\oint (x+2y, x-2y) \cdot d\gamma = \iint_K -1 \cdot dx\,dy \quad \Big| \quad K = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le x \text{ e } 0 \le x \le 1\}$$

$$\therefore \iint_K -1\,dx\,dy = \int_0^1\int_{x^2}^{x} -1\,dy\,dx = -\int_0^1 (x - x^2)\,dx = -\left(\frac{1}{2} - \frac{1}{3}\right) = -\frac{1}{6}$$

c) $\oint_\gamma (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy = \oint (y + e^{\sqrt{x}}, 2x + \cos y^2)\,d\gamma$

$$\gamma: \begin{cases} y = x^2 \\ x = y^2 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ y = \pm\sqrt{x} \end{cases} \Leftrightarrow$$



anti-horário

$K = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le \sqrt{x},\ 0 \le x \le 1\}$

$\vec{F}(x,y) = (y + e^{\sqrt{x}}, 2x + \cos y^2)$; $\vec{F}: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 - \{(x,y) \mid x < 0\}$

Logo $K \subset \Omega$ e, portanto, é válido o Teorema de Green

$$\text{rot}(\vec{F}) = \vec{\nabla} \wedge \vec{F} = \left(\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right)\hat{k} = 1\,\hat{k}$$

$$\oint (y + e^{\sqrt{x}}, 2x + \cos y^2)\,d\gamma = \iint_K 1\,dx\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} 1\,dy\,dx = \int_0^1 (\sqrt{x} - x^2)\,dy = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Perceba que, neste caso, o teorema ainda remove a parte horrível de se integrar $e^{\sqrt{x}}$ e $\cos y^2$

d) $\oint x^2dx + y^2dy = \oint (x^2, y^2)\, d\gamma$; $\gamma: (x^3)^2 + (y^3)^2 = 1$

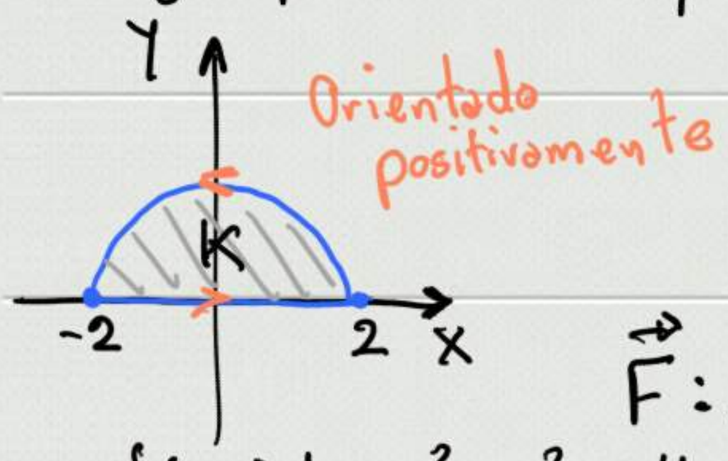
$\vec{F}(x,y): \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2$, Percebe-se que, qualquer que seja K, $K \subset \Omega$. Antes de se fazer uma substituição por coordenadas polares para se descrever K mais facilmente. Obtenha $rot(\vec{F})$, pois, se $rot(\vec{F}) = \vec{0}$, então $\oint \vec{F}.d\gamma = 0$

$rot(\vec{F}) = \left(\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x^2\right)\hat{k} = \vec{0}$. Portanto, $\oint \vec{F}.d\gamma = \iint_K 0.dxdy = 0$

Obs: $rot(\vec{F}) = \vec{0} \Rightarrow \oint \vec{F}.d\gamma = 0$

Obs 2: não significa necessariamente que $\vec{F}$ seja conservativo

e) $\oint xy\,dx + 2x^2dy = \oint (xy, 2x^2)\, d\gamma$; $\gamma: \begin{cases} x^2+y^2 = 4 \\ y \geq 0 \end{cases}$



$rot(\vec{F}) = \left(\frac{\partial}{\partial x} 2x^2 - \frac{\partial}{\partial y} xy\right)\hat{k} = 3x\,\hat{k}$

$\vec{F}: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 \quad \therefore K \subset \Omega$ vale o Green

$K = \{(x,y) \mid x^2+y^2 \leq 4 \text{ e } y \geq 0\}$, em coordenadas polares:

$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \Leftrightarrow \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$; $K_{r\theta} = \{(r,\theta) \in \mathbb{R}_+ \times [0,2\pi] \mid r \leq 2 \text{ e } 0 \leq \theta \leq \pi\}$

$\oint (xy, 2x^2) \cdot d\gamma = \iint_K 3x\, dxdy = \iint_{K_{r\theta}} 3r\cos\theta . r . drd\theta$

$= \int_0^{\pi}\int_0^2 3r^2\cos\theta . drd\theta = \int_0^{\pi} 3.\frac{8}{3}.\cos\theta . d\theta = 8.0 = 0$

f) $\oint 2xy\,dx + x^2dy = \oint (2xy, x^2)\, d\gamma$; $\gamma: r = 1 + \cos\theta$

$\vec{F}: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 \quad \therefore K \subset \Omega$

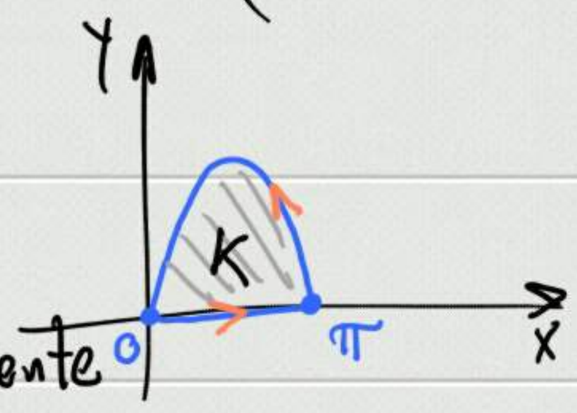
$rot(\vec{F}) = \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} 2xy\right)\hat{k} = \vec{0}$, $rot(\vec{F}) = \vec{0} \Rightarrow \oint \vec{F}d\gamma = 0$

$\therefore \oint (2xy, x^2)\, d\gamma = 0$

g) $\oint (xy + e^{x^2})dx + (x^2 - \ln(1+y))dy = \oint (xy + e^{x^2}, x^2 - \ln(1+y))\, d\gamma$

$\gamma: \begin{cases} y = \text{sen}\, x \\ \text{ou} \\ x = 0 \end{cases}$

orientada positivamente



$F: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 - \{(x,y) \mid y > -1\}$

$K = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \text{sen}\, x \text{ e } 0 \leq x \leq \pi\}$

$\therefore K \subset \Omega$

$rot(\vec{F}) = \left(\frac{\partial}{\partial x}(x^2 - \ln(1+y)) - \frac{\partial}{\partial y}(xy + e^{x^2})\right)\hat{k} = x\,\hat{k}$

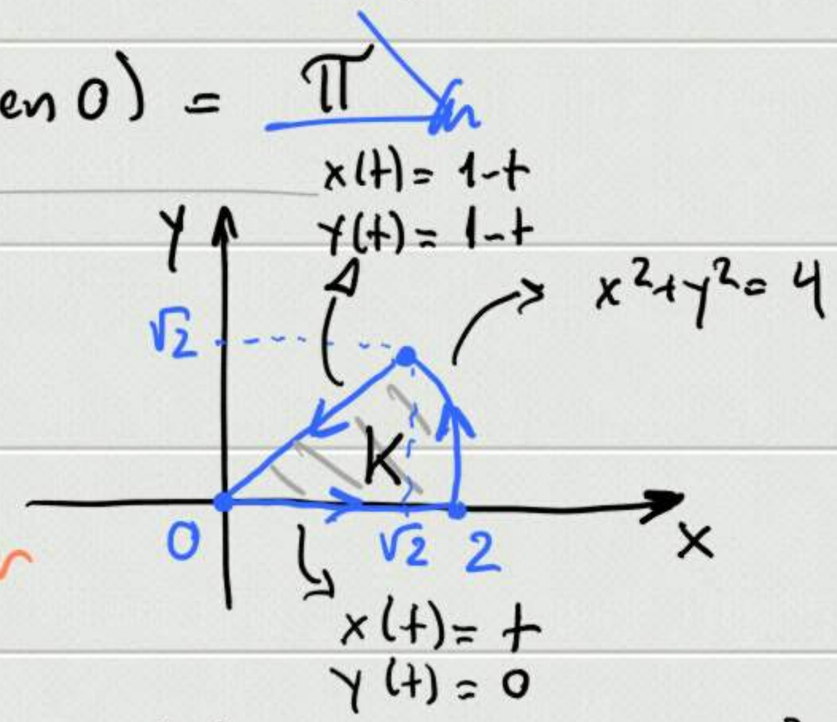
$$\oint \left(xy+e^{x^2},\, x^2-\ln(1-y)\right) d\gamma = \iint_K x\,dx\,dy = \int_0^{\pi}\int_0^{\operatorname{sen}x} x\,dy\,dx = \int_0^{\pi} x.\operatorname{sen}x\,dx =$$

$$\left[x.(-\cos x)\right]_0^{\pi} - \int_0^{\pi} 1.(-\cos x).dx = \pi + (\operatorname{sen}\pi - \operatorname{sen}0) = \pi$$

h) $\oint \vec{F}.d\vec{r}$, $\vec{F}(x,y) = (y^2 - x^2y,\, xy^2)$

$F: \Omega \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 \quad \therefore \quad K \subset \Omega$

γ:



Percebe-se que K é mais facil de se descrever em coordenadas polares

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \end{cases} \Leftrightarrow \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r \;;\; K_{r\theta} = \left\{(r,\theta) \in \mathbb{R}_+ \times [0,2\pi] \mid r \le 2,\; 0 \le \theta \le \frac{\pi}{4}\right\}$$

$$\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x} xy^2 - \frac{\partial}{\partial y}(y^2 - x^2y)\right)\hat{k} = (y^2 - 2y + x^2)\,\hat{k}$$

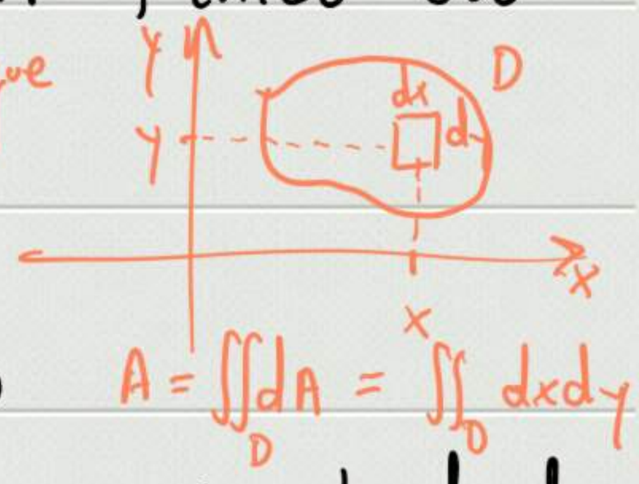
$$\oint (y^2 - x^2y,\, xy^2).d\gamma = \iint_K (y^2 - 2y + x^2)\,dx\,dy = \iint_{K_{r\theta}} (r^2\operatorname{sen}^2\theta - 2r\operatorname{sen}\theta + r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_0^2 (r^3 - 2r^2\operatorname{sen}\theta)\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{2^4}{4} - 2.\frac{2^3}{3}.\operatorname{sen}\theta\right) d\theta = \pi - \frac{16}{3}\left[-\cos\theta\right]_0^{\frac{\pi}{4}} =$$

$$\pi + \frac{16}{3}\left(\frac{\sqrt{2}}{2} - 1\right) = \pi + \frac{16}{3}\left(\frac{1}{\sqrt{2}} - 1\right)$$

7) a) Se D é um lugar geométrico de $\mathbb{R}^2$, então sua área é dada por : $\text{Área}(D) = \iint_D dx\,dy$

verifique



$A = \iint_D dA = \iint_D dx\,dy$

D satisfaz a hipótese de Green e existe algum $\vec{F}(x,y)$ cujo domínio $\Omega \subset \mathbb{R}^2$ contém D, então, sendo $\vec{F} = (P, Q)$, $\oint \vec{F}\,d\gamma = \iint_D \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right) dx\,dy$, tal que γ é a borda de D

Se $\boxed{\oint \vec{F}\,d\gamma = \iint_D 1\,dx\,dy}$ (Eq.0) $\Leftrightarrow$ $\boxed{\frac{\partial Q}{\partial x} - \frac{\partial}{\partial y}P = 1}$ (Eq.1)

Existe $\vec{F_1}$ tal que $\frac{\partial}{\partial y}P = 0$, então $\frac{\partial Q}{\partial x} = 1$ pela Eq.1 e $Q = x + K$, observe que $\oint (x+K)\,dy = \oint x\,dy \Leftrightarrow \oint \vec{F_1}\,d\gamma = \oint x\,dy$

Existe $\vec{F_2}$ tal que $\frac{\partial}{\partial x}Q = 0$, então $\frac{\partial P}{\partial y} = -1$ pela Eq.1 e $P = -y + K$, sendo que $\oint (-y+K)\,dx = \oint -y\,dx \Leftrightarrow \oint \vec{F_2}\,d\gamma = \oint -y\,dx$

Portanto:

$$\oint_D \vec{F}\,d\gamma = \oint_D \vec{F_1}\,d\gamma = \oint_D \vec{F_2}\,d\gamma = \iint_D 1\,dx\,dy \quad \therefore \quad \boxed{\text{Área}(D) = \oint_D x\,dy = \oint_D -y.dx}$$

b) i) $D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$

Parametrizando-se a borda de D $\gamma: \begin{cases} x = a.\cos t \\ y = b.\operatorname{sen} t \end{cases}, \; t \in [0, 2\pi]$

$\gamma(t) = (a\cos t, b\operatorname{sen} t)$

$$\text{Área}(D) = \oint_\gamma x\,dy = \int_\gamma (0, x)\cdot d\gamma = \int_0^{2\pi} (0, a\cos t)\cdot(-a\operatorname{sen} t, b\cos t)\,dt$$

$$= \int_0^{2\pi} ab\cos^2 t\,.dt = ab\left(\left[\frac{\operatorname{sen} t.\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right) = \pi ab \text{ u.a.}$$

ii) $D = \{(x,y) \in \mathbb{R}^2 \mid (x^{1/3})^2 + (y^{1/3})^2 \leq (a^{1/3})^2\}$

Parametrizando-se a borda de D $\gamma: \begin{cases} x^{1/3} = a^{1/3}.\cos t \\ y^{1/3} = a^{1/3}.\operatorname{sen} t \end{cases} \Leftrightarrow$

$\begin{cases} x = a.\cos^3 t \\ y = a.\operatorname{sen}^3 t \end{cases}, \; t \in [0, 2\pi]$ Obs: perceba que o período de $\cos^n t$ ou $\operatorname{sen}^n t$ com n ímpar e 2π!

$$\text{Área}(D) = \oint_\gamma x\,dy = \int_\gamma (0,x)\cdot d\gamma = \int_0^{2\pi} (0, a.\cos^3 t)\cdot(a\,3\cos^2 t.(-\operatorname{sen} t), a.3\operatorname{sen}^2 t.\cos t)\,dt$$

$$= \int_0^{2\pi} a^2.3.\cos^4 t.\operatorname{sen}^2 t.dt = 3a^2\int_0^{2\pi}\left(\frac{\cos 2t + 1}{2}\right)^2\cdot\left(\frac{1-\cos 2t}{2}\right).dt$$

$w = 2t$
$\frac{dw}{2} = dt$

$$= \frac{3a^2}{8}\cdot\int_0^{2\pi}(\cos 2t + 1)(1-\cos^2 2t).dt = \frac{3a^2}{8}\left(\int_0^{2\pi}\cos 2t.\operatorname{sen}^2 2t.dt + \int_0^{2\pi}\operatorname{sen}^2 2t.dt\right)$$

$$= \frac{3a^2}{8}.\left(\int_0^0 \cos 2t\,.u^2\,\frac{-du}{2\cos 2t} + \frac{1}{2}\int_0^{4\pi}\operatorname{sen}^2 w.dw\right)$$

$u = \operatorname{sen} 2t$
$du = 2(-\cos 2t)dt$
$dt = \frac{-du}{2\cos 2t}$
$t = 2\pi \Rightarrow u = 0$
$t = 0 \Rightarrow u = 0$

$$= \frac{3a^2}{8}.\left(0 + 0 + \frac{1}{2}\left(\left[\frac{-\cos w.\operatorname{sen} w}{2}\right]_0^{4\pi} + \frac{1}{2}\int_0^{4\pi} 1.dt\right)\right)$$

$$= \frac{3a^2}{8}.\pi = \frac{3\pi a^2}{8} \text{ u.a.}$$

(Eq.1)

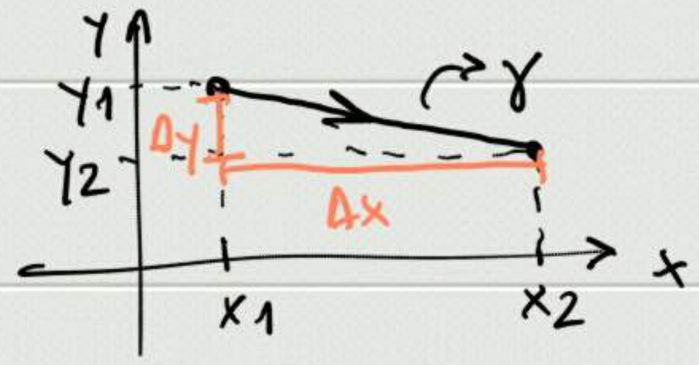
c) $\vec{r}(t) = (\cos^3 t, \operatorname{sen}^3 t)$, $t \in [0, 2\pi]$, sendo $\vec{r}(t)$ o contorno de D

$$\text{Área}(D) = \oint_\gamma x\,dy = \oint_\gamma (0,x)\cdot d\gamma = \int_0^{2\pi}(0, \cos^3 t)\cdot(3\cos^2 t.(-\operatorname{sen} t), 3\operatorname{sen}^2 t.\cos t).dt$$

$$= \int_0^{2\pi} 3\cos^4 t.\operatorname{sen}^2 t.dt = \int_0^{2\pi} 3.1^2.\cos^4 t.\operatorname{sen}^2 t.dt$$, segundo Eq.1

$$\boxed{\text{Área}(D) = \frac{3\pi}{8} \text{ u.a.}}$$

8 a)



$$\int_\gamma x\,dy - y\,dx = ?$$

Parametrizando

$$\gamma: \begin{cases} (x_2-x_1)t \\ (y_2-y_1)t \end{cases}, \quad t \in [0,1]$$

$$\gamma(t) = (x_1 + (x_2-x_1)t,\ y_1 + (y_2-y_1)t)$$

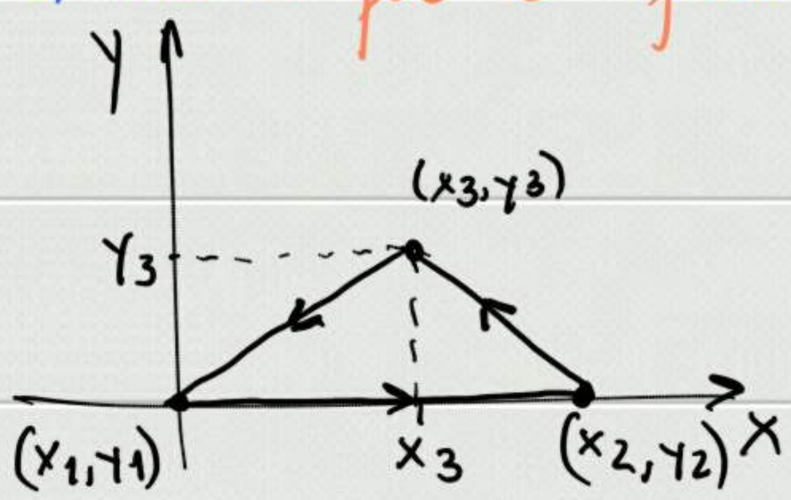
$$\int_\gamma x\,dy - y\,dx = \int_\gamma (-y, x)\,d\gamma = \int_0^1 \big(-(y_1+(y_2-y_1)t),\ x_1+(x_2-x_1)t\big)\cdot(x_2-x_1,$$

$$y_2-y_1)\,dt = \int_0^1 \big(y_1(x_1-x_2) + \cancel{(y_2-y_1)(x_1-x_2)t} + x_1(y_2-y_1) + \cancel{(x_2-x_1)(y_2-y_1)t}\big)\,dt$$

$$\int_0^1 (\cancel{x_1y_1} - x_2y_1 + x_1y_2 - \cancel{x_1y_1})\,dt = x_1y_2 - x_2y_1$$

(Eq. 1) $\int_\gamma x\,dy - y\,dx = x_1y_2 - x_2y_1$ c.q.d.

b) Note que a fórmula funciona:



$$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

$$A = \frac{1}{2}\left[(0.0 - x_2 0) + (x_2y_3 - x_3.0) + (x_3.0 - 0.y_3)\right]$$

$$\boxed{A = \frac{1}{2}\,x_2y_3}$$

$x_1 = y_1 = y_2 = 0$

Sendo K um conjunto contido em $\mathbb{R}^2$, com interior não vazio e sendo γ as bordas de K (os lados do polígono), tem-se, pelo Teorema de Green, que:

$$\oint_\gamma x\,dy - y\,dx = \oint_\gamma (-y, x)\,d\gamma = \iint_K 1+1\,dx\,dy = 2\iint_K 1\,dx\,dy = 2A$$

$$\therefore \boxed{A = \frac{1}{2}\cdot\oint_\gamma x\,dy - y\,dx}$$ (Eq. 2) Tal que A é a área do polígono determinado pelo conjunto K

Se γ é a curva que representa os lados de um polígono de n lados e seja γ_i o i-ésimo lado do polígono, que liga os pontos (x_i, y_i) e (x_{i+1}, y_{i+1}) tal que $1 \le i < n \mid i \in \mathbb{N}^*$ e quando $i = n$, γ_n liga (x_n, y_n) a (x_1, y_1), fechando o polígono.

Tem-se então que:

$$\oint_\gamma x\,dy - y\,dx = \oint_{\gamma_1} x\,dy - y\,dx + \oint_{\gamma_2} x\,dy - y\,dx + \dots + \oint_{\gamma_{n-1}} x\,dy - y\,dx + \oint_{\gamma_n} x\,dy - y\,dx$$

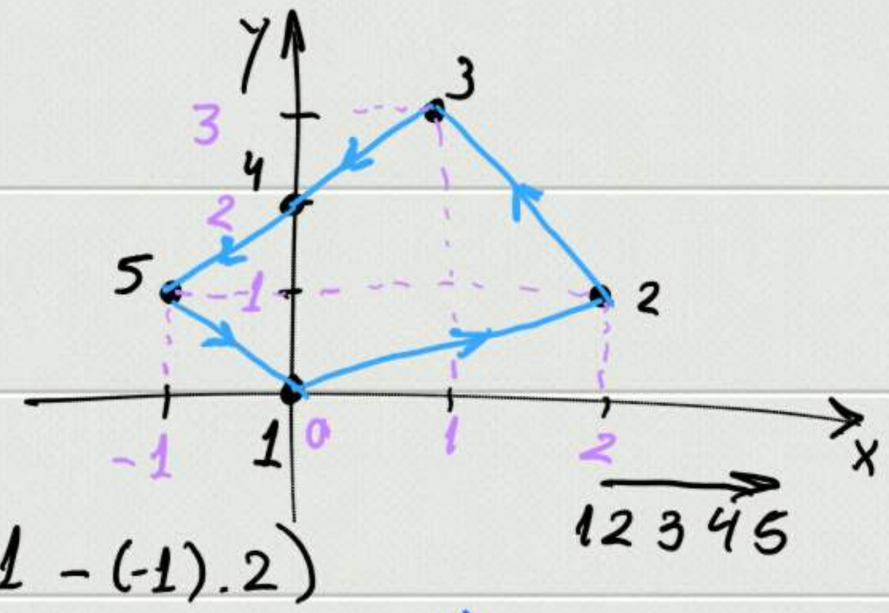
Portanto, segundo a Equação 1 demonstrada no item a:

$$\boxed{\oint_\gamma x\,dy - y\,dx = (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \dots + (x_{n-1}\cdot y_n - x_n\cdot y_{n-1}) + (x_ny_1 - x_1y_n)}$$ (Eq. 3)

De Eq. 2 e Eq. 3: $\boxed{A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + \dots + (x_{n-1}\cdot y_n - x_n y_{n-1}) + (x_ny_1 - x_1y_n)\right]}$

c.q.d.

c) Aplicando a equação obtida no item b no sentido anti-horário:



$$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5)\right]$$

$$A = \frac{1}{2}\left[(0.1 - 2.0) + (2.3 - 1.1) + (1.2 - 0.3) + (0.1 - (-1).2) + (-1.0 - 0.1)\right] = \frac{1}{2}\left[0 + 5 + 2 + 2 + 0\right] = \frac{9}{2} \text{ u.a.}$$

9 a) $\int_\gamma x^2(5y\,dx + 7x\,dy) + e^y dy = \int_\gamma 5x^2y\,dx + (7x^3 + e^y)\,dy$

$$\int_\gamma (5x^2y,\ 7x^3 + e^y)\,d\gamma \quad ; \quad \gamma : \begin{cases} 16x^2 + 25y^2 = 100 \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2}{25/4} + \dfrac{y^2}{4} = 1 \\ x \geq 0 \end{cases}$$

Parametrizando de $(0,-2)$ até $(0,2)$

$$\gamma : \begin{cases} x = \frac{5}{2}\cos t \\ y = 2\,\text{sen}\,t \end{cases}, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow \gamma(t) = \left(\frac{5}{2}\cos t,\ 2\,\text{sen}\,t\right)$$

$$\int_\gamma \vec{F}\cdot d\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (5x^2y,\ 7x^3 + e^y) \cdot \left(\frac{5}{2}(-\text{sen}\,t),\ 2\cos t\right) \cdot dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(5\left(\frac{5}{2}\cos t\right)^2 . 2\,\text{sen}\,t,\ 7\left(\frac{5}{2}\cos t\right)^3 + e^{2\,\text{sen}\,t}\right) \cdot \left(\frac{5}{2}(-\text{sen}\,t),\ 2\cos t\right) \cdot dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{125}{4}\cos^2 t . 2\,\text{sen}\,t . \left(-\frac{5}{2}\,\text{sen}\,t\right) + 7 . \frac{125}{8} . \cos^3 t . 2\cos t + 2\cos t . e^{2\,\text{sen}\,t}\right) . dt$$

$$= -\frac{625}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t . \text{sen}^2 t . dt + 7 . \frac{125}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t\,dt + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t . e^{2\,\text{sen}\,t} . dt$$

$$= -\frac{625}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} . \frac{1 - \cos 2t}{2} . dt$$

$u = \text{sen}\,t \Leftrightarrow du = -\cos t . dt \Leftrightarrow dt = \dfrac{du}{-\cos t}$

$t = \frac{\pi}{2} \Rightarrow u = 1 ; t = -\frac{\pi}{2} \Rightarrow u = -1$

$w = 2t \Leftrightarrow dt = \dfrac{dw}{2}$

$t = \frac{\pi}{2} \Rightarrow w = \pi ; t = -\frac{\pi}{2} \Rightarrow w = -\pi$

$$+ 7 . \frac{125}{4}\left(\left[\frac{\text{sen}\,t . \cos^3 t}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt\right) + 2\int_{-1}^{1} \cancel{\cos t} . e^{2u} \frac{du}{-\cancel{\cos t}}$$

$$= -\frac{625}{4} . \frac{1}{4} . \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos^2 2t)\,dt + 7 . \frac{125}{4} . \left(0 + \frac{3}{4}\left(\left[\frac{\text{sen}\,t . \cos t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dt\right)\right)$$

$$+ \frac{2}{2}\left[e^{2u}\right]_{-1}^{1} = -\frac{625}{16} . \left(\pi - \int_{-\pi}^{\pi} \cos^2 w \frac{dw}{2}\right) + 7 . \frac{125}{4} . \frac{3\pi}{8} + e^2 - \frac{1}{e^2} =$$

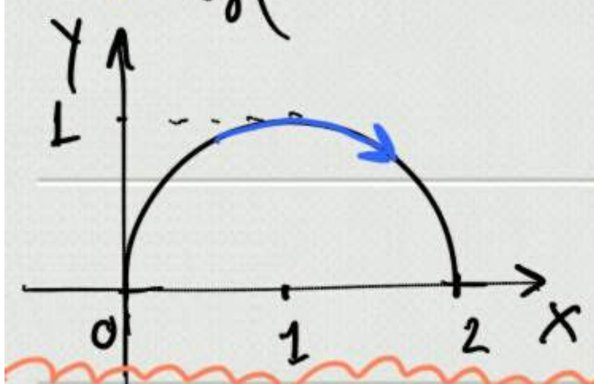
$$-\frac{625}{16}\left(\pi - \frac{1}{2}\left(\left[\frac{\text{sen}\,w . \cos w}{2}\right]_{-\pi}^{\pi} + \frac{1}{2}\int_{-\pi}^{\pi} 1\,dw\right)\right) + 7 . \frac{125}{4} . \frac{3\pi}{8} + e^2 - \frac{1}{e^2}$$

$$-\frac{625}{16}\left(\pi - \frac{\pi}{2}\right) + 7\,\frac{125}{4} . \frac{3\pi}{8} + e^2 - \frac{1}{e^2} = -5 . \frac{125}{4} . \frac{\pi}{8} + 7\,\frac{125}{4}\,\frac{3\pi}{8} + e^2 - \frac{1}{e^2}$$

$$= \frac{125\pi}{2} + e^2 - \frac{1}{e^2}$$

b) $\int_\gamma \left(2x e^y - x^2 y - \frac{y^3}{3}\right)dx + \left(x^2 e^y + \operatorname{sen} y\right) dy$; Percebe-se que este campo vetorial não é conservativo pois $\operatorname{rot}(\vec{F}) \neq \vec{0}$

$$\operatorname{rot}(\vec{F}) = \left(2x e^y - 2x e^y + x^2 + y^2\right)\hat{k} = (x^2+y^2)\hat{k} \neq \vec{0}$$



$$\gamma: \begin{cases} x^2+y^2-2x = 0 \\ y \geq 0 \end{cases} \Leftrightarrow \begin{cases} x^2-2x+1+y^2 = 1 \\ y \geq 0 \end{cases} \Leftrightarrow \begin{cases} (x-1)^2+y^2 = 1 \\ y \geq 0 \end{cases}$$

Ou faz t percorrer o sentido horário ou integra no anti-horário e multiplica por -1

$$\gamma: \begin{cases} x = 1+\cos(-t) \\ y = \operatorname{sen}(-t) \end{cases} \Leftrightarrow \gamma(t) = (1+\cos t, -\operatorname{sen} t) \mid t \in [0,\pi]$$

$$\frac{dx(t)}{dt} = -\operatorname{sen} t, \quad \frac{dy(t)}{dt} = -\cos t$$

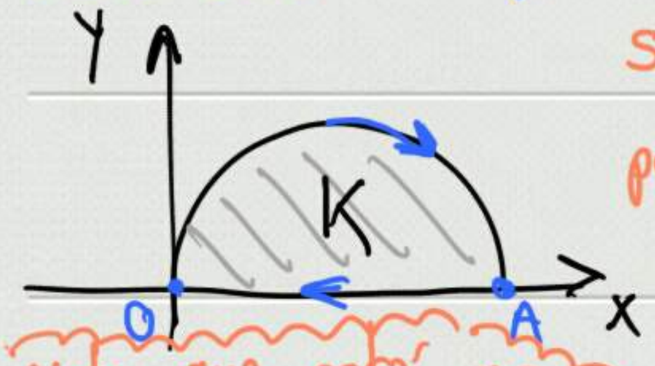
$$\therefore \int_\gamma \left(2x e^y - x^2 y - \frac{y^3}{3}\right)dx + \left(x^2 e^y + \operatorname{sen} y\right)dy =$$

$$\int_0^\pi \left[2(1+\cos t).e^{-\operatorname{sen} t} - (1+\cos t)^2(-\operatorname{sen} t) - \frac{(-\operatorname{sen} t)^3}{3}\right](-\operatorname{sen} t)\,dt + \left[(1+\cos t)^2 . e^{-\operatorname{sen} t} + \operatorname{sen}(-\operatorname{sen} t)\right](-\cos t)\,dt$$

$$= \int_0^\pi -2(\operatorname{sen} t + \operatorname{sen} t . \cos t).e^{-\operatorname{sen} t}\,dt - \int_0^\pi (1+\cos t)^2 \operatorname{sen}^2 t . dt - \int_0^\pi \frac{\operatorname{sen}^4 t}{3} . dt$$

$$- \int_0^\pi (1+\cos t)^2 . e^{-\operatorname{sen} t} . \cos t - \int_0^\pi \cos t . \operatorname{sen}(-\operatorname{sen} t) . dt$$

Melhor parar por aqui, até dá para resolver esta integral, mas será um trabalho braçal monstruoso pior do que o da questão anterior. Veja o valor do rotacional de $\vec{F}$ é muito bonito para ser verdade. A próxima ideia é o melhor esquema de resolução:



Se fechar o caminho, ligando com o segmento $\overleftarrow{OA}$, pode se fazer a integral pelo Teorema de Green e depois desconta a integral de linha do segmento $\overleftarrow{OA}$ no sentido horário (que é igual a somar a integral no sentido oposto).

Note que está no sentido anti-horario logo deve se multiplicar por -1

$$\oint_{\gamma_1} \vec{F} . d\gamma_1 = \iint_K x^2+y^2 . dx dy \text{ ; coordenadas polares}$$

$$\begin{cases} x = 1+r\cos\theta \\ y = r.\operatorname{sen}\theta \end{cases} ; \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r \qquad \begin{matrix} 0 \leq \theta \leq \pi \\ 0 \leq r \leq 1 \end{matrix}$$

$$\therefore \oint_{\gamma_1} \vec{F}\, d\gamma_1 = -\oint_{\gamma_1} \vec{F} . d\gamma_1 = \iint_{K_{r,\theta}} (1+2r\cos\theta + r^2\cos^2\theta + r^2\operatorname{sen}^2\theta) . r . dr d\theta$$

$$= \int_0^\pi \int_0^1 (r + 2r^2\cos\theta + r^3)\,dr\,d\theta = \int_0^\pi \left(\frac{1}{2} + \frac{2}{3}\cos\theta + \frac{1}{4}\right)d\theta = \frac{\pi}{2} + \frac{\pi}{4} + \frac{2}{3}\left[\operatorname{sen}\theta\right]_0^\pi$$

$$= \frac{3\pi}{4} \Leftrightarrow \boxed{\oint_{\gamma_1} \vec{F} . d\gamma_1 = -\frac{3\pi}{4}}$$

Fazendo a integral de linha de $\overrightarrow{OA}$: $\gamma_{OA}: \begin{cases} x = t \\ y = 0 \end{cases} \Leftrightarrow \gamma_{\overrightarrow{OA}} = (t,0) \mid t \in [0,2]$

$$\int_{\gamma_{\overrightarrow{OA}}} \left(2xe^y - x^2y - \frac{y^3}{3}, x^2e^y + \operatorname{sen} y\right) \cdot d\gamma = \int_0^2 \left(2.t.e^0 - t^2.0 - \frac{0^3}{3}, t^2e^0 + \operatorname{sen} 0\right) \cdot (1,0)\,dt$$

$$\int_0^2 2t.dt = \frac{2}{2}\left[t^2\right]_0^2 = \boxed{4}$$

Sendo $\gamma_1 = \gamma + \gamma_{\overrightarrow{AO}}$ e $\gamma_{\overrightarrow{AO}} = -\gamma_{\overrightarrow{OA}}$, então $\gamma_1 = \gamma - \gamma_{\overrightarrow{OA}} \Leftrightarrow \gamma = \gamma_1 + \gamma_{\overrightarrow{OA}} \Rightarrow \int_\gamma \vec{F} . d\gamma = \oint_{\gamma_1} \vec{F} . d\gamma + \int_{\gamma_{\overrightarrow{OA}}} \vec{F} . d\gamma$

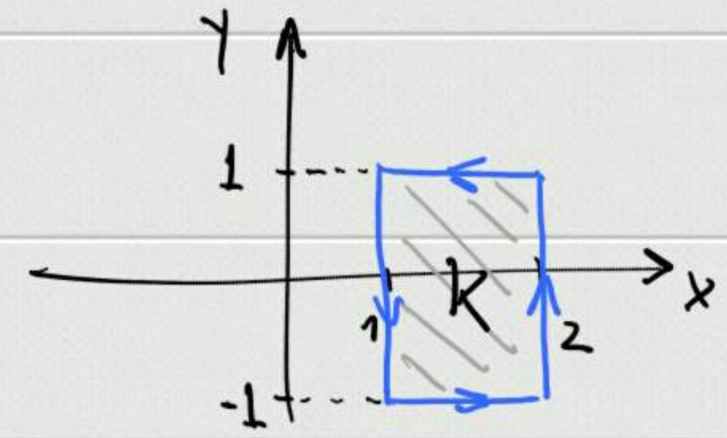
$$\therefore \boxed{\int_\gamma \vec{F} . d\gamma = -\frac{3\pi}{4} + 4}$$

c) $\oint_\gamma \vec{v}\cdot d\vec{r}$ γ é a fronteira do retângulo $[1,2]\times[-1,1]$

e $\vec{v}(x,y) = 2\,\text{arctg}\left(\frac{y}{x}\right)\hat{i} + \left(\ln(x^2+y^2) + 2x\right)\hat{j}$

Sendo $\vec{v}: \Omega \to \mathbb{R}^2$; $\text{arctg}\left(\frac{y}{x}\right) \to x \neq 0$; $\ln(x^2+y^2) \to x^2+y^2 \neq 0$

$\Leftrightarrow y \neq 0$. Portanto $\Omega = \mathbb{R}^2 - \{(0,0)\}$



Sendo que $K \subset \Omega$ e K satisfaz as condições do Teorema de Green, então:

$$\oint_\gamma \vec{v}\cdot d\vec{r} = \iint_K \text{rot}(\vec{v})\cdot\hat{k}\,.\,dxdy$$

Sendo que $\text{rot}(\vec{v}) = \left(\frac{\partial}{\partial x}\left(\ln(x^2+y^2)+2x\right) - \frac{\partial}{\partial y} 2.\text{arctg}\left(\frac{y}{x}\right)\right)\hat{k}$

$$\iint_K \left[\frac{\partial}{\partial x}\left(\ln(x^2+y^2)+2x\right) - \frac{\partial}{\partial y}\left(2\,\text{arctg}\,\frac{y}{x}\right)\right]dx\,dy = \int_{-1}^{1}\int_{1}^{2}\left(\frac{1}{x^2+y^2}\cdot 2x + 2 - \frac{2}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}\right)dx\,dy$$

$$= \int_{-1}^{1}\int_{1}^{2}\left(\frac{2x}{x^2+y^2} + 2 - \frac{2x}{x^2+y^2}\right)dx\,dy = \int_{-1}^{1} 2\,dy = 4$$

10 a) Lembrando que o gradiente de uma função $f: \mathbb{R}^n \to \mathbb{R}$ (uma função escalar de várias variáveis) tem a característica de vetorializar uma função escalar com suas derivadas parciais:

Sendo $f: D_{om.} \to \mathbb{R} \mid f(x,y) = \text{arctg}\,\frac{y}{x}$ e $D_{om.} \subset \mathbb{R}^2$

$$\nabla\,\text{arctg}\,\frac{y}{x} = \left(\frac{\partial}{\partial x}\text{arctg}\,\frac{y}{x}, \frac{\partial}{\partial y}\text{arctg}\,\frac{y}{x}\right) = \left(\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-y\cdot x^{-2}\right), \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\right)$$

$$= \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Sendo $g: B \to \mathbb{R} \mid g(x,y) = \text{arctg}\,\frac{x}{y}$ e $B \subset \mathbb{R}^2$

$$\nabla\,\text{arctg}\,\frac{x}{y} = \left(\frac{\partial}{\partial x}\text{arctg}\,\frac{x}{y}, \frac{\partial}{\partial y}\text{arctg}\,\frac{x}{y}\right) = \left(\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}, \frac{1}{1+\frac{x^2}{y^2}}\cdot\left(-xy^{-2}\right)\right)$$

$$= \left(\frac{y}{y^2+x^2}, \frac{-x}{y^2+x^2}\right)$$

b) $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ e $f_1, f_2, f_3 \in \mathcal{C}^1$. Seja $\vec{F} = (f_1(x), f_2(y), f_3(z))$ $\mid \vec{F}: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$

1º Maneira de se demonstrar

Então $\vec{F} \in \mathcal{C}^1$ e seu rotacional:

$$\text{rot}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\cancelto{0}{\frac{\partial}{\partial y}f_3(z)} - \cancelto{0}{\frac{\partial}{\partial z}f_2(y)}\right)\hat{i} + \left(\cancelto{0}{\frac{\partial}{\partial z}f_1(x)} - \cancelto{0}{\frac{\partial}{\partial x}f_3(z)}\right)\hat{j} + \left(\cancelto{0}{\frac{\partial}{\partial x}f_2(y)} - \cancelto{0}{\frac{\partial}{\partial y}f_1(x)}\right)\hat{k} = \vec{0}$$

Se $\vec{F}(x,y,z)$ é conservativo, então há $\varphi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ |

$\nabla\varphi(x,y,z) = \vec{F}(x,y,z) \Leftrightarrow \nabla\varphi(x,y,z) = (f_1(x), f_2(y), f_3(z)) \Leftrightarrow$

$$\begin{cases} \frac{\partial}{\partial x}\varphi = f_1(x) & \text{(Eq.1)} \\ \frac{\partial}{\partial y}\varphi = f_2(y) & \text{(Eq.2)} \\ \frac{\partial}{\partial z}\varphi = f_3(z) & \text{(Eq.3)} \end{cases} \Rightarrow \varphi(x,y,z) = \int f_1(x).dx = F_1(x) + K(y,z) \quad \text{(Eq.4)} \Leftrightarrow$$

Sejam $F_1(x)$, $F_2(y)$ e $F_3(z)$ as primitivas de $f_1(x)$, $f_2(y)$ e $f_3(z)$ respectivamente

$$\frac{\partial}{\partial y}\varphi = \frac{\partial}{\partial y}\left(F_1(x) + K(y,z)\right) = \frac{\partial}{\partial y}K(y,z) \quad \text{(Eq.5)}$$

De Eq.2 e de Eq4, tem-se que:

$$\frac{\partial}{\partial y}K(y,z) = f_2(y) \Leftrightarrow K(y,z) = \int f_2(y).dy = F_2(y) + K(z) \quad \text{(Eq.6)}$$

De Eq.4 e de Eq.6:

$$\varphi(x,y,z) = F_1(x) + F_2(y) + K(z) \quad \text{(Eq.7)}$$

Derivando em relação a z:

$$\frac{\partial}{\partial z}\varphi = \frac{\partial}{\partial z}\left(F_1(x) + F_2(y) + K(z)\right) = \frac{\partial}{\partial z}K(z) \quad \text{(Eq.8)}$$

De Eq.3 e de Eq.8:

$$\frac{\partial}{\partial z}K(z) = f_3(z) \Leftrightarrow K(z) = \int f_3(z).dz = F_3(z) + K$$

De Eq.7 e de Eq.9:

$$\boxed{\varphi(x,y,z) = F_1(x) + F_2(y) + F_3(z) + K}$$

, sendo K uma constante qualquer

É com este K que se pode arbitrar que $\varphi(P) = 0$, $\forall P \in \mathbb{R}^3$

É daí que vem a ideia de potencial elétrico ser uma medida relativa e não absoluta, a mesma ideia vale p/ $\forall$ energia potencial

Sendo φ um campo escalar que condiz com $\nabla\varphi = \vec{F}$, então $\vec{F}$ é um campo conservativo c.q.d.

2ª Maneira de se demonstrar

Há uma outra maneira de se demonstrar que $\vec{F}$ é conservativo. Consiste em se demonstrar que $rot(\vec{F}) = \vec{0}$ e que Ω (domínio do campo vetorial) é simplesmente conexo, pois, sendo $\vec{F}(x,y,z) = (f_1(x), f_2(y), f_3(z))$ | f_1, f_2 e $f_3 \in C^1 \Leftrightarrow f_1, f_2$ e f_3 são funções contínuas $\forall$ x, y ou z $\in \mathbb{R}$. Consequentemente $\Omega = \mathbb{R}^3$

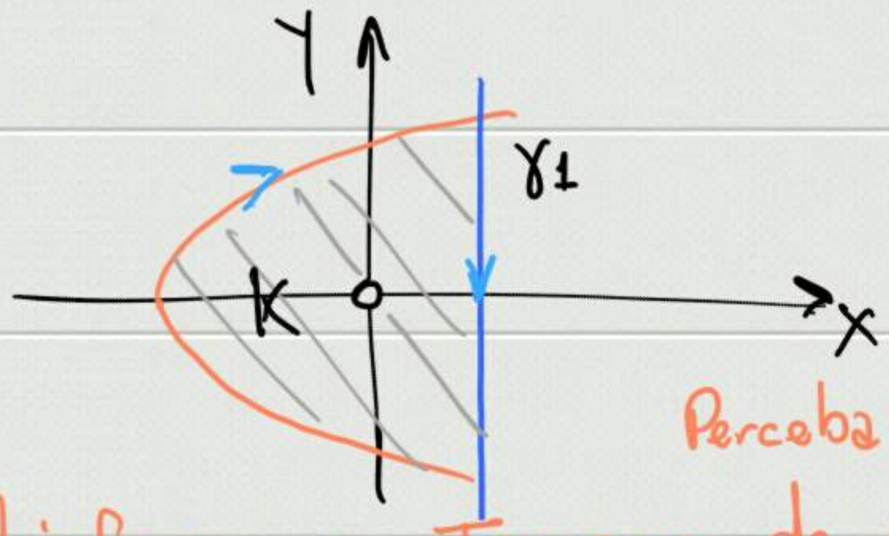
Sendo $\mathbb{R}^3$ um conjunto simplesmente conexo, pois não existe uma curva plana e fechada γ_i que contenha uma região K_i

tal que $\gamma^i \subset \Omega$ e $K_i \not\subset \Omega$

Portanto, sendo $rot(\vec{F}) = \vec{0}$ e Ω, domínio de $\vec{F}$, um conjunto simplesmente conexo, necessariamente $\vec{F}$ é um campo conservativo.

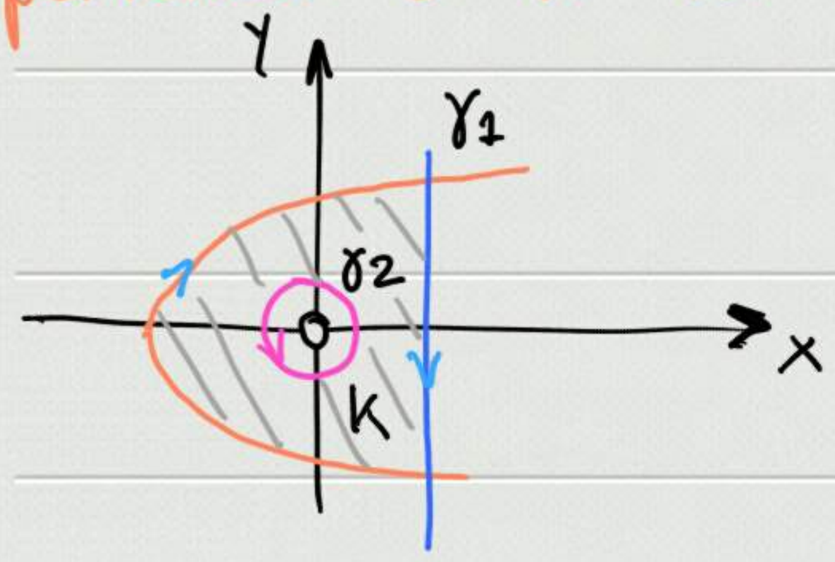
11 a) $\begin{cases} y^2 = 2x+4 \\ x = a \end{cases} \Leftrightarrow \begin{cases} \frac{y^2}{2} - 2 = x \\ x = a \end{cases} \quad | \quad a > 0$

$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$



$\vec{F} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \quad | \quad \Omega = \mathbb{R}^2 - \{(0,0)\}$

Perceba que este Lugar Geométrico K não satisfaz o Teorema de Green, pois há um ponto no interior dele que não está em Ω ($K \not\subset \Omega$). Ou se faz a parametrização de γ e se faz a integral de linha da maneira tradicional, ou se faz um condição de contorno para que o Teorema de Green seja válido: faça uma curva γ_2 que rode no sentido oposto a γ e focalize γ_1 a excluir a região que não deve pertencer a K tal que K fique contido em Ω.



$\gamma_2(t) = (r\cos t, r\,\text{sen}\,t) \quad | \quad r \to 0 \Rightarrow K_2 = \{(0,0)\}$

Sendo K_1 a região limitada por γ_1 e K_2 a região limitada por γ_2

$K = K_1 - K_2$ e seja $A = \gamma_1 \cup \gamma_2$

$$\int_A \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_{\gamma_1} \frac{-y\,dx + x\,dy}{x^2+y^2} + \int_{\gamma_2} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_K rot(\vec{F}) \cdot \hat{k}\, dx\,dy$$

$$rot(\vec{F}) = \left(\frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} - \left(\frac{-1.(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} \right) \right) \hat{k}$$

$$rot(\vec{F}) = \frac{x^2+y^2 - 2x^2 + x^2+y^2 - 2y^2}{(x^2+y^2)^2} . \hat{k} = \vec{0}$$

Eq. 1

$$\therefore \int_{\gamma_1} \frac{-y\,dx + x\,dy}{x^2+y^2} + \int_{\gamma_2} \frac{-y\,dx + x\,dy}{x^2+y^2} = 0 \Leftrightarrow \boxed{\int_{\gamma_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = -\int_{\gamma_2} \frac{-y\,dx + x\,dy}{x^2+y^2}}$$

$t \in [0, 2\pi]$

$\gamma_2(t) = (r\cos t, r\,\text{sen}\,t) \Leftrightarrow \frac{d}{dt}\gamma_2(t) = (-r\,\text{sen}\,t, r.\cos t) = \left(\frac{dx(t)}{dt}, \frac{d}{dt}y(t)\right)$

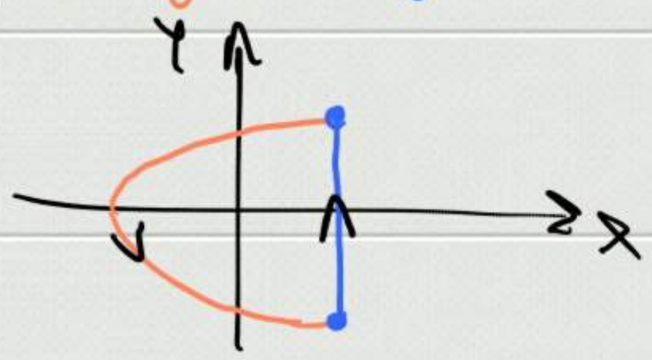
$$\therefore \int_{\gamma_2} \frac{-y.dx + x dy}{x^2+y^2} = \int_0^{2\pi} \frac{r.\text{sen}\,t.(-r\,\text{sen}\,t.dt) + r.\cos t.(r\cos t.dt)}{r^2}$$

$$= \int_0^{2\pi} \frac{r^2}{r^2}.dt = 2\pi \quad \therefore \boxed{\int_{\gamma_1} \frac{-y dx + x dy}{x^2+y^2} = -2\pi}$$

Fazendo pela descrição de γ_1:

γ_1 no sentido anti-horário (depois é só multiplicar por -1)

$\gamma_1 = \gamma_{1a} \cup \gamma_{1b}$ $\quad \gamma_{1a} = \begin{cases} x = \frac{t^2}{2} - 2 \\ y = t \end{cases}$ $\quad \gamma_{1b} = \begin{cases} y = \sqrt{2a+4}.t \\ x = a \end{cases}$



$a = \frac{t^2}{2} - 2$ $\quad t \in [-1, 1]$

$t = \pm\sqrt{2a+4}$

$t \in [-\sqrt{2a+4}, \sqrt{2a+4}]$

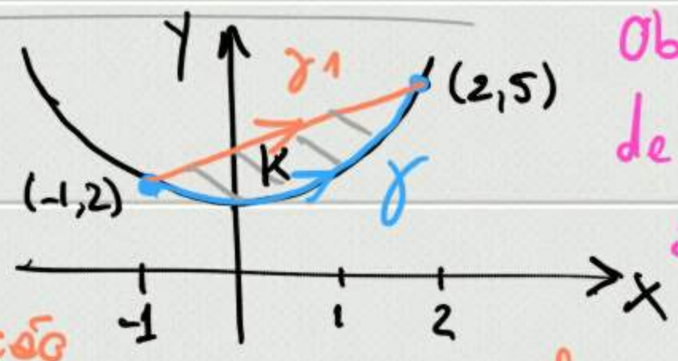
$$\oint_{\gamma_1} \frac{-y dx + x dy}{x^2+y^2} = \int_{\gamma_{1a}} \frac{-y dx + x dy}{x^2+y^2} + \int_{\gamma_{1b}} \frac{-y dx + x dy}{x^2+y^2}$$

$$\int_{\gamma_{1a}} \frac{-y dx + x dy}{x^2+y^2} = \int_{-\sqrt{2a+4}}^{\sqrt{2a+4}} \frac{-t.t.dt + \left(\frac{t^2}{2}-2\right).dt}{\left(\frac{t^2}{2}-2\right)^2 + t^2} = \int_{-\sqrt{2a+4}}^{\sqrt{2a+4}} \frac{-\frac{t^2}{2}-2 \;.dt}{\frac{t^4}{4} - 2t^2 + 4 + t^2}$$

$$= \int_{-\sqrt{2a+4}}^{\sqrt{2a+4}} -\frac{\frac{t^2}{2}+2}{\frac{t^4}{4} - t^2 + 4}.dt$$

Acho que já dá para perceber qual é o melhor jeito de se resolver

b) $\int_\gamma \frac{x dx + y dy}{x^2+y^2}$ $\quad \gamma: \begin{cases} y = x^2+1 \\ x \in [-1, 2] \end{cases}$



Obs γ_1 é fácil de parametrizar e aplicar

Pode se resolver fazendo a parametrização de γ e aplicando a integral de linha. Ou se faz o Teorema de Green e soma a integral de linha de γ_1 (sentido horário)

$$\int_\gamma \frac{x dx + y dy}{x^2+y^2} = \iint_K \text{rot}(\vec{F}) \cdot \hat{k}\, dx dy + \int_{\gamma_1} \frac{x dx + y dy}{x^2+y^2} \quad ; \gamma_1 : [-1, 2] \to \mathbb{R}^2 |$$

$\gamma_1 : \begin{cases} y = t+3 \\ x = t \end{cases} \Leftrightarrow \gamma_1(t) = (t, t+3)$; $K = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 2,\ x^2+1 \le y \le x+3\}$

$$\text{rot}(\vec{F}) = \left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2yx}{(x^2+y^2)^2}\right)\hat{k} = \vec{0} \quad \therefore \int_\gamma \frac{x dx + y dy}{x^2+y^2} = \int_{\gamma_1} \frac{x dx + y dy}{x^2+y^2}$$

$$\int_{\gamma_1} \frac{x dx + y dy}{x^2+y^2} = \int_{-1}^{2} \frac{t.dt + (t+3)dt}{t^2 + (t+3)^2} = \int_{-1}^{2} \frac{2t+3}{2t^2+6t+9}.dt = \int_{-1}^{2} \frac{2t+3}{2\left(t+\frac{3}{2}\right)^2 + \frac{9}{2}}.dt$$

$u = t + \frac{3}{2}$; $u\Big|_{1/2}^{7/2}$

$du = dt$

$$\int_{1/2}^{7/2} \frac{2(u-\frac{3}{2})+3}{2u^2+\frac{9}{2}}\,du = \int_{1/2}^{7/2} \frac{2u}{2u^2+\frac{9}{2}}\,du = \int_{1/2}^{7/2} \frac{u}{u^2+\frac{9}{4}}\,du$$

$w = u^2 + \frac{9}{4}$

$dw = 2u\,du$; $w\Big|_{10/4}^{58/4}$

$u\,du = \frac{dw}{2}$

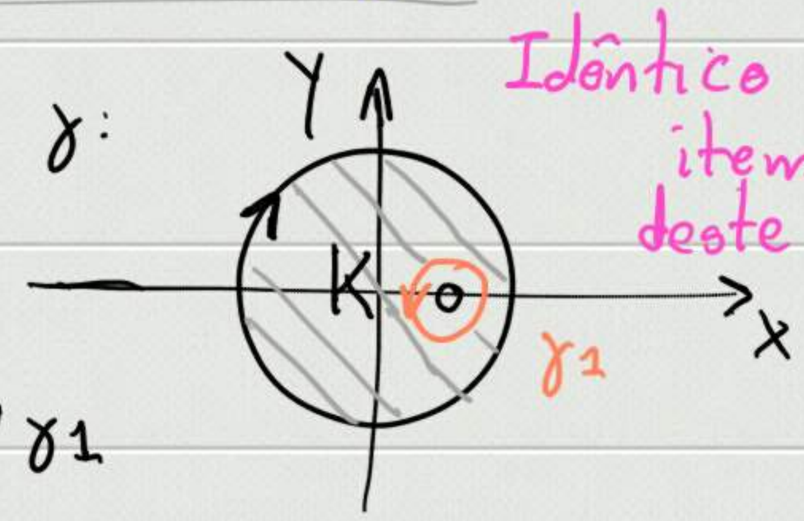
$$= \int_{10/4}^{58/4} \frac{1}{2}\,\frac{dw}{w} = \frac{1}{2}\ln\frac{58/4}{10/4} = \frac{1}{2}\ln\frac{29}{5}$$

$$\therefore \boxed{\int_\gamma \frac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}\ln\frac{29}{5}}$$

c) $\displaystyle\int_\gamma \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2}$ $\quad \gamma: x^2+y^2=4$ $\quad \gamma:$

$t \in [0, 2\pi]$

Idêntico ao item a deste exercício



$\gamma_1 = (1 + r\cos t,\ r\,\text{sen}\,t)$, $r \to 0$

$A = \gamma \cup \gamma_1$

$$\int_A \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2} = \int_\gamma \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2} + \int_{\gamma_1} \frac{y\,dx - (x-1)\,dx}{(x-1)^2+y^2} = \iint_K \text{rot}(\vec{F})\cdot\hat{k}\,dx\,dy$$

$$\text{rot}(\vec{F})\cdot\hat{k} = \left(\frac{\partial}{\partial x}\frac{-(x-1)}{(x-1)^2+y^2} - \frac{\partial}{\partial y}\frac{y}{(x-1)^2+y^2}\right)\hat{k}\cdot\hat{k} = \frac{-1((x-1)^2+y^2)+(x-1)\,2(x-1)}{((x-1)^2+y^2)^2}$$

$$-\frac{(x-1)^2+y^2 - y\cdot 2y}{((x-1)^2+y^2)^2} = \frac{-(x-1)^2-y^2+2(x-1)^2-(x-1)^2-y^2+2y^2}{((x-1)^2+y^2)^2} = 0$$

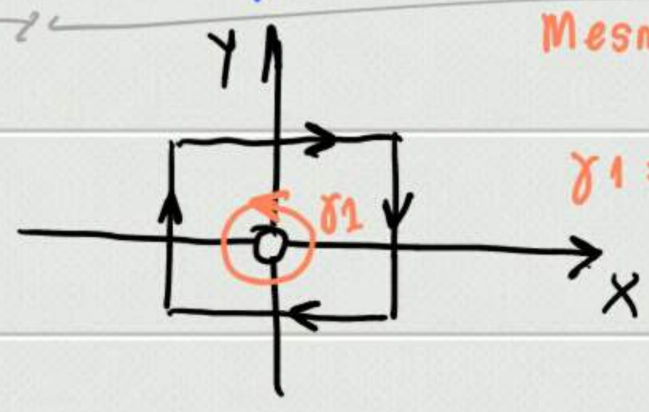
$$\therefore \boxed{\int_\gamma \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2} = -\int_{\gamma_1} \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2}}\ ;$$

$$\int_{\gamma_1} \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2} = \int_0^{2\pi} \frac{r\,\text{sen}\,t\,(-r\,\text{sen}\,t)\,dt - (1+r\cos t-1)(r\cos t\,dt)}{r^2} = \int_0^{2\pi} -1\,dt$$

$= -2\pi$

$$\therefore \boxed{\int_\gamma \frac{y\,dx - (x-1)\,dy}{(x-1)^2+y^2} = 2\pi}$$

d) $\displaystyle\int_\gamma \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2}$ $\quad \gamma:$



Mesma ideia do item a

$\gamma_1 = (r\cos t,\ r\,\text{sen}\,t)$, $t \in [0, 2\pi]$

$r \to 0$

$A = \gamma \cup \gamma_1$

$$\text{rot}(\vec{F}) = \left[\frac{\partial}{\partial x}\left(\frac{-x^3}{(x^2+y^2)^2}\right) - \frac{\partial}{\partial y}\left(\frac{x^2y}{(x^2+y^2)^2}\right)\right]\hat{k}$$

$$\text{rot}(\vec{F}) = \left[\frac{-3x^2(x^2+y^2)^2 + 2x^3(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} - \frac{x^2(x^2+y^2)^2 - x^2y\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}\right]\hat{k}$$

$$\text{rot}(\vec{F}) = -4x^6 - 8x^4y^2 - 4x^2y^4 + 4x^6 + 4x^4y^2 + 4x^4y^2 + 4x^2y^4 = \vec{0}$$

$$\int_A \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2} = \int_\gamma \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2} + \int_{\gamma_1} \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2} = \iint_K 0\,dx\,dy$$

Eq. 1

$$\int_{\gamma 1} \frac{x^2y\,dx - x^3dy}{(x^2+y^2)^2} = \int_0^{2\pi} \frac{(r\cos t)^2 \cdot r\,\text{sen}\,t \cdot (-r\,\text{sen}\,t\,dt) - (r\cos t)^3 \cdot (r\cos t \cdot dt)}{r^4}$$

$$= -\int_0^{2\pi} \frac{r^4\cos^2 t\,\text{sen}^2 t + r^4\cos^2 t \cdot \cos^2 t}{r^4} \cdot dt = -\int_0^{2\pi} \cos^2 t \cdot dt = -\left[\frac{\text{sen}\,t \cdot \cos t}{2}\right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{2}\,dt$$

$$= -\pi$$

Portanto, pela equação 1

$$\boxed{\int_\gamma \frac{x^2y\,dx - x^3dy}{(x^2+y^2)^2} = \pi}$$

---

(12) $\int_\gamma 2x\,\text{sen}\,y\,dx + (x^2\cos y - 3y^2)\,dy$ $\quad \gamma$: liga os pontos $(-1,0)$ a $(5,1)$

Se independe do caminho, é um campo conservativo, logo existe um campo potencial φ cujo gradiente recai em $\vec{F}$ e $2x\,\text{sen}\,y\,dx + (x^2\cos y - 3y^2)\,dy$ é a forma diferencial exata de φ

$$\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(x^2\cos y - 3y^2) - \frac{\partial}{\partial y}(2x\,\text{sen}\,y)\right)\hat{k} = (2x\cos y - 2x\cos y)\hat{k} = \vec{0}$$

e $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 \therefore \Omega$ é um conjunto simplesmente conexo e, portanto, $\vec{F}$ é um campo conservativo.

$$\frac{\partial}{\partial x}\varphi = 2x\,\text{sen}\,y \iff \varphi = \int 2x\,\text{sen}\,y \cdot dx \iff \varphi(x,y) = x^2\text{sen}\,y + K(y)$$

$$\iff \frac{\partial}{\partial y}\varphi = x^2 \cdot \cos y + \frac{\partial}{\partial y}K(y) \iff x^2\cos y - 3y^2 = x^2\cos y + \frac{\partial}{\partial y}K(y)$$

$$\iff \frac{\partial}{\partial y}K(y) = -3y^2 \iff K(y) = -\int 3y^2\,dy \iff K(y) = -y^3 + K$$

K é uma constante de integração $\therefore \boxed{\varphi(x,y) = x^2\text{sen}\,y - y^3 + K}$ (Eq. 1)

campo potencial

Portanto

$$\int_\gamma 2x\,\text{sen}\,y\,dx + (x^2\cos y - 3y^2)\,dy = \int_{(-1,0)}^{(5,1)} d\varphi = 25 \cdot \text{sen}\,1 - 1 - ((-1)^2 \cdot \text{sen}\,0 - 0)$$

$$= 25\,\text{sen}\,1 - 1$$

---

(13) $\gamma 1: \quad x^2+y^2 = \frac{1}{16}$

$$P(x,y) = \boxed{\frac{-y}{(x-1)^2+y^2}} + \boxed{\frac{-y}{x^2+y^2}} + \boxed{\frac{-y}{(x+1)^2+y^2}}$$

$$Q(x,y) = \boxed{\frac{x-1}{(x-1)^2+y^2}} + \boxed{\frac{x}{x^2+y^2}} + \boxed{\frac{x+1}{(x+1)^2+y^2}}$$

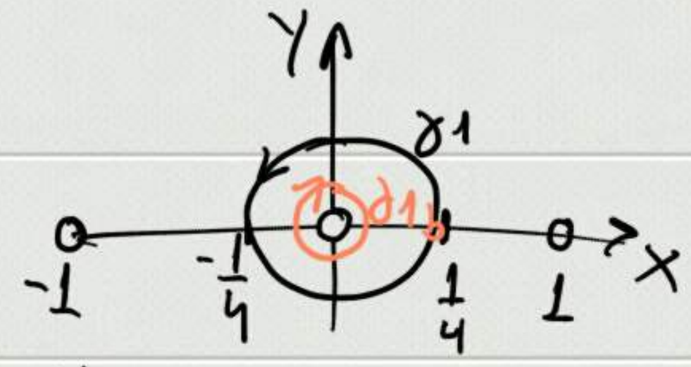
O rotacional do laranja e do azul já foram calculados no ex. 11 item c e item a, ambos dando zero.

E o rotacional do vermelho é igual ao do laranja, veja a semelhança entre eles.

Cálculo de I_1

$\gamma_{1b} = \gamma_{1a}$ no sentido horário

$\gamma_{1a} = (r\cos t, r\operatorname{sen} t)$
$r \to 0^+$ sentido anti-horário.
$t \in [0, 2\pi]$



$A = \gamma \cup \gamma_{1b}$

$$\int_A P(x,y).dx + Q(x,y).dy = \int_{\gamma_1} P(x,y).dx + Q(x,y).dy + \int_{\gamma_{1b}} P(x,y)dx + Q(x,y)dy = 0$$

$$\int_{\gamma_1} P(x,y)dx + Q(x,y)dy = \int_{\gamma_{1a}} P(x,y)dx + Q(x,y)dy$$

$$\int_{\gamma_{1a}} P(x,y)dx + Q(x,y)dy = \int_{\gamma_{1a}} \frac{-y\,dx + (x-1)dy}{(x-1)^2+y^2} + \frac{-y\,dx + x\,dy}{x^2+y^2} + \frac{-y\,dx + (x+1)dx}{(x+1)^2+y^2}$$

$$\int_{\gamma_{1a}} \frac{-y\,dx + (x-1)dy}{(x-1)^2+y^2} + \int_{\gamma_{1a}} \frac{-y\,dx + x\,dy}{x^2+y^2} + \int_{\gamma_{1a}} \frac{-y\,dx + (x+1)dy}{(x+1)^2+y^2}$$

$$\int_0^{2\pi} \frac{-r\operatorname{sen}t.(-r\operatorname{sen}t.dt) + (r\cos t - 1).(r\cos t.dt)}{(r\cos t-1)^2 + (r\operatorname{sen}t)^2} + 2\pi + \int_0^{2\pi} \frac{-r\operatorname{sen}t.(-r\operatorname{sen}t.dt) + (r\cos t+1)(r\cos t\,dt)}{(r\cos t+1)^2+(r\operatorname{sen}t)^2}$$

$$\lim_{r\to 0}\left(\int_0^{2\pi} \frac{r^2 - r\cos t}{r^2 - 2r\cos t + 1}.dt + 2\pi + \int_0^{2\pi} \frac{r^2 + r\cos t}{r^2 + 2r\cos t + 1}.dt\right)$$

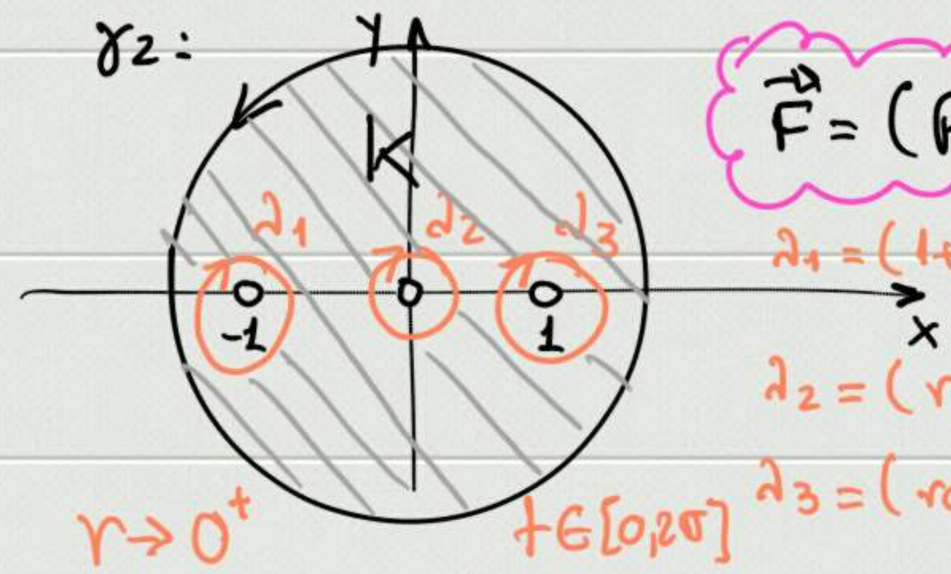
Calculado no Ex. 11.a.

$$= \int_0^{2\pi} \lim_{r\to 0} \frac{r^2 - r\cos t}{r^2 - 2r\cos t + 1}.dt + 2\pi + \int_0^{2\pi} \lim_{r\to 0} \frac{r^2 + r\cos t}{r^2 + 2r\cos t + 1}.dt\Big) = 0 + 2\pi + 0$$

$$= 2\pi$$

Cálculo de I_2

γ_2:



$\vec{F} = (P, Q)$

$\gamma_1 = (1 + r\cos t, r\operatorname{sen} t)$

$\gamma_2 = (r\cos t, r\operatorname{sen} t)$

$\gamma_3 = (r\cos t - 1, r\operatorname{sen} t)$

$\gamma_1' \to \gamma_1$ no sentido anti-horário
$\gamma_2' \to \gamma_2$ no sentido anti-horário
$\gamma_3' \to \gamma_3$ no sentido anti-horário

Sendo $A = \gamma_2 \cup \gamma_1' \cup \gamma_2' \cup \gamma_3'$

$$\int_A \vec{F}.d\vec{r} = \oint_{\gamma_2} \vec{F}.d\vec{r} + \oint_{\gamma_1'} \vec{F}.d\vec{r} + \oint_{\gamma_2'} \vec{F}.d\vec{r} + \oint_{\gamma_3'} \vec{F}.d\vec{r} = \iint_K 0\,dx\,dy = 0$$

(Eq. 1)

$$\therefore \boxed{\oint_{\gamma_2} \vec{F}.d\vec{r} = -\oint_{\gamma_1} \vec{F}.d\vec{r} - \oint_{\gamma_2} \vec{F}.d\vec{r} - \oint_{\gamma_3} \vec{F}.d\vec{r} = \oint_{\gamma_1'} \vec{F}.d\vec{r} + \oint_{\gamma_2'} \vec{F}.d\vec{r} + \oint_{\gamma_3'} \vec{F}.d\vec{r}}$$

$$\int_{\gamma_1} \vec{F}.d\vec{r} = \lim_{r\to 0}\left(\oint_{\gamma_1'} \frac{-y\,dx + (x-1)dy}{(x-1)^2+y^2} + \oint_{\gamma_1'} \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{\gamma_2'} \frac{-y\,dx + (x+1)dy}{(x+1)^2+y^2}\right)$$

$$\int_{\gamma_2} \vec{F}.d\vec{r} = \lim_{r\to 0}\left(\oint_{\gamma_2'} \frac{-y\,dx + (x-1)dy}{(x-1)^2+y^2} + \oint_{\gamma_2'} \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{\gamma_2'} \frac{-y\,dx + (x+1)dy}{(x+1)^2+y^2}\right)$$

$$\int_{\gamma_3} \vec{F}.d\vec{r} = \lim_{r\to 0}\left(\oint_{\gamma_3'} \frac{-y\,dx + (x-1)dy}{(x-1)^2+y^2} + \oint_{\gamma_3'} \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{\gamma_3'} \frac{-y\,dx + (x+1)dy}{(x+1)^2+y^2}\right)$$

De forma idêntica ao cálculo anterior

$\oint_{\gamma_1} \vec{F}.d\vec{r} = 2\pi + 0 + 0$

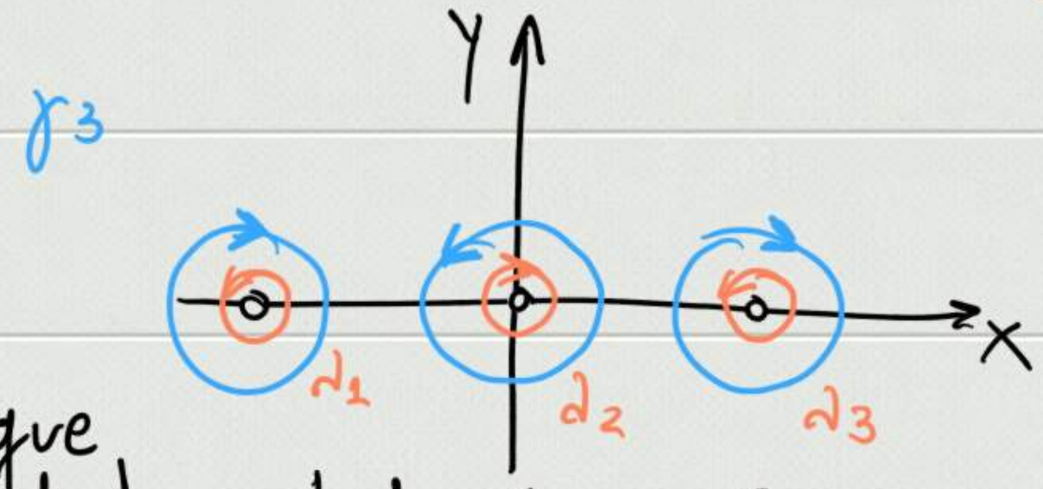
$\oint_{\gamma_2} \vec{F}.d\vec{r} = 0 + 2\pi + 0$

$\oint \vec{F}.d\vec{r} = 0 + 0 + 2\pi$

De eq. 1

$$\therefore \oint_{\gamma_2} \vec{F}.d\vec{r} = 6\pi$$

Calculo de I_3



O cálculo de I_3 é idêntico ao de I_2, a única diferença é que λ_1 e λ_3 agora estão no sentido anti-horário. Consequentemente, a equação 1, para este caso, ficaria:

$$\int_{\gamma_3} \vec{F}\cdot d\vec{r} = -\oint_{\lambda_1} \vec{F}\cdot d\vec{r} + \oint_{\lambda_2} \vec{F}\cdot d\vec{r} - \oint_{\lambda_3} \vec{F}\cdot d\vec{r} = -2\pi + 2\pi - 2\pi = \underline{-2\pi}$$

---

(14) Uma coisa importante que esqueci de falar: A curva λ que isola o ponto de singularidade (que não pertence ao Ω, domínio de $\vec{F}$) deve ser uma parametrização semelhante ao Denominador de $P(x,y,z)$, $Q(x,y,z)$ e de $R(x,y,z)$ (Quando $\vec{F}: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$). Pois a curva λ deve convergir para o ponto $P \mid P \notin \Omega$ na mesma proporção do denominador, não possibilitando que se tire algo a mais ou algo a menos da região interior a curva fechada γ |

$$\int_{\gamma_1} \vec{F}\cdot d\vec{r} = \int_{\gamma} \vec{F}\cdot d\vec{r} + \int_{\lambda} \vec{F}\cdot d\vec{r}$$

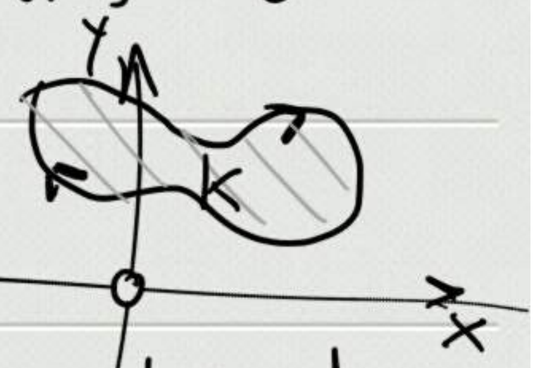
a) $\oint_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2}$; sendo $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ |

$\Omega = \mathbb{R}^2 - \{(0,0)\}$

Se γ é tal que a região em seu interior K esteja contida em Ω, então pode se utilizar o Teorema de Green

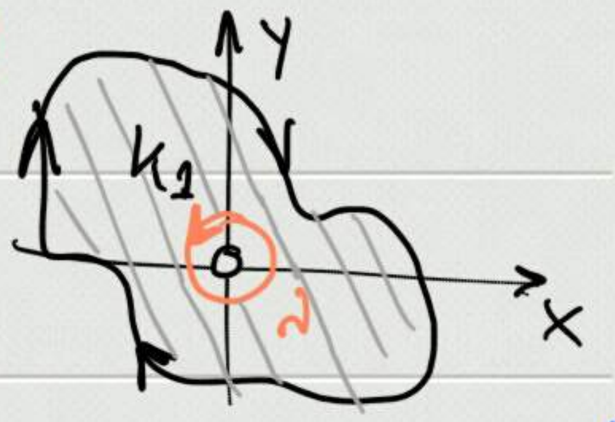
$$\oint_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2} = -\iint_K \text{rot}(\vec{F})\cdot\hat{k}\,dx\,dy \quad , \text{ só que } \text{rot}(\vec{F}) = \vec{0}$$

Então $\boxed{\oint_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2} = 0 \ , \ \forall\, \gamma \mid K \subset \Omega}$



Se γ é tal que K não esteja contida em Ω, então deve se isolar a região $D \mid D \subset K$ e $D \notin \Omega$ com uma curva λ que envolva a região D.

$\lambda(t) = (r.\cos t, r\,\text{sen}\,t)$
$t \in [0, 2\pi]$
$r \to 0^+$



$K_1 = K - D$; $\gamma_1 = \gamma \cup \lambda$

$$\oint_{\gamma_1} \vec{F}\cdot d\vec{r} = \oint_\gamma \vec{F}\cdot d\vec{r} + \oint_\lambda \vec{F}\cdot d\vec{r} = \iint_{K_1} 0.\,dx\,dy = 0 \iff$$

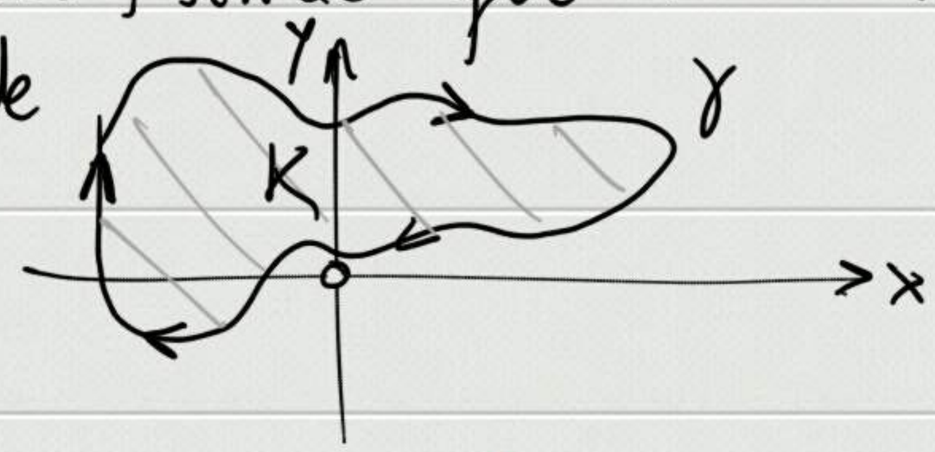
$$\oint_\gamma \vec{F}\cdot d\vec{r} = -\oint_\lambda \vec{F}\cdot d\vec{r} = -\oint_\lambda \frac{-y\,dx + x\,dy}{x^2+y^2} = -\int_0^{2\pi} \frac{-r\,\text{sen}\,t.(-r\,\text{sen}\,t.dt) + r\cos t.(r\cos t.dt)}{r^2} = \underline{-2\pi}$$

Portanto

$$\boxed{\oint \frac{-y\,dx + x\,dy}{x^2+y^2} = -2\pi \ , \ \forall\, \gamma \ \text{cuja}\ K \not\subset \Omega}$$

b) $\oint_\gamma \frac{-y\,dx + x\,dy}{4x^2+9y^2} = ?\ \text{rot}(\vec{F}) = \left(\frac{1(4x^2+9y^2) - x\cdot 8x}{(4x^2+9y^2)^2} - \frac{-(4x^2+9y^2) + y\cdot 18y}{(4x^2+9y^2)^2} \right)\hat{k} = \vec{0}$

Se γ é uma curva fechada e plana tal que a região K, interna à γ, é tal que $K \subset \Omega$, sendo que $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\Omega = \mathbb{R}^2 - \{(0,0)\}$. Então, pelo Teorema de Green:

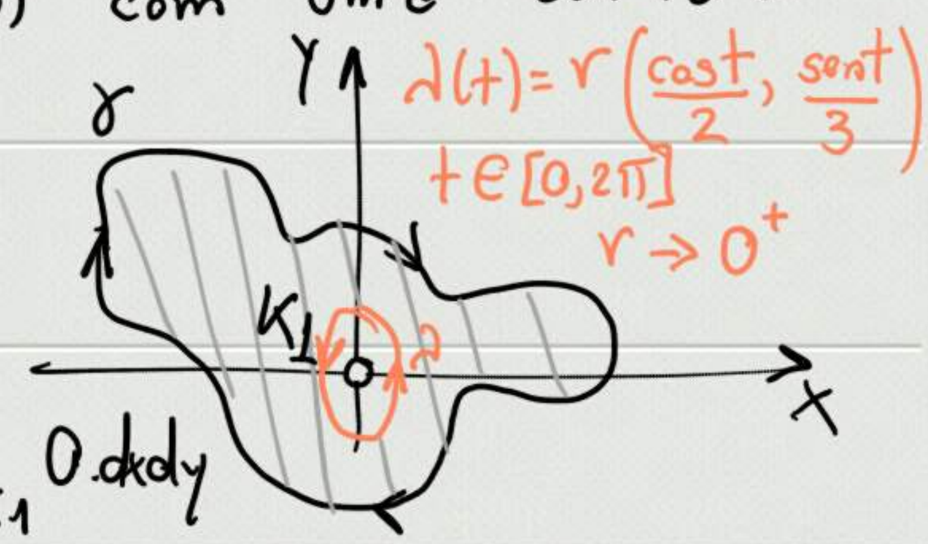


$$\oint \frac{-y\,dx + x\,dy}{4x^2+9y^2} = \iint_K 0 \cdot dx\,dy$$

$$\therefore \quad \boxed{\oint \frac{-y\,dx + x\,dy}{4x^2+9y^2} = 0 \ , \ \forall\, \gamma \mid K \subset \Omega}$$

Se γ é uma curva fechada e plana tal que $K \not\subset \Omega$, pois $(0,0) \in K$ e $(0,0) \notin \Omega$. Então, antes de se utilizar o Teorema de Green, deve se isolar $(0,0)$ com uma curva λ suficientemente pequena tal que:

$\gamma_1 = \gamma \cup \lambda$ e $K_1 = K - \underbrace{\{(0,0)\}}_{\text{Região interna à } \lambda}$



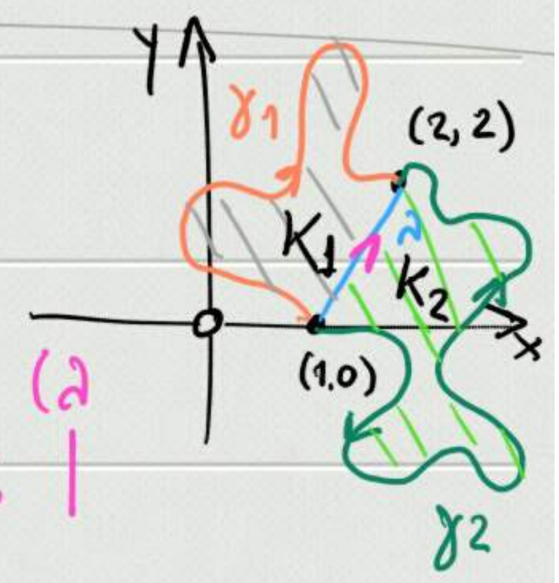
$$\oint_{\gamma_1} \frac{-y\,dx + x\,dy}{4x^2+9y^2} = \oint_\gamma \frac{-y\,dx + x\,dy}{4x^2+9y^2} + \oint_\lambda \frac{-y\,dx + x\,dy}{4x^2+9y^2} = \iint_{K_1} 0 \cdot dx\,dy$$

$$\Leftrightarrow \quad \oint_\gamma \frac{-y\,dx + x\,dy}{4x^2+9y^2} = -\oint_\lambda \frac{-y\,dx + x\,dy}{4x^2+9y^2} = -\int_0^{2\pi} \frac{-r\,\text{sen}\,t\left(-r\,\text{sen}\,t\cdot dt/2\right) + r\cos t\left(r\cos t\cdot dt/3\right)}{r^2}$$

$$= -\int_0^{2\pi} \frac{1}{6}\frac{r^2}{r^2}\,dt = -\frac{\pi}{3} \qquad \therefore \ \boxed{\oint \frac{-y\,dx + x\,dy}{4x^2+9y^2} = -\frac{\pi}{3} \ , \ \forall\, \gamma \mid K \not\subset \Omega}$$

15) $\displaystyle\int_{(1,0)}^{(2,2)} \frac{-y\,dx + x\,dy}{x^2+y^2}$; Seja $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$

$\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 - \{(0,0)\}$; $\text{rot}(\vec{F}) = \vec{0}$



Se γ_i caracteriza uma curva cuja união com $\bar{\lambda}$ (a percorrida no sentido contrário) forma uma região fechada $K_i \mid (0,0) \notin K_i$, então, pelo Teorema de Green:

$$\oint_{\bar{\lambda}\cup\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_{\bar{\lambda}} \frac{-y\,dx+x\,dy}{x^2+y^2} + \int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = -\iint_{K_i} 0 \cdot dx\,dy \Leftrightarrow$$

$$\int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = -\int_{\bar{\lambda}} \frac{-y\,dx+x\,dy}{x^2+y^2} \Leftrightarrow \boxed{\int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_{\lambda} \frac{-y\,dx+x\,dy}{x^2+y^2}} \text{ (Eq. 1)}$$

$\lambda(t) = (1+t,\ 2t) \mid t \in [0,1]$

$$\int_{\lambda} \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_0^1 \frac{-2t\cdot dt + (1+t)\cdot 2\,dt}{(1+t)^2+(2t)^2} = \int_0^1 \frac{2}{5t^2+2t+1}\cdot dt = \int_0^1 \frac{2}{\left(\sqrt{5}t+\frac{1}{\sqrt{5}}\right)^2+\frac{4}{5}}\,dt$$

$u = \sqrt{5}t + \frac{1}{\sqrt{5}}$

$du = \sqrt{5}\,dt$

$u\Big|_{\frac{1}{\sqrt{5}}}^{\sqrt{5}+\frac{1}{\sqrt{5}}}$

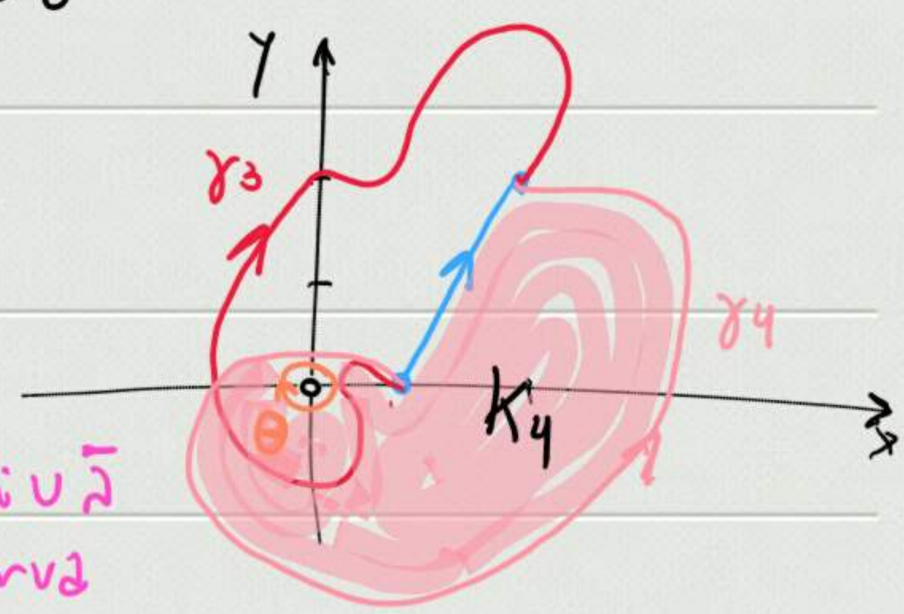
$$= \int_{\frac{1}{\sqrt{5}}}^{\sqrt{5}+\frac{1}{\sqrt{5}}} \frac{2}{u^2+\frac{4}{5}}\cdot\frac{1}{\sqrt{5}}\,du = \frac{2}{\sqrt{5}}\cdot\frac{5}{4}\cdot\int_{\frac{1}{\sqrt{5}}}^{\sqrt{5}+\frac{1}{\sqrt{5}}} \frac{1}{\frac{5}{4}u^2+1}\,du = \frac{\sqrt{5}}{2}\cdot\left[\frac{2}{\sqrt{5}}\cdot \operatorname{arctg}\frac{\sqrt{5}}{2}u\right]_{\frac{1}{\sqrt{5}}}^{\sqrt{5}+\frac{1}{\sqrt{5}}}$$

$$= \left(\operatorname{arctg}\left(\frac{1}{2}(5+1)\right) - \operatorname{arctg}\left(\frac{1}{2}\right)\right) = \overbrace{\left(\operatorname{arctg} 3 - \operatorname{arctg}\tfrac{1}{2}\right)}^{\operatorname{arctg} 1} = \frac{\pi}{4}$$

Obs: $\operatorname{tg}(\operatorname{arctg} 3 - \operatorname{arctg}\tfrac{1}{2}) = \dfrac{3-\frac{1}{2}}{1+3\cdot\frac{1}{2}} = 1$

$\forall \gamma_i \mid K_i \subset \Omega$

Se γ_i caracteriza uma curva plana cuja união com $\bar{\lambda}$ forma uma curva fechada tal que a região em seu interior K_i é tal que $K_i \not\subset \Omega$, então, se terá 2 casos: quando a curva $\gamma_i \cup \bar{\lambda}$ possuir um sentido horário (γ_3), então a curva laranja θ estará no anti-horário.



$\theta =$

$\gamma_i \cup \bar{\lambda}$ anti-horário $\Rightarrow$ θ horário

A exemplo de K_4, deve se isolar $(0,0)$ para se aplicar a Teorema de Green.

- $\gamma_i \cup \bar{\lambda}$ sentido horário:

$$\oint_{\theta\cup\gamma_i\cup\bar{\lambda}} \frac{-y\,dx+x\,dy}{x^2+y^2} = \oint_{\theta} \frac{-y\,dx+x\,dy}{x^2+y^2} + \int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} + \int_{\bar{\lambda}} \frac{-y\,dx+x\,dy}{x^2+y^2} = -\iint_{K_i} 0\,dx\,dy$$

$$\Leftrightarrow \int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = -\oint_{\theta} \frac{-y\,dx+x\,dy}{x^2+y^2} - \int_{\bar{\lambda}} \frac{-y\,dx+x\,dy}{x^2+y^2}$$

$$\Leftrightarrow \int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = -\oint_{\theta} \frac{-y\,dx+x\,dy}{x^2+y^2} + \int_{\lambda} \frac{-y\,dx+x\,dy}{x^2+y^2}$$

Ex 11. item a

$\operatorname{arctg} 1 - \operatorname{arctg} 0$

A função potencial de Ω, quando $x>0$ $\operatorname{arctg} \frac{y}{x}$

$$\therefore \boxed{\int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = -2\pi + \left(\operatorname{arctg} 3 - \operatorname{arctg}\tfrac{1}{2}\right)} = -\frac{7}{4}\pi$$

- $\gamma_i \cup \bar{\lambda}$ sentido anti-horário

Mesmo processo do anterior, só muda que θ está no sentido horário

$$\therefore \boxed{\int_{\gamma_i} \frac{-y\,dx+x\,dy}{x^2+y^2} = 2\pi + \operatorname{arctg} 3 - \operatorname{arctg}\tfrac{1}{2}}$$

c.q.d.

16 $\vec{F}(x,y) = cxy\,\hat{i} + x^6y^2\,\hat{j}$, $(0,0) \longrightarrow (1,a)$

$\gamma(t) = (t, at^b) \mid t \in [0,1]$ $\quad a,b,c \in \mathbb{R}^*_+$

$$\int_\gamma \vec{F}\cdot d\gamma = \vec{0} \Leftrightarrow \int_\gamma (cxy,\ x^6y^2)\cdot d\gamma = \int_\gamma (ct\cdot at^b,\ t^6\cdot a^2t^{2b})\cdot(1,\ abt^{b-1})\,dt$$

$$= \int_0^1 (c\cdot a\,t^{b+1} + a^3b\cdot t^{3b+5})\,dt = 0 \Leftrightarrow \frac{c\cdot a}{b+2} + \frac{a^3b}{3b+6} = 0 \Leftrightarrow$$

$$\boxed{c = -\frac{a^2b}{3}}$$

O que parece demonstrar que é impossível o ter trabalho nulo para c e b positivos.

17 $\int_\gamma \vec{F}\cdot d\vec{r}$, $\vec{F} = \left(\frac{-y}{x^2+\frac{y^2}{9}} + y,\ \frac{x}{x^2+\frac{y^2}{9}} + 3x\right)$

Perceba que a lógica trabalhada aqui é igual a do 11, com pequenas ressalvas

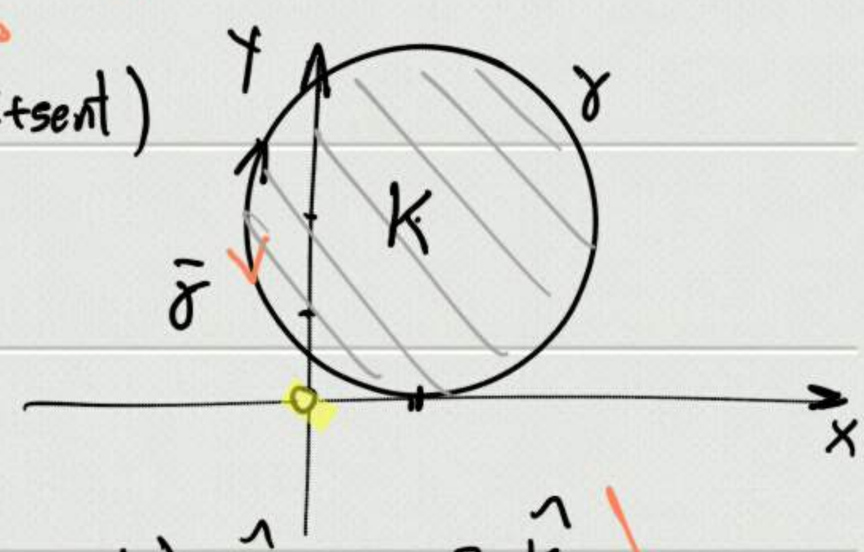
a) γ é $(x-1)^2 + (y-2)^2 = \underline{4}$

$r = 2$

sentido horário

$\bar{\gamma}(t) = (1+\cos t,\ 2+\operatorname{sen} t)$ → sentido inverso

$t \in [0, 2\pi]$



$$\oint_\gamma \vec{F}\cdot d\gamma = -\iint_K \operatorname{rot}(\vec{F})\cdot\hat{k}\,dxdy$$

$$\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+\frac{y^2}{9}} + 3x\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+\frac{y^2}{9}} + y\right)\right)\hat{k} = (3-1)\,\hat{k} = \underline{2\hat{k}}$$

$$\oint_\gamma \vec{F}\cdot d\gamma = -\iint_K 2\,dxdy = -2\iint_K dxdy = -2\cdot \text{Área}(K) = -2\cdot\pi\cdot 2^2 = \underline{-8\pi}$$

b) γ é $(x-1)^2 + y^2 = 4$, sentido horário

$\vec{F} = \left(\frac{-y}{x^2+\frac{y^2}{9}} + y,\ \frac{x}{x^2+\frac{y^2}{9}} + 3x\right)$

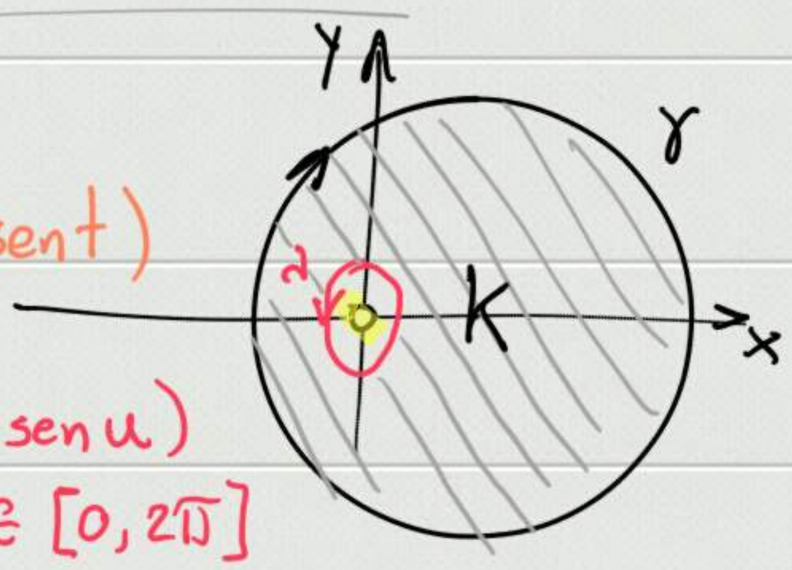
Antes de utilizar Green, deve se isolar o ponto de singularidade ($(0,0) \notin \Omega$)

$\bar{\gamma}(t) = (1+\cos t,\ \operatorname{sen} t)$

$t \in [0, 2\pi]$

$\partial(t) = (r\cos u,\ 3r\operatorname{sen} u)$

$\lim r \to 0$, $u \in [0, 2\pi]$



$$\oint_{\gamma\cup\partial} \vec{F}\cdot d\vec{r} = \oint_\gamma \vec{F}\cdot d\vec{r} + \oint_\partial \vec{F}\cdot d\vec{r} = -\iint_K 2\cdot dxdy \Leftrightarrow \boxed{\oint_\gamma \vec{F}\cdot d\vec{r} = \underline{-8\pi} - \oint_\partial \vec{F}\cdot d\partial} \quad (Eq.1)$$

(a área do ponto isolado é zero)

tem-se que:

$$\oint_\partial \vec{F}\cdot d\partial = \oint_\partial \left(\frac{-y}{x^2+\frac{y^2}{9}} + y,\ \frac{x}{x^2+\frac{y^2}{9}} + 3x\right)\cdot d\partial =$$

$$= \int_0^{2\pi} \lim_{r\to 0}\left[\left(\frac{-3r\operatorname{sen} u}{r^2} + 3r\operatorname{sen} u,\ \frac{r\cos u}{r^2} + 3r\cos u\right)\cdot(-r\operatorname{sen} u,\ 3r\cos u)\,du\right]$$

$$= \int_0^{2\pi} \lim_{r\to 0}\left(3 + 3r^2(\cos^2 u - \operatorname{sen}^2 u)\right)du = \int_0^{2\pi} 3\,du = \underline{6\pi}$$

De eq.1 : $\boxed{\oint_\gamma \vec{F}\cdot d\vec{r} = -14\pi}$

18 $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $g: \mathbb{R} \to \mathbb{R}$ $\quad \vec{F}(x,y) = g(|\vec{r}|)\vec{r}$

$\vec{r} = x\hat{i} + y\hat{j}$, $g \in \mathcal{C}^1$

$g(|\vec{r}|) = g(\sqrt{x^2+y^2})$; $\vec{F}(x,y) = g(\sqrt{x^2+y^2})\,x\,\hat{i} + g(\sqrt{x^2+y^2})\,y\,\hat{j}$

Se $\vec{F}$ for conservativo, tem-se que, no mínimo:

$\text{rot}(\vec{F}) = \vec{0} \iff \frac{\partial}{\partial x}\left(g(\sqrt{x^2+y^2})\cdot y\right) - \frac{\partial}{\partial y}\left(g(\sqrt{x^2+y^2})\cdot x\right) = 0 \iff$

$\frac{d}{dr}g(r) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x \cdot y - \frac{d}{dr}g(r) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y \cdot x = 0$ $\qquad r = \sqrt{x^2+y^2}$

$\frac{d}{dr}g(r)\,\frac{xy}{\sqrt{x^2+y^2}} - \frac{d}{dr}g(r) \cdot \frac{xy}{\sqrt{x^2+y^2}} = 0$

$\therefore \boxed{\text{rot}(\vec{F}) = \vec{0}}$ (Eq. 1)

Obs: este exercície seria muito complicado, tendo que se considerar vários casos de funções de $g(\sqrt{x^2+y^2})$ (mais especificamente, os vários casos de descontinuidades), se $g(r)$ não fosse $g \in \mathcal{C}^1$. Obviamente g é contínua se sua primeira derivada é contínua. Portanto:

Sabe-se que $\Omega = \mathbb{R}^2$, pois $g \in \mathcal{C}^1$ e $\mathbb{R}^2$ é um conjunto simplesmente conexo, consequentemente, se $\text{rot}(\vec{F}) = \vec{0}$ e se Ω é simplesmente conexo, dado as condições de $g(\sqrt{x^2+y^2})$, então $\vec{F}$ é um campo conservativo

19 Utilizarei um esquema prático do domínio simplesmente conexo para determinar se o campo é conservativo ou não.

a) $\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}x\right)\hat{k} = 1\hat{k} \neq \vec{0}$

Portanto $\vec{F}$ não é conservativo

b) $\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(x^2e^y + x - 2y) - \frac{\partial}{\partial y}(2xe^y + y)\right)\hat{k}$

$\text{rot}(\vec{F}) = \left(2xe^y + 1 - (2xe^y + 1)\right)\hat{k} = \vec{0}$

$\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\Omega = \mathbb{R}^2$ e não há pontos de singularidade

Logo Ω é um conjunto simplesmente conexo

$\therefore$ $\boxed{\vec{F} \text{ é conservativo}}$

Determinando-se o campo escalar $\varphi(x,y): \Omega \to \mathbb{R}$

$$\nabla\varphi(x,y) = \vec{F}(x,y) \Leftrightarrow \begin{cases} \frac{\partial\varphi}{\partial x} = 2xe^y + y & \text{(Eq.1)} \\ \frac{\partial\varphi}{\partial y} = x^2e^y + x - 2y & \text{(Eq.2)} \end{cases}$$

De eq.1:

$\varphi(x,y) = \int 2xe^y + y \cdot dx = x^2e^y + xy + k(y)$ (Eq.3)

Derivando-se parcialmente Eq.3 em y e igualando com Eq.2:

$x^2e^y + x + \frac{\partial}{\partial y}k(y) = x^2e^y + x - 2y \Leftrightarrow \frac{\partial}{\partial y}k(y) = -2y \Leftrightarrow$

$k(y) = \int -2y \cdot dy \Leftrightarrow k(y) = -y^2 + K$ (Eq.4)

De Eq.3 e Eq.4

$$\boxed{\varphi(x,y) = (x^2e^y + xy - y^2 + K)}$$

c) $\vec{F}(x,y,z) = (2x^2 + 8xy^2,\ 3x^3y - 3xy,\ 4z^2y^2 + 2x^3z)$ com $\Omega = \mathbb{R}^3$

$$\text{rot}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2+8xy^2 & 3x^3y-3xy & 4z^2y^2+2x^3z \end{vmatrix} = \left(\frac{\partial}{\partial y}(4z^2y^2+2x^3z) - \frac{\partial}{\partial z}(3x^3y-3xy)\right)\hat{i}$$

$$+\left(\frac{\partial}{\partial z}(2x^2+8xy^2) - \frac{\partial}{\partial x}(4z^2y^2-2x^3z)\right)\hat{j} + \left(\frac{\partial}{\partial y}(2x^2+8xy^2) - \frac{\partial}{\partial x}(3x^3y-3xy)\right)\hat{k}$$

$$= (8z^2y - 0)\hat{i} + (0 - 6x^2z)\hat{j} + (16xy - 9x^2y + 3y)\hat{k} \neq \vec{0}$$

$\therefore$ $\vec{F}$ não é conservativo

d) $\vec{F}(x,y,z) = (x+z, -y-z, x-y)$ $\quad \Omega = \mathbb{R}^3$

$$\text{rot}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & -y-z & x-y \end{vmatrix} = (-1-(-1))\hat{i} + (1-(+1))\hat{j} + (0+0)\hat{k}$$

$$= \vec{0}$$

Sendo $\mathbb{R}^3$ um conjunto simplesmente conexo, então $\vec{F}$ é conserv.

Obtendo-se $\varphi : \Omega \to \mathbb{R}$

$$\vec{F} = \nabla\varphi \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = x+z & \text{(Eq.1)} \\ \frac{\partial}{\partial y}\varphi = -y-z & \text{(Eq.2)} \\ \frac{\partial}{\partial z}\varphi = x-y & \text{(Eq.3)} \end{cases}$$

De Eq.1:

$$\varphi = \int (x+z).dx = \frac{x^2}{2} + xz + K(y,z) \quad \text{(Eq.4)}$$

Derivando Eq.4 em relação a y:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}K(y,z)$$

E igualando com Eq.2:

$$-y-z = \frac{\partial}{\partial y}K(y,z) \Leftrightarrow K(y,z) = \int(-y-z).dy = -\frac{y^2}{2} - yz + K(z) \quad \text{(Eq.5)}$$

De Eq.4 e Eq.5

$$\varphi = \frac{x^2}{2} + xz - \frac{y^2}{2} - yz + K(z) \quad \text{(Eq.6)}$$

Derivando Eq.6 em relação a z e igualando com Eq.3

$$x-y = x-y + \frac{\partial}{\partial z}K(z) \Leftrightarrow K(z) = K$$

$$\therefore \boxed{\varphi(x,y,z) = \frac{x^2}{2} + xz - \frac{y^2}{2} - yz + K}$$

e) $\vec{F}(x,y,z) = \underbrace{(y^2\cos x + z^3)}_{P}\hat{i} + \underbrace{(2y.\text{sen}x - 4)}_{Q}\hat{j} + \underbrace{(3xz^2+2)}_{R}\hat{k}$

$\Omega = \mathbb{R}^3$

$$\text{rot}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$

$$\left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right)\hat{i} + \left(\frac{\partial}{\partial z}P - \frac{\partial}{\partial x}R\right)\hat{j} + \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k}$$

$$= (0-0)\hat{i} + (3z^2 - 3z^2)\hat{j} + (2y\cos x - 2y\cos x)\hat{k} = \vec{0}$$

O Domínio Ω de $\vec{F}$ é simplesmente conexo, portanto, $\vec{F}$ é um campo vetorial, logo:

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = y^2\cos x + z^3 \\ \frac{\partial}{\partial y}\varphi = 2y\,\text{sen}x - 4 \\ \frac{\partial}{\partial z}\varphi = 3xz^2 + 2 \end{cases}$$

$$\varphi = \int(y^2\cos x + z^3)\,dx$$

$$\varphi = y^2\text{sen}x + xz^3 + K(y,z)$$

$$\frac{\partial}{\partial y}\varphi = 2y\,\text{sen}x + \frac{\partial}{\partial y}K(y,z) \Leftrightarrow$$

$$2y\,\text{sen}x + \frac{\partial}{\partial y}K(y,z) = 2y\,\text{sen}x - 4 \Leftrightarrow K(y,z) = -4y + K(z) \Rightarrow$$

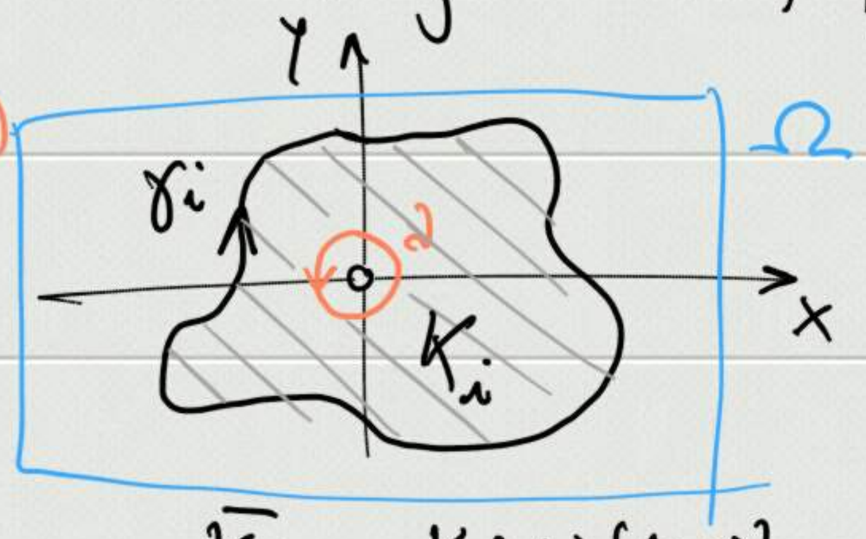
$$\varphi = y^2\text{sen}x + xz^3 - 4y + K(z) \Leftrightarrow \frac{\partial\varphi}{\partial z} = 3xz^2 + \frac{\partial}{\partial z}K(z) \Leftrightarrow 3xz^2 + 2 = 3xz^2 + \frac{\partial}{\partial z}K(z)$$

$$\Leftrightarrow K(z) = 2z + K \qquad \therefore \boxed{\varphi(x,y,z) = y^2\text{sen}x + xz^3 - 4y + 2z + K}$$

f) $\vec{F}(x,y) = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$, $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$

$\text{rot}(\vec{F}) = \vec{0}$ → veja o 11. a e Ω não é um conjunto simplesmente conexo, logo, $\vec{F}$ so será conservativo se $\oint_{\gamma_i} \vec{F}.d\vec{r} = 0$, $\forall \gamma_i$ cuja a região interior $\bar{K_i}$ contenha o ponto de singularidade, que, neste caso, é o $(0,0)$.

$\partial(t) = (r\cos t, r\,\text{sen}\,t)$
$\lim r \to 0^+$
$t \in [0, 2\pi]$



$\bar{K_i} = K_i \cup \{(0,0)\}$

$$\oint_{\gamma_i \cup \partial} \vec{F}.d\vec{r} = \oint_{\gamma_i} \vec{F}.d\vec{r} + \oint_{\partial} \vec{F}.d\vec{r} = -\iint_{K_i} 0.\,dxdy$$

$$\Leftrightarrow \boxed{\oint_{\gamma_i} \vec{F}.d\vec{r} = -\oint_{\partial} \vec{F}.d\vec{r}}$$

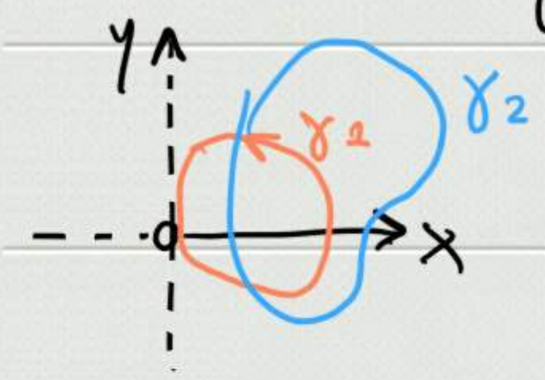
$$\oint_{\partial} \vec{F}.d\vec{r} = \oint_{\partial} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \lim_{r\to 0^+} \left(\frac{r^2\text{sen}^2 t + r^2\cos^2 t}{r^2}\right) d\theta = 2\pi$$

$\therefore$ $\vec{F}$ não é conservativo, pois $\text{rot}(\vec{F}) = \vec{0}$, mas $\oint_{\gamma_i} \vec{F}.d\vec{r} = -2\pi \neq 0$, $\forall \gamma_i$ cuja $\bar{K_i} \not\subset \Omega$

g) $\vec{F}(x,y,z) = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$, $\Omega = \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$

$\text{rot}(\vec{F}) = \vec{0}$ e Ω é um conjunto simplesmente conexo, pois $\nexists\, \gamma_i \mid \exists\, K_i \not\subset \Omega$, sendo K_i a região interior à γ_i, uma curva fechada e plana.

arctg $-\frac{2}{3}$



$\therefore$ $\vec{F}$ é conservativo

Então:

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = \frac{-y}{x^2+y^2} & \text{(Eq.1)} \\ \frac{\partial}{\partial y}\varphi = \frac{x}{x^2+y^2} & \text{(Eq.2)} \end{cases}$$

$$\frac{\partial}{\partial x}\varphi = \frac{-y}{x^2+y^2} \Leftrightarrow \varphi = \int \frac{-y}{x^2+y^2}dx \Leftrightarrow \varphi = \int \frac{-y}{y^2\left(\frac{x^2}{y^2}+1\right)}dx = -\frac{1}{y}\int \frac{1}{\left(\frac{x^2}{y^2}+1\right)}dx$$

$$-\frac{1}{y}\cdot\frac{1}{1/y}\,\text{arctg}\frac{x}{y} + K(y) = -\text{arctg}\frac{x}{y} + K(y) \Leftrightarrow \varphi = -\text{arctg}\frac{x}{y} + K(y)$$

$$\Leftrightarrow \frac{\partial}{\partial y}\varphi = \frac{-1}{\frac{x^2}{y^2}+1}\cdot -\frac{x}{y^2} + \frac{\partial}{\partial y}K(y) \Leftrightarrow \boxed{\frac{\partial}{\partial y}\varphi = \frac{x}{x^2+y^2} + \frac{\partial}{\partial y}K(y)} \text{ (Eq.3)}$$

$$\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}K(y) = \frac{x}{x^2+y^2} \Leftrightarrow \frac{\partial}{\partial y}K(y) = 0 \quad \therefore \varphi(x,y) = -\text{arctg}\frac{x}{y} + K$$

Obs: $-\text{arctg}\frac{x}{y} + K = \text{arctg}\frac{-x}{y} + K$

(35)

$$\operatorname{arctg}\frac{-x}{y} + K = \frac{\pi}{2} + \operatorname{arctg}\frac{-x}{y} + K' = \operatorname{arctg}\frac{y}{x} + K'$$

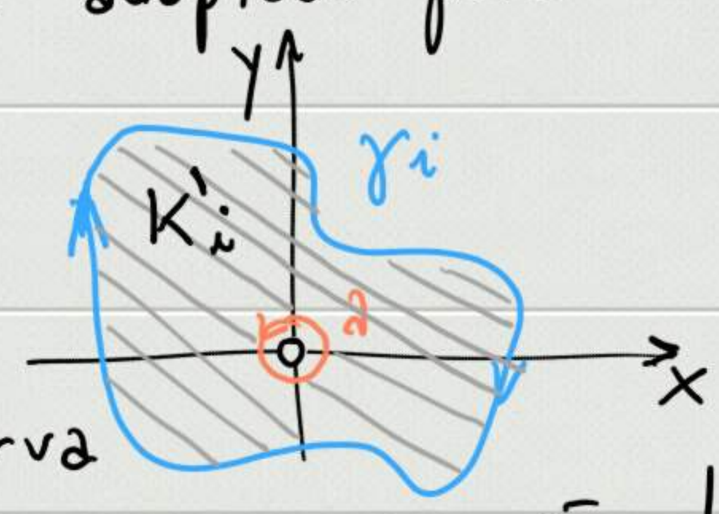
essa integral possui essa peculiaridade que pode ser demonstrada por trigonometria convencional

Obs²: se começar pela derivada parcial $\frac{\partial}{\partial y}$, vai obter a mesma resposta do gabarito só que de maneira direta.

h) $F(x,y) = \dfrac{x\hat{i} + y\hat{j}}{x^2+y^2}$ Perceba que este satisfaz a condição do Ex. 18: $\dfrac{1}{x^2+y^2}(x\hat{i} + y\hat{j}) = g(\sqrt{x^2+y^2}).\vec{r}$

$$\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right)\hat{k} = \left(\frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2}\right)\hat{k} = \vec{0}$$

$\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$, Ω não é um conjunto simplesmente conexo, então, pelo Teorema de Green adaptado para regiões vazadas:



Sejam γ_i, uma curva plana e fechada qualquer | $\gamma_i \subset \Omega$, K_i, a região interior à γ_i | $K_i \not\subset \Omega$, e λ a curva que isola o ponto de singularidade na proporção do do denominador de $\vec{F}$, tem-se que: $\lambda = r(\cos t, \operatorname{sen} t)$, $\lim r \to 0^+$, $t \in [0, 2\pi]$

$$\oint_{\gamma_i \cup \lambda} \vec{F}.d\vec{r} = \oint_{\gamma_i} \vec{F}.d\vec{r} + \oint_{\lambda} \vec{F}.d\vec{r} = -\iint_{K'_i} 0.dxdy$$

$$\Leftrightarrow \oint_{\gamma_i} \vec{F}.d\vec{r} = -\oint_{\lambda} \vec{F}.d\vec{r} = -\oint_{\lambda} \vec{F}.d\lambda \quad \text{(Eq.1)}$$

$$\oint_{\lambda} \vec{F}.d\lambda = \oint_{\lambda} \frac{1}{x^2+y^2}(x,y)\cdot d\lambda = \int_0^{2\pi} \lim_{r\to 0^+}\left[\frac{1}{r^2}.(r.\cos t, r\operatorname{sen} t)\cdot(-r\operatorname{sen} t, r\cos t).dt\right]$$

$$\int_0^{2\pi} \lim_{r\to 0^+}\left[\frac{1}{r^2}(-r^2.\cos t\operatorname{sen} t + r^2\cos t\operatorname{sen} t).dt\right] = \int_0^{2\pi} \lim_{r\to 0^+} 0.dt = 0$$

Portanto, sendo $\operatorname{rot}(\vec{F}) = \vec{0}$ e $\oint_{\gamma_i} \vec{F}.d\vec{r} = 0$, $\forall \gamma_i \mid K_i \not\subset \Omega$, $\vec{F}$ é um campo conservativo

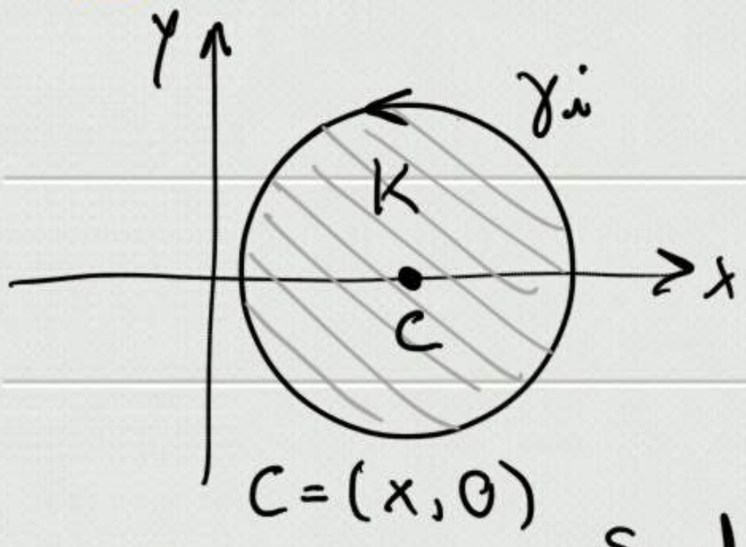
$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = \frac{x}{x^2+y^2} \\ \frac{\partial}{\partial y}\varphi = \frac{y}{x^2+y^2} \end{cases} \Leftrightarrow \varphi = \int \frac{x}{x^2+y^2}.dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \frac{1}{2}\ln u + K(y)$$

$$= \frac{1}{2}\ln(x^2+y^2) + K(y)$$

$$\frac{\partial}{\partial y}\varphi = \frac{1}{2}\frac{1}{x^2+y^2}.2y + \frac{\partial}{\partial y}K(y) = \frac{y}{x^2+y^2} + \frac{\partial}{\partial y}K(y) \Leftrightarrow \frac{y}{x^2+y^2} + \frac{\partial}{\partial y}K(y) = \frac{y}{x^2+y^2}$$

$$\Leftrightarrow \frac{\partial}{\partial y}K(y) = 0 \qquad \therefore \boxed{\varphi(x,y) = \frac{1}{2}\ln(x^2+y^2) + K}$$

20 $\vec{F}(x,y) = x\hat{i} + xy\hat{j}$; $\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}x\right)\hat{k} = y.\hat{k} \neq \vec{0}$ Eq. 0



$\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2$

Pelo Teorema de Green:

$$\oint_{\gamma_i} \vec{F}.d\gamma_i = \iint_K y.dxdy \quad \text{Eq. 1}$$

Sendo a ordenada de C igual a zero para todo γ_i, circunferência de raio qualquer cujo centro está contido no eixo das abcissas, tem-se que:

$$\oint_{\gamma_i} \vec{F}.d\vec{r} = 0, \ \forall \gamma_i \subset \Omega$$

$\vec{F}$ não é conservativa, pois $\text{rot}(\vec{F}) \neq 0$

21 a) $\int_\gamma (-2xy + x^2)dx + \sqrt{8-y^7}.dy$, $\gamma(x) = (x, \cos x) \mid x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

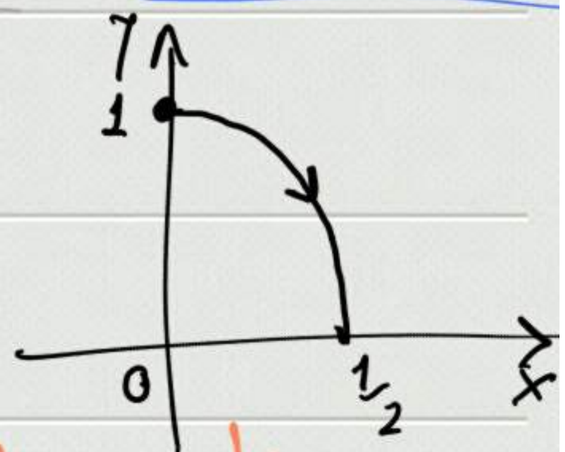
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2x\cos x + x^2).dx + \left(\sqrt{8-\cos^7 x}.\right)-\text{sen}x.dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2x\cos x.dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2.dx + 0$$

$$= \underbrace{\left[-2x\,\text{sen}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{0} - \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2.\text{sen}x\,dx}_{0} + \left[\frac{x^3}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^3}{24} - \frac{-\pi^3}{24} = \frac{\pi^3}{12}$$

0 função par

$u = \cos x$
$du = -\text{sen}x\,dx$
$x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \to u\Big|_0^0$

b) $\int_\gamma \frac{2xy^2}{x^2+1}dx + 2y\ln(x^2+1)dy$, $\gamma(t) = (\frac{1}{2}\cos t, \text{sen}t)$
$t \in [0, \frac{\pi}{2}]$
t indo de $\frac{\pi}{2}$ à 0



$$\int_{\frac{\pi}{2}}^{0} \left(\frac{2\frac{1}{2}\cos t\,\text{sen}^2 t}{\frac{\cos^2 t}{4}+1} . -\frac{1}{2}\text{sen}t.dt + 2\,\text{sen}t\ln\left(\frac{\cos^2 t}{4}+1\right).\cos t\,dt \right)$$

... parece ter um jeito mais fácil

$$\text{rot}(\vec{F}) = \left(\frac{2y}{x^2+1}.2x - \frac{2x.2y}{x^2+1}\right)\hat{k} = \vec{0}$$

$\therefore$ $\vec{F}$ é conservativo, pois Ω é simplesmente conexo.

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = \frac{2xy^2}{x^2+1} \\ \frac{\partial}{\partial y}\varphi = 2y\ln(x^2+1) \end{cases}$$

integra o mais fácil!

$$\Leftrightarrow \varphi = \int 2y\ln(x^2+1).dy = y^2\ln(x^2+1) + k(x)$$

$$\frac{\partial}{\partial x}\varphi = \frac{y^2}{x^2+1}.2x + \frac{\partial}{\partial x}k(x) = \frac{2xy^2}{x^2+1} \Leftrightarrow \frac{\partial k(x)}{\partial x} = 0 \quad \therefore \boxed{\varphi(x,y) = y^2\ln(x^2+1) + k}$$

$$\therefore \int_\gamma \vec{F}.d\vec{r} = \int_\gamma \nabla\varphi\,d\gamma = \int_{(0,1)}^{(\frac{1}{2},0)} d\varphi = \varphi(\tfrac{1}{2},0) - \varphi(0,1) = \underbrace{0.\ln(\tfrac{1}{4}+1)}_{0} - \underbrace{1.\ln 1}_{0}$$

Eq. 1

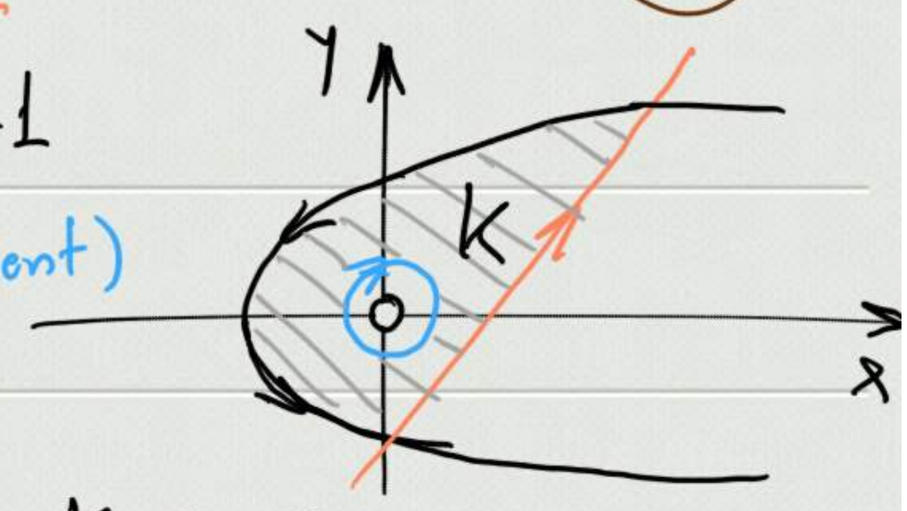
$$\therefore \boxed{\int_\gamma \vec{F}.d\vec{r} = 0}$$

c) $\int_\gamma \frac{-y}{x^2+y^2}dx + \left(\frac{x}{x^2+y^2} + xy\right)\cdot dy$ | $\gamma: \begin{cases} y \geq x-1 \\ x \geq y^2-1 \end{cases}$

$\lambda(t) = r(\cos t, \operatorname{sen} t)$

$r \to 0^+$

$t \in [0, 2\pi]$



$\operatorname{rot}(\vec{F}) = y\,\hat{k}$

Pela Teorema de Green em relação à região $K \subset \Omega$:

$$\oint_{\gamma\cup\lambda} \vec{F}\cdot d\vec{r} = \oint_\gamma \vec{F}\cdot d\vec{r} + \oint_\lambda \vec{F}\cdot d\vec{r} = \iint_K y\cdot dx\,dy \Leftrightarrow \boxed{\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_K y\,dx\,dy + \oint_\lambda \vec{F}\cdot d\vec{r}} \quad \text{(Eq. 1)}$$

Mais fácil descrever o conjunto em relação ao eixo x

$y^2-1 \leq x \leq y+1$; $y^2-1 = y+1 \Leftrightarrow y^2-y-2=0 \Leftrightarrow (y-2)(y+1)=0$

$-1 \leq y \leq 2$

$K = \{(x,y) \in \mathbb{R}^2 \mid y^2-1 \leq x \leq y+1 \text{ e } -1 \leq y \leq 2\}$

$$\iint_K y\,dx\,dy = \int_{-1}^{2}\int_{y^2-1}^{y+1} y\,dx\,dy = \int_{-1}^{2}(y+1-(y^2-1))y\,dy = \int_{-1}^{2}(-y^3+y^2+2y)\,dy$$

$$= -\frac{1}{4}(2^4-(-1)^4) + \frac{1}{3}(2^3-(-1)^3 + \frac{2}{2}(2^2-(-1)^2) = -\frac{15}{4} + 3 + 3 = \frac{9}{4} \quad \text{(Eq. 2)}$$

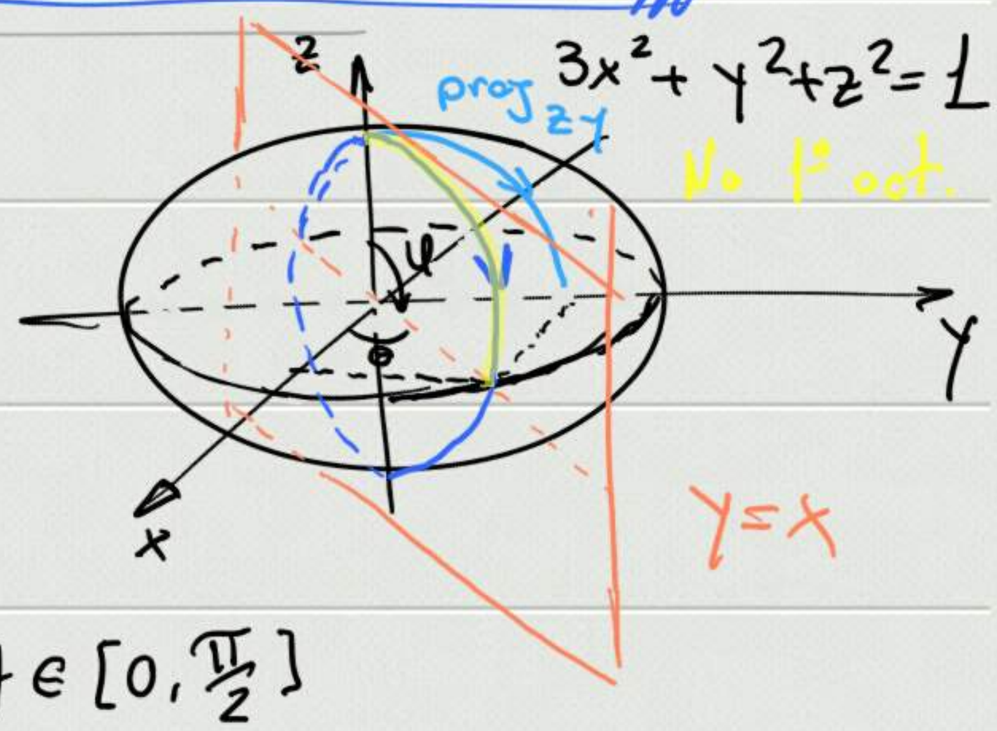
$$\oint_\lambda \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \lim_{r\to 0^+}\left[\frac{-r\operatorname{sen}t\cdot(-r\cdot\operatorname{sen}t)\,dt}{r^2} + \left(\frac{r\cos t}{r^2} + r^2\cos t\cdot\operatorname{sen}t\right)\cdot r\cdot\cos t\cdot dt\right]$$

$$= \int_0^{2\pi} \lim_{r\to 0^+}\left(\frac{r^3}{r^2} + r^3\cos^2 t\cdot\operatorname{sen}t\right)dt = \int_0^{2\pi} 1\cdot dt = 2\pi \quad \text{(Eq. 3)}$$

Portanto, de Eq. 1, 2 e 3:

$$\boxed{\oint_\gamma \vec{F}\cdot d\vec{r} = \frac{9}{4} + 2\pi}$$

d) $\int_\gamma (2xy + \operatorname{sen} y)\,dx + x\cos y\,dy + x^2\,dz$ | γ



$\begin{cases} 3x^2+y^2+z^2=1 \\ y=x \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases} \Leftrightarrow 4y^2+z^2=1$ $x=0$

projeção zy

$y = \frac{\operatorname{sen}t}{2}$, $z = \cos t$

$\therefore \gamma(t) = \left(\frac{\operatorname{sen}t}{2}, \frac{\operatorname{sen}t}{2}, \cos t\right)$ $t \in [0, \frac{\pi}{2}]$

$\frac{d\gamma(t)}{dt} = \left(\frac{\cos t}{2}, \frac{\cos t}{2}, -\operatorname{sen}t\right) \Leftrightarrow d\gamma(t) = \left(\frac{\cos t}{2}, \frac{\cos t}{2}, -\operatorname{sen}t\right)\cdot dt \Leftrightarrow$

$dx = \frac{\cos t}{2}dt$, $dy = \frac{\cos t}{2}\cdot dt$, $dz = -\operatorname{sen}t\cdot dt$

$u = \operatorname{sen}t$ $u|_0^1$

$du = \cos t\cdot dt$

$dt = \frac{du}{\cos t}$

$$\therefore \int_\gamma (2xy+\operatorname{sen}y)dx + x\cos y\,dy + x^2dz = \int_0^{\frac{\pi}{2}}\left(2\frac{\operatorname{sen}t}{2}\cdot\frac{\operatorname{sen}t}{2} + \operatorname{sen}\left(\frac{\operatorname{sen}t}{2}\right)\right)\frac{\cos t}{2}dt$$

$$+ \frac{\operatorname{sen}t}{2}\cos\left(\frac{\operatorname{sen}t}{2}\right)\cdot\frac{\cos t}{2}\cdot dt + \frac{\operatorname{sen}^2 t}{4}\cdot(-\operatorname{sen}t\,dt)$$

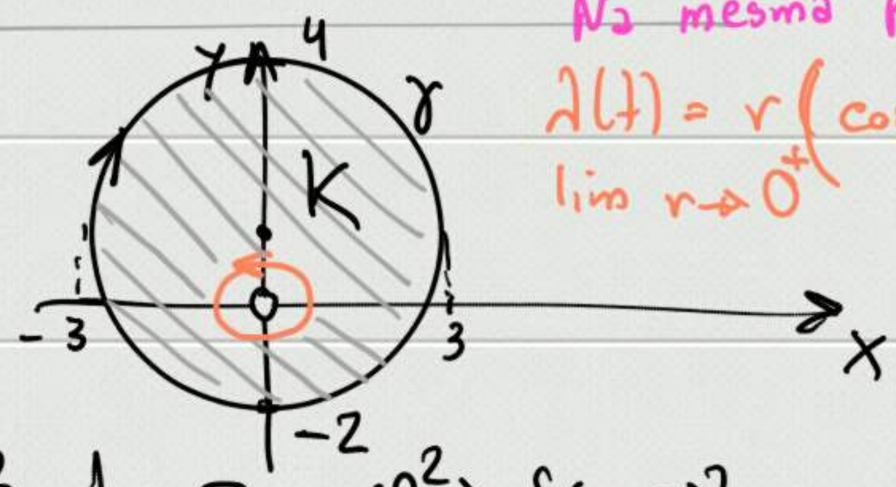
$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{sen}^2 t\cdot\cos t}{4}dt + \int_0^{\frac{\pi}{2}} \frac{\cos t}{2}\cdot\operatorname{sen}\left(\frac{\operatorname{sen}t}{2}\right)dt + \int_0^{\frac{\pi}{2}} \frac{\operatorname{sen}t\cdot\cos t}{4}\cdot\cos\left(\frac{\operatorname{sen}t}{2}\right)dt - \int_0^{\frac{\pi}{2}} \frac{\operatorname{sen}^2 t\operatorname{sen}t}{4}dt$$

$$= \int_0^1 \frac{u^2 \cdot \cos t}{4} \cdot \frac{du}{\cos t} + \int_0^1 \frac{\cos t}{2} \operatorname{sen}\left(\frac{u}{2}\right) \cdot \frac{du}{\cos t} + \int_0^1 \frac{u \cdot \cos t}{4} \cdot \cos\left(\frac{u}{2}\right) \cdot \frac{du}{\cos t} - \frac{1}{4}\left(\left[\frac{-\cos t \operatorname{sen}^2 t}{3}\right]_0^{\frac{\pi}{2}} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \operatorname{sen} t \, dt\right)$$

$$\frac{1}{12} + \int_0^1 \frac{1}{2} \operatorname{sen}\left(\frac{u}{2}\right) \cdot du + \left[\frac{u \cdot 2 \operatorname{sen}\left(\frac{u}{2}\right)}{4}\right]_0^1 - \int_0^1 \frac{1}{2} \operatorname{sen}\left(\frac{u}{2}\right) du - \frac{1}{4} \cdot \frac{2}{3}\left(-\cos\frac{\pi}{2} - (-\cos 0)\right)$$

$$\frac{1}{12} + \frac{1}{2} \operatorname{sen}\left(\frac{1}{2}\right) - \frac{1}{6} = \frac{1}{12}\left(6 \operatorname{sen}\left(\frac{1}{2}\right) - 1\right)$$

e) $\int_\gamma \frac{-y}{x^2+2y^2} dx + \frac{x}{x^2+2y^2} dy$



Na mesma proporção do denominador

$\lambda(t) = r\left(\cos t, \frac{\operatorname{sen} t}{\sqrt{2}}\right)$

$\lim_{r \to 0^+}$

$t \in [0, 2\pi]$

$\vec{F} = \left(\frac{-y}{x^2+2y^2}, \frac{x}{x^2+2y^2}\right)$

$\operatorname{rot}(\vec{F}) = \vec{0}$, $\vec{F} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2 \setminus \{(0,0)\}$

$$\oint_{\gamma_i \cup \lambda} \vec{F} \cdot d\vec{r} = \oint_{\gamma_i} \vec{F} \cdot d\vec{r} + \oint_\lambda \vec{F} \cdot d\vec{r} = \iint_K 0 \cdot dx\, dy \Leftrightarrow \boxed{\oint_{\gamma_i} \vec{F} \cdot d\vec{r} = -\oint_\lambda \vec{F} \cdot d\vec{r}} \quad (Eq.1)$$

$$\oint_\lambda \vec{F} \cdot d\vec{r} = \oint_\lambda \frac{-y}{x^2+2y^2} dx + \frac{x}{x^2+2y^2} dy = \int_0^{2\pi} \lim_{r \to 0^+}\left[\frac{-r \operatorname{sen} t}{\sqrt{2}\, r^2}(-r \operatorname{sen} t\, dt) + \frac{r \cdot \cos t}{r^2}\left(\frac{r \cos t}{\sqrt{2}} dt\right)\right]$$

$$= \int_0^{2\pi} \frac{1}{\sqrt{2}} \cdot dt = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$$

$$\therefore \boxed{\oint_{\gamma_i} \vec{F} \cdot d\vec{r} = -\sqrt{2}\pi}$$

Acho que o gabarito pode estar errado.

Em seu domínio Ω mais amplo, $\vec{F}$ não é conservativo embora $\operatorname{rot}(\vec{F}) \neq \vec{0}$, pois Ω não é simplesmente conexo e $\oint_{\gamma_i} \vec{F} \cdot d\vec{r} = \vec{0}$, $\forall\, \gamma_i$ tal que a região interna à γ_i contenha o ponto $(0,0)$.

22) $\vec{F}(x,y) = \frac{x\hat{i} + y\hat{j}}{x^2+y^2}$, $\gamma(t) = (e^t, \operatorname{sen} t)$, $0 \leq t \leq \pi$

Percebe-se que $\gamma(0) = (1, 0)$ está numa região do Domínio onde $\vec{F}$ é um campo conservativo. Portanto, utilizando-se a função potencial de $\vec{F}(x,y)$

Na dúvida, dne o Ex. 19 item h

$\gamma(0) = (1,0)$ $\quad \gamma(\pi) = (e^\pi, 0)$ $\quad \mid \quad \varphi(x,y) = \frac{1}{2}\ln(x^2+y^2) + K$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma \nabla\varphi \cdot d\gamma = \int_{(1,0)}^{(e^\pi,0)} d\varphi = \varphi(e^\pi, 0) - \varphi(1,0) = \frac{1}{2}(\ln e^{2\pi} - \ln 1)$$

$$= \frac{2\pi}{2}$$

$$\therefore \boxed{\int_\gamma \vec{F} \cdot d\vec{r} = \pi}$$

23 a) $\int_\gamma 7x^6y\,dx + x^7dy$, $\gamma(t) = (t, e^{t^2-1})$ $t \in [0,1]$

tem uma carinha de conservativo $\vec{F} = (7x^6y, x^7)$ | $\Omega = \mathbb{R}^2$

$\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x} x^7 - \frac{\partial}{\partial y} 7x^6y\right)\hat{k} = \vec{0}$ e, sendo Ω o domínio de F, Ω é um conjunto simplesmente conexo, portanto, $\vec{F}$ é um campo conservativo.

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = 7x^6y \Leftrightarrow \varphi = x^7y + K(y) \\ \frac{\partial}{\partial y}\varphi = x^7 \Leftrightarrow \varphi = x^7y + K(x) \end{cases} \quad \therefore \varphi(x,y) = x^7y + K \quad \text{(Eq.1)}$$

$\gamma(0) = (0, \frac{1}{e})$ e $\gamma(1) = (1,1)$

$$\therefore \int_\gamma \vec{F}.d\vec{r} = \varphi(1,1) - \varphi(0,\tfrac{1}{e}) = 1 - 0 \Leftrightarrow \boxed{\int_\gamma \vec{F}\cdot d\vec{r} = 1}$$

b) $\int_\gamma (\ln(x+y^2) - y)dx + (2y\ln(x+y^2) - x)dy$ Outro com cara de conservativo no intervalo trabalhado

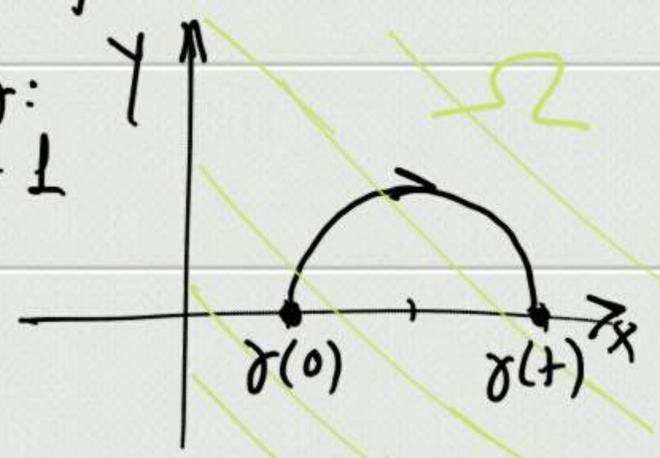
$\vec{F} = (\ln(x+y^2) - y, 2y\ln(x+y^2) - x)$

$\text{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(2y\ln(x+y^2) - x) - \frac{\partial}{\partial y}(\ln(x+y^2) - y)\right)\hat{k}$

$\text{rot}(\vec{F}) = \left[\frac{2y}{x+y^2}\cdot 1 - 1 - \left(\frac{1}{x+y^2}\cdot 2y - 1\right)\right]\hat{k} = \vec{0}$

Sendo que o caminho traçado pela curva está numa região em que $x > 0$, o domínio Ω de $\vec{F}$ pode ser reduzido para $\Omega = \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$, tornando-se assim um conjunto simplesmente conexo. Consequentemente $\vec{F}$ é conservativo

γ: $(x-2)^2 + y^2 = 1$



$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = \ln(x+y^2) - y & \text{(Eq.1)} \\ \frac{\partial}{\partial y}\varphi = 2y\ln(x+y^2) - x & \text{(Eq.2)} \end{cases}$$

$\Leftrightarrow \varphi = \int(\ln(x+y^2) - x).dx$

$\Leftrightarrow \varphi = (x+y^2)\ln(x+y^2) - (x+y^2) - \frac{x^2}{2} + K(y)$ (Eq.3)

$\int \ln(z).dz = z.\ln(z) - \int z\,\frac{1}{z}.dz$

$\int \ln(z)dz = z.\ln(z) - z$

$\frac{\partial\varphi}{\partial y} = 2y.\ln(x+y^2) + (x+y^2)\frac{1}{x+y^2}.2y - 2y + \frac{\partial}{\partial y}K(y)$

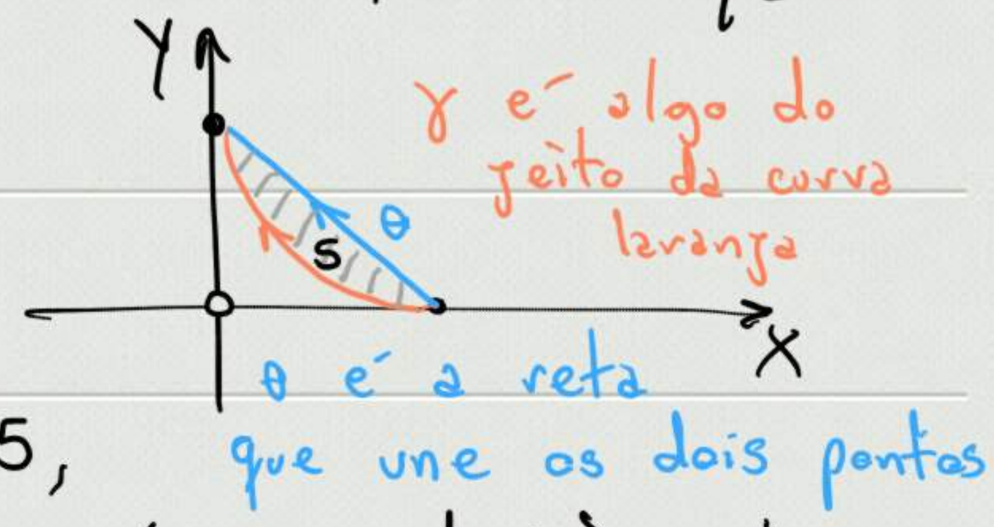
$\frac{\partial}{\partial y}\varphi = 2y\ln(x+y^2) + \frac{\partial}{\partial y}K(y) = 2y\ln(x+y^2) - y \Leftrightarrow \frac{\partial}{\partial y}K(y) = -y \Leftrightarrow K(y) = -\frac{y^2}{2} + K$

$\therefore \varphi(x,y) = (x+y^2)\ln(x+y^2) - (x+y^2) - \frac{x^2}{2} - \frac{y^2}{2} + K$ (Eq.4)

$\gamma(0) = (1,0) \rightarrow \gamma(t) = (3,0)$

$\int_\gamma \vec{F}.d\vec{r} = \varphi(3,0) - \varphi(1,0) = 3.\ln(3) - 3 - \frac{9}{2} - \left(1.\ln 1 - 1 - \frac{1}{2}\right) = 3\ln 3 - 6$

c) $\int_\gamma \frac{y\,dx - x\,dy}{x^2+y^2} = \int_\gamma -\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)$, $\gamma(t) = (\cos^3 t, \operatorname{sen}^3 t)$ $y \geq 0$



$\operatorname{rot}(\vec{F}) = \vec{0}$ e sabe-se que este campo vetorial é conservativo para um Ω simplesmente conexo.

Segundo a demonstração do Ex. 15, tem-se que: $\varphi(x,y) = -\operatorname{arctg}\frac{y}{x} + K = -\left(\frac{\pi}{2} - \operatorname{arctg}\frac{x}{y}\right) + K'$.

Como a região S não contém $(0,0)$, conclui-se, pelo Teorema de Green e pela demo do Ex. 15, que:

$$\boxed{\int_\gamma \vec{F}\cdot d\vec{r} = \int_\theta \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} d\varphi} \quad \text{(Eq. 1)}$$

$$\therefore \int_\gamma \vec{F}\cdot d\vec{r} = \varphi(0,1) - \varphi(1,0) = -\operatorname{arctg}\left(\frac{1}{0^+}\right) + \operatorname{arctg}\left(\frac{0}{1}\right)$$

$\rightarrow$ se aproxima pela direita

$$\int_\gamma \vec{F}\cdot d\vec{r} = -\operatorname{arctg}(+\infty) \Leftrightarrow \boxed{\int_\gamma \vec{F}\cdot d\vec{r} = -\frac{\pi}{2}}$$

24 a) $\int_{(1,1)}^{(a,b)} \underbrace{2xy}_{Q}\,dx + \underbrace{(x^2-y^2)}_{P}\,dy$; $\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(2xy)\right)\hat{k}$

$\vec{F} = (Q, P)$

$$\operatorname{rot}(\vec{F}) = (2x - 2x)\hat{k} = \vec{0}$$

O Domínio Ω deste campo vetorial é $\Omega = \mathbb{R}^2$, constituindo-se um conjunto simplesmente conexo. Portanto, o campo vetorial $\vec{F}$ é conservativo.

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \frac{\partial}{\partial x}\varphi = 2xy \\ \frac{\partial}{\partial y}\varphi = x^2-y^2 \end{cases} \Leftrightarrow \varphi = \int 2xy\,dx \Leftrightarrow \varphi = x^2y + K(y) \Leftrightarrow$$

$$\frac{\partial}{\partial y}\varphi = x^2 + \frac{\partial}{\partial y}K(y) \Leftrightarrow x^2 - y^2 = x^2 + \frac{\partial}{\partial y}K(y)$$

$$\Leftrightarrow \frac{\partial}{\partial y}K(y) = -y^2 \Leftrightarrow K(y) = \int -y^2\,dy \Leftrightarrow K(y) = -\frac{1}{3}y^3 + K$$

$$\therefore \boxed{\varphi(x,y) = x^2y - \frac{1}{3}y^3 + K} \quad \text{(Eq. 1)}$$

Então $\int_{(1,1)}^{(a,b)} 2xy\,dx + (x^2-y^2)\,dy = \int_{(1,1)}^{(a,b)} d\varphi$

$$= \varphi(a,b) - \varphi(1,1) = a^2b - \frac{1}{3}b^3 - \left(1^2\cdot 1 - \frac{1}{3}\cdot 1^3\right) = a^2b - \frac{b^3}{3} - \frac{2}{3}$$

$$\therefore \boxed{\int_{(1,1)}^{(a,b)} 2xy\,dx + (x^2-y^2)\,dy = a^2b - \frac{b^3}{3} - \frac{2}{3}}$$

b) $\int_{(0,0)}^{(a,b)} \operatorname{sen} y\,dx + x\cos y\,dy$; $\vec{F} = (\operatorname{sen} y, x\cos y)$; $\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}x\cos y - \frac{\partial}{\partial y}\operatorname{sen} y\right)\hat{k}$

$$\operatorname{rot}(\vec{F}) = (\cos y - \cos y)\hat{k} = \vec{0}$$

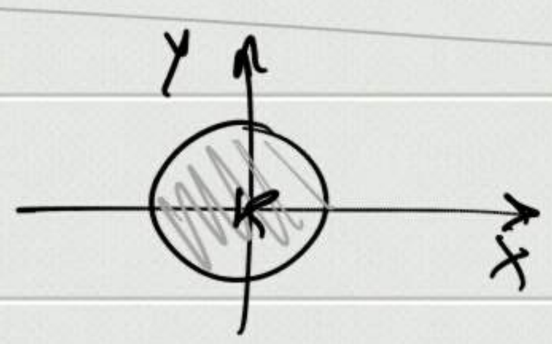
O domínio Ω de $\vec{F}$ é simplesmente conexo, pois $\Omega = \mathbb{R}^2$. Portanto, $\vec{F}$ é conservativo já que $\operatorname{rot}(\vec{F}) = \vec{0}$

$$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \partial/\partial x\,\varphi = \operatorname{sen} y \\ \partial/\partial y\,\varphi = x\cos y \end{cases} \Leftrightarrow \boxed{\varphi = x\operatorname{sen} y + K} \quad \text{(Eq. 1)}$$

Portanto $\boxed{\int_{(0,0)}^{(a,b)} \operatorname{sen} y\,dx + x\cos y\,dy = a \operatorname{sen} b - 0 . \operatorname{sen} 0}$

25 a) $\vec{F}(y^2+xy,\ 3y^2)$, $\operatorname{rot}(\vec{F}) = 0-(2y+x)$

$\operatorname{rot}(\vec{F}) = -x-2y$



$\gamma = (\cos t\ ,\ \operatorname{sen} t)\ |\ t \in [0, 2\pi]$

$\oint_\gamma \vec{F}.d\gamma = \oint_0^{2\pi} (\operatorname{sen}^2 t + \cos t \operatorname{sen} t,\ 3\operatorname{sen}^2 t).(-\operatorname{sen} t, \cos t).dt =$

$\int_0^{2\pi} (-\operatorname{sen}^3 t - \cos t \operatorname{sen}^2 t + 3\cos t \operatorname{sen}^2 t).dt$ → dá zero

Pelo Teorema de Green:

$\oint_\gamma \vec{F}.d\gamma = \iint_K -(x+2y)\,dx\,dy$ | $x = r.\cos\theta$ $\quad 0 \le r \le 1$

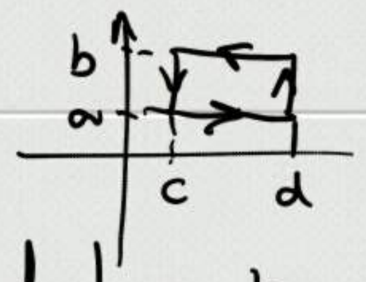
$y = r.\operatorname{sen}\theta$ $\quad 0 \le \theta \le 2\pi$

$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

$= \iint_{K_{r,\theta}} -(r\cos\theta + 2r\operatorname{sen}\theta)\,r\,dr\,d\theta$

$= -\int_0^{2\pi}\int_0^1 (r^2\cos\theta + 2r^2\operatorname{sen}\theta)\,dr\,d\theta$ → sempre irá da zero para circunferências ou elipses, pois vai de zero a 2π

O esquema é ser γ as bordas de um retângulo:



$\oint_\gamma \vec{F}.d\gamma = \iint_K -(x+2y)\,dx\,dy = \int_a^b\int_c^d -(x+2y)\,dx\,dy$

$\boxed{\oint_\gamma \vec{F}.d\gamma = -\left[\left(\frac{d^2-c^2}{2}\right)(b-a) + (b^2-a^2)(d-c)\right]} \neq 0 \quad \forall\, a,b,c,d \mid a,b,c,d$ formam as coordenadas de um retângulo

$\operatorname{rot}(\vec{F}) \neq \vec{0}$, ∴ $\vec{F}$ não é conservative

b) $\vec{F} = (e^{xy}, e^{-xy})$; $\operatorname{rot}(\vec{F}) = \left(\frac{\partial}{\partial x}(e^{-xy}) - \frac{\partial}{\partial y}(e^{xy})\right)\hat{k} = (-y e^{-xy} - x e^{xy})\hat{k}$

$F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \mid \Omega = \mathbb{R}^2$ $\quad \neq \vec{0}$

$\oint_{\gamma_i} \vec{F}.d\vec{r} = \iint_K (-y e^{-xy} - x e^{xy})\,dx\,dy$, seja γ_i as bordas de um retângulo em $\mathbb{R}^2$.

$\int_{-1}^{1}\int_0^1 (-y e^{-xy} - x e^{xy})\,dx\,dy = \int_{-1}^{1}\left(\left[-y \cdot \frac{1}{-y} e^{-xy}\right]_0^1 - \left[\frac{x}{y} \cdot e^{xy} - \frac{1}{y} e^{xy}\right]_0^1\right) dy$

$= \int_{-1}^{1}\left[e^{-1y} - 1 - \left(\frac{e^{1y}}{y} - 0 - \frac{1}{y}e^{1y} + \frac{1}{y}\right)\right] dy = \int_{-1}^{1}\left(e^{-y} - 1 - \frac{1}{y}\right) dy =$

$\left[-e^{-y} - y - \ln|y|\right]_{-1}^{1} = -e^{-1} + e^{1} - 1 - 1 - \ln|1| + \ln|-1|$

$= e - \frac{1}{e} - 2 \neq 0$ $\quad$ Não é conservative

26 a) $\displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$; $\text{rot}(\vec{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$

$\text{rot}(\vec{F}) = (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k} = \vec{0}$

e $\Omega = \mathbb{R}^3$ é simplesmente conexo, logo $\vec{F}$ é um campo conservativo

$$\nabla\varphi = \vec{F} \iff \begin{cases} \frac{\partial}{\partial x}\varphi = yz \\ \frac{\partial}{\partial y}\varphi = xz \\ \frac{\partial}{\partial z}\varphi = xy \end{cases} \iff \boxed{\varphi(x,y,z) = xyz + K} \quad (Eq.1)$$

$$\therefore \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = \varphi(3,5,0) - \varphi(1,1,2) = 0 - 2 = -2$$

b)

$$\int_\gamma \text{sen}(yz)\,dx + xz\cos(yz)\,dy + xy\cos(yz)\,dz$$

$\gamma(t) = (\cos t, \text{sen}\, t, t) \quad t \in [0, \frac{\pi}{4}]$

Essa integral saí tanto pelo método clássico da integral de linha, quanto verificando que $\text{rot}(\vec{F}) = \vec{0}$ e que $\Omega = \mathbb{R}^3$ é um conjunto simplesmente conexo e, por conseqüência, obtém-se o campo escalar φ relacionado a $\vec{F}$ e insere-se os pontos final, $\gamma(\frac{\pi}{4})$, e inicial, $\gamma(0)$.

Dá pra se perceber mentalmente que:

$\varphi(x,y,z) = x\,\text{sen}(yz) + K$, verifique fazendo as derivadas parciais

$$\int_\gamma \vec{F}\cdot d\vec{r} = \varphi\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) - \varphi(1, 0, 0) = \frac{\sqrt{2}}{2}\,\text{sen}\left(\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}\right) - 1\cdot\text{sen}\,0.$$

$$\therefore \boxed{\int_\gamma \vec{F}\cdot d\vec{r} = \frac{\sqrt{2}}{2}\,\text{sen}\left(\frac{\sqrt{2}\pi}{8}\right)}$$

a demo do Ex. 18 também

27 Perceba que este exercício é só uma extensão do Ex.19 item h e que a superfície potencial varia de acordo com o raio das esferas concêntricas à origem do espaço de coordenadas cartesianas. É igualzinho a um exercício de P1 de Física III que envolva cargas pontuais

$$\int_A^B \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2} = \varphi(B) - \varphi(A), \text{ onde } \varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$$

sendo $r = \sqrt{x^2+y^2+z^2}$ $\therefore A \to B \Rightarrow r = 1 \to r = 2$

$$\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) = \frac{1}{x^2+y^2+z^2}\cdot 2x$$

$$\therefore \int_A^B \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2} = \frac{1}{2}\ln 2$$