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Feito por Caio Alcântara

# 1ª lista de exercícios Cálculo III

## Pequena revisão de fórmulas:

- Teorema de Fubini (ou Cálculo de Integral Dupla/tripla)

Seja $f(x,y)$ integrável em $R=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d \mid a<b,\ c<d \atop a,b,c,d\in\mathbb{R}\}$ e se $\exists \int_a^b f(x,y)\,dx$ e $\int_c^d f(x,y)\,dy$ em seus respectivos intervalos, então:

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right).dy = \int_a^b \left(\int_c^d f(x,y).dy\right).dx$$

- Determinar a Área de uma figura

$$A = \iint_E dA = \iint_E dx\,dy$$

- Determinar o Volume de uma figura

$$V = \iiint_E dV = \iiint_E dx\,dy\,dz$$

- Determinar a massa

Para duas variáveis

$$m = \iint_E dm = \iint_E \delta(x,y).\,dA = \iint_E \delta(x,y)\,dx\,dy \mid \delta(x,y): \text{Densidade areolar}$$

Para três variáveis

$$m = \iiint_E dm = \iiint_E \delta(x,y,z).dV \mid \delta(x,y,z): \text{Densidade Volumétrica}$$

- Determinar o Centro de Massa

$$\vec{R} = \left(\frac{\iint_E x.dm}{m}, \frac{\iint_E y.dm}{m}\right)$$

$$\vec{R} = \left(\frac{\iiint_E x\,dm}{m}, \frac{\iiint_E y.dm}{m}, \frac{\iiint_E z.dm}{m}\right)$$

Demonstração destas nos Exercícios 12 e 17

## • Determinar o Momento de Inércia

caso geral $\boxed{dI = r^2 . dm}$ | r é a distância de dm em relação ao eixo ou ao ponto que se calcula o momento de inércia

Casos especiais

2 variáveis $\boxed{I_x = \iint_E y^2 . dm \quad e \quad I_y = \iint_E x^2 . dm}$ | $I_x$: M.I. em relação ao eixo x; $I_y$: M.I. " " " eixo y

3 variáveis $\boxed{I_x = \iiint_E (y^2+z^2) dm \;,\; I_y = \iiint_E (x^2+z^2) . dm \;,\; I_z = \iiint_E (x^2+y^2) . dm}$ | $I_z$: M.I. " " " eixo z

Em relação à origem

$$\boxed{I_o = \iint_E (x^2+y^2) . dm = I_x + I_y}$$

$$\boxed{I_o = \iiint_E (x^2+y^2+z^2) . dm = \frac{I_x + I_y + I_z}{2}}$$

Demonstrações nos Exercícios (13) e (18)

## • Substituições de Variáveis

{Fórmula na pág. seguinte}

– Coordenadas quaisquer

$E = \{(x,y,z) \in \mathbb{R}^2 \mid$ " Descrição do Lugar Geométrico em $x,y,z$"$\}$

$E_{u,v,w}$ $\{(u,v,w) \in \mathbb{R}^2 \mid$ " Nova descrição do L.G., mais simples que a antiga"$\}$

1º Determina-se as relações DIRETAS de $x, y, z$ com $u, v, w$

$$\begin{cases} u(x,y,z) = \text{"alguma função } f(x,y,z)\text{"} \\ v(x,y,z) = \text{" " " "} \\ w(x,y,z) = \text{" " " "} \end{cases} \Longleftrightarrow \begin{cases} x(u,v,w) = \text{" } g(u,v,w)\text{"} \\ y(u,v,w) = \text{" " } \\ z(u,v,w) = \text{" " } \end{cases}$$

Exemplo: exercício (33)

2º Determinado $x(u,v,w)$, $y(u,v,w)$ e $z(u,v,w)$, determina-se o Jacobiano, utilizando-se os gradientes de $x, y, z$

Obs: Lembrando que o gradiente é um vetor, logo um vetor de vetores é, matematicamente, uma matriz.

$$J = \begin{bmatrix} \nabla x(u,v,w) \\ \nabla y(u,v,w) \\ \nabla z(u,v,w) \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Vale o mesmo algoritmo para duas variáveis

(3)

3º Determinando o determinante jacobiano

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det J$ Irá utilizar o módulo disto, ok!?

4º Fazendo-se a substituição

$$\iiint_E F(x,y,z)\,dxdydz = \iiint_{E_{u,v,w}} F(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

Fórmula suprema da substituição

- Coordenadas Polares (Demonstração no exercício (9))

para problemas no $\mathbb{R}2$ que envolvam circunferências como lugar geométrico

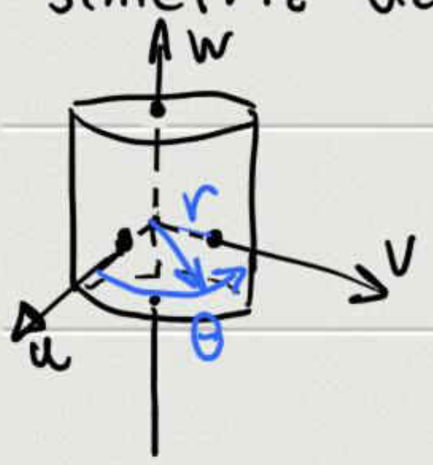
$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \end{cases} ; \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

tal que $r \in \mathbb{R}_+$ e $\theta \in [0, 2\pi]$

- Coordenadas Cilíndricas (Demonstração Ex

para problemas que possam ser envolvidos por um cilíndro (aborda qualquer cônica no geral; pode ser mais fácil que esféricas mesmo se envolvendo esferos no problema.)

Para eixo axial, escolhe-se o eixo que é paralelo ao de simetria das cônicas:



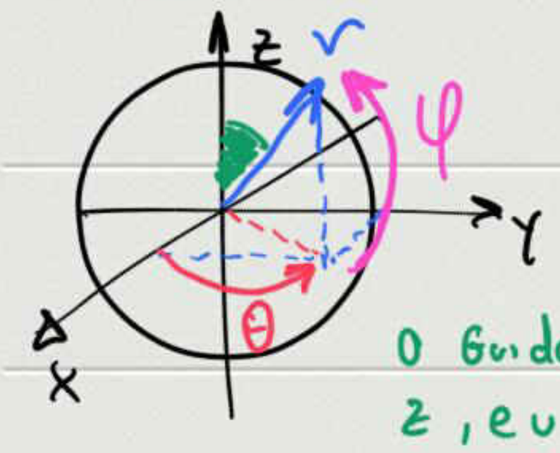

$$\begin{cases} u = r.\cos\theta \\ v = r.\operatorname{sen}\theta \\ w = w \end{cases} \qquad \left|\frac{\partial(u,v,w)}{\partial(r,\theta,w)}\right| = r$$

tal que $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi]$ e $w \in \mathbb{R}$

- Coordenadas Esféricas (Demonstração Ex. (22))

para problemas envelvendo cônicas, de preferência, esferos.

Antes que alguém de um nó na cabeça, eu uso uma referência de sentido para $\varphi$ diferente do convencional, mas tudo dá no mesmo no final, é só um modelamento matemático diferente (o dos livros russos).




$$\begin{cases} x = r.\cos\varphi.\cos\theta \\ y = r.\cos\varphi.\operatorname{sen}\theta \\ z = r.\operatorname{sen}\varphi \end{cases} \qquad \left|\frac{\partial(u,v,w)}{\partial(r,\theta,\varphi)}\right| = r^2.\cos\varphi$$

tal que $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi]$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

O Guidorizzi põe a referência do $\varphi$ começando no eixo z, eu coloco no plano xy

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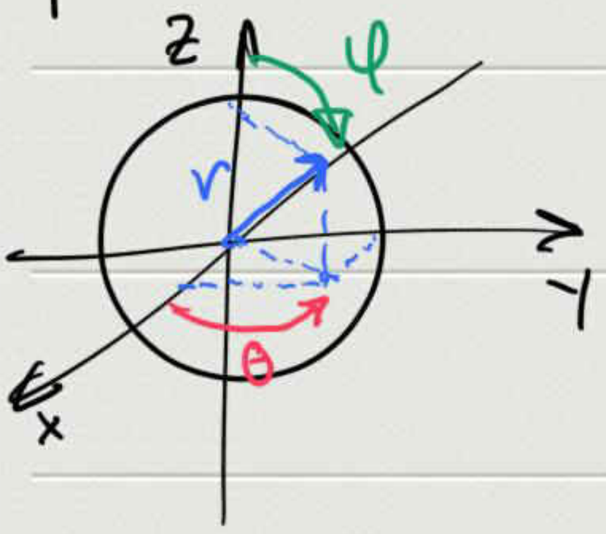
ou se preferir pelo do Guidorizzi (só se lembre que $\varphi$ começa no eixo z e roda no sentido horário)




$$\begin{cases} x = r.\operatorname{sen}\varphi.\cos\theta \\ y = r.\operatorname{sen}\varphi.\operatorname{sen}\theta \\ z = r.\cos\varphi \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2.\operatorname{sen}\varphi$$

$r \in \mathbb{R}_+$, $\theta \in [0, 2\pi]$

$\varphi \in [0, \pi]$

Lembrando que todos os exercícios com esféricas eu fiz do outro jeito

## • Integrais de Linha

Seja $\gamma(t) \in \mathbb{R}^n \mid t \in [a,b]$ $(n = 2$ ou $3)$

e $\vec{F}$, um campo vetorial contínuo, $\gamma'(t) = \frac{d}{dt}\gamma(t)$

$$\int_\gamma \vec{F}.ds = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t).dt$$

Acho que essa nem cai na prova, afinal, não se teve campos vetoriais ainda

## • Integral de Linha relativa ao Comprimento de Arco

Seja $\gamma : [a,b] \to \mathbb{R}^{3 \text{ ou } 2}$, $\gamma \in \mathcal{C}^1$

$$\int_\gamma f(x,y,z).ds = \int_a^b f(\gamma(t)) \,\|\gamma'(t)\|.dt$$

onde $ds = \|\gamma'(t)\|.dt$ → ds é uma parcelinha do comprimento total do arco

O resto é técnica, vá treinar, Cafe.

(5)

(1) a) $\iint_R (2y^2 - 3xy^3)\,dx\,dy = \int_0^3 \left( \int_1^2 (2y^2 - 3xy^3)\,dx \right) . dy$

$= \int_0^3 \left[ 2xy^2 - \frac{3}{2}x^2y^3 + k \right]_1^2 . dy = \int_0^3 \left( 2y^2 - \frac{9}{2}y^3 \right) . dy = \left[ \frac{2}{3}y^3 - \frac{9}{8}y^4 + k \right]_0^3$

$= \frac{18.8 - 9.3^4}{8} = \frac{144 - 729}{8} = -\frac{585}{8}$

b) $\iint_R x . \operatorname{sen} y . dx\,dy \quad | \quad R = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 4 \text{ e } 0 \le y \le \frac{\pi}{6}\}$

$\int_0^{\pi/6} \left( \int_1^4 x \operatorname{sen} y . dx \right) . dy = \int_0^{\pi/6} \left[ \frac{x^2}{2} \operatorname{sen} y + k \right]_1^4 dy = \int_0^{\pi/6} \frac{15}{2} \operatorname{sen} y . dy =$

$\left[ -\frac{15}{2} \cos y + k \right]_0^{\pi/6} = -\frac{15}{2} \left( \frac{\sqrt{3}}{2} - 1 \right) = \frac{15}{4} (2 - \sqrt{3})$

(2) $z = x\sqrt{x^2+y}$, $0 \le x \le 1$ e $0 \le y \le 1$, análogo ao cálculo I, se não há função (abaixo de $f(x,y) = x\sqrt{x^2+y}$) que limite z, o cálculo da integral automaticamente já se limita ao plano $z = 0$. Portanto, o volume é dado por:

$$V = \int_0^1 \left( \int_0^1 x\sqrt{x^2+y}\,.dx \right).dy$$

$\int x\sqrt{x^2+y}\,.dx \; ; \quad x^2+y = a \Leftrightarrow 2x.dx = da \Leftrightarrow \boxed{dx = \frac{da}{2x}}$

$\int x\sqrt{a} \,.\frac{da}{2x} = \int \frac{a^{1/2}}{2}.da = \frac{2}{3}\frac{a^{3/2}}{2} + K = \frac{\sqrt{(x^2+y)^3}}{3} + K$

$V = \int_0^1 \left( \int_0^1 x\sqrt{x^2+y}\,.dx \right).dy = \int_0^1 \left[ \frac{(x^2+y)^{3/2}}{3} + K \right]_0^1 . dy$

$V = \int_0^1 \left( \frac{(y+1)^{3/2}}{3} - \frac{y^{3/2}}{3} \right).dy = \int_0^1 \frac{(y+1)^{3/2}}{3}.dy - \int_0^1 \frac{y^{3/2}}{3}.dy$

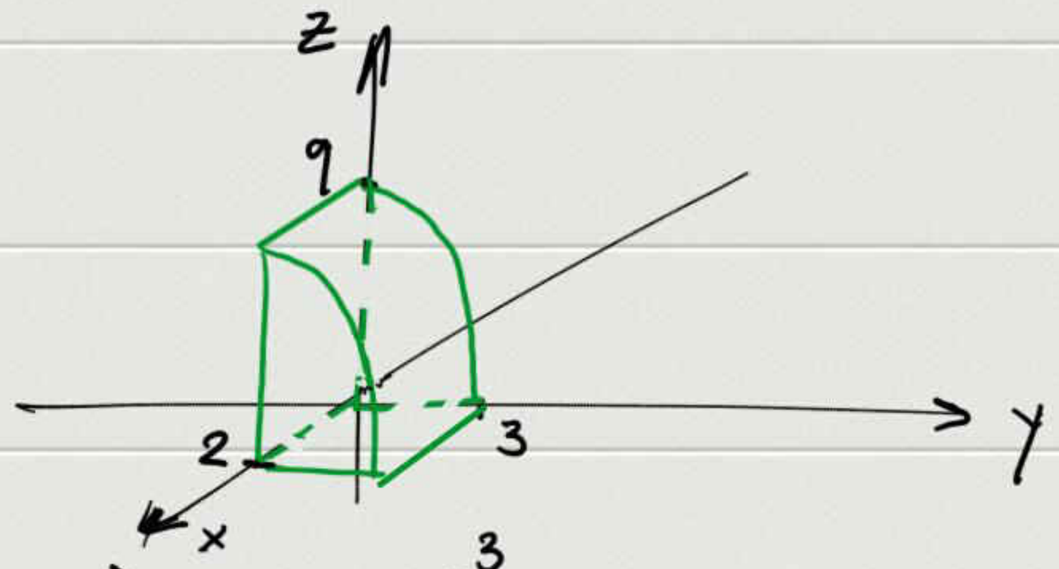
$\int \frac{(y+1)^{3/2}}{3}.dy \; ; \quad b = y+1 \Leftrightarrow db = dy \; ; \; \int \frac{b^{3/2}}{3}.db = \frac{2}{5}.\frac{b^{5/2}}{3} + k =$

$\frac{2}{15}(y+1)^{5/2} + k$

$V = \left[ \frac{2}{15}(y+1)^{5/2} + k \right]_0^1 - \left[ \frac{2}{15} y^{5/2} + k \right]_0^1 = \frac{4}{15}(2\sqrt{2} - 1)$

(6)

(3) $z = 9 - y^2$




$0 \le x \le 2$ e $0 \le y \le 3$

$$V = \int_0^3 \left( \int_0^2 9 - y^2 \, dx \right) dy = \int_0^3 \left[ (9 - y^2)x + K \right]_0^2 dy = \int_0^3 2(9 - y^2)\, dy$$

$$V = \left[ 2\left(9y - \frac{1}{3}y^3\right) \right]_0^3 = 2\left(9 \cdot 3 - \frac{1}{3} \cdot 3^3\right) = 36$$

(4) $$\int_0^1 \int_0^1 \frac{x - y}{(x+y)^3} \, dy \, dx = \int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3} \, dy \right) dx$$

$$\int \frac{x-y}{(x+y)^3} \, dy = \int \frac{x}{(x+y)^3} \, dy - \int \frac{y}{(x+y)^3} \, dy ; \quad a = x + y \iff da = dy$$

$$\int \frac{x}{a^3} \, da = -\frac{1}{2} \frac{x}{a^2} + K = -\frac{1}{2} \frac{x}{(x+y)^2} + K$$

$$\int \frac{y}{(x+y)^3} \, dy = \int \frac{a - x}{a^3} \, da = \int \frac{1}{a^2} \, da - \int \frac{x}{a^3} \, da = -\frac{1}{x+y} + \frac{1}{2} \frac{x}{(x+y)^2} + K$$

$$\therefore \int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3} \, dy \right) dx = \int_0^1 \left[ \frac{1}{x+y} - \frac{x}{(x+y)^2} + K \right]_0^1 dx = \int_0^1 \left( \frac{1}{1+x} - \frac{x}{(x+1)^2} \right) dx$$

$$= \int_0^1 \frac{1}{1+x} \, dx - \int_0^1 \frac{x}{(x+1)^2} \, dx ; \quad b = x + 1 \iff db = dx ; \int \frac{x}{(x+1)^2} \, dx = \int \frac{b-1}{b^2} \, db$$

$$= \int \frac{1}{b} \, db - \int \frac{1}{b^2} \, db = \ln|b| + \frac{1}{b} + K = \ln|x+1| + \frac{1}{x+1} + K$$

$$\therefore \int_0^1 \frac{1}{1+x} \, dx - \int_0^1 \frac{x}{(x+1)^2} \, dx = \left[ \ln|x+1| - \left( \ln|x+1| + \frac{1}{x+1} + K \right) \right]_0^1 = -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

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$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy = \int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3}\,dx \right) dy \quad ; \quad \begin{matrix} a = x+y \\ da = dx \end{matrix}$$

$$\int \frac{x-y}{(x+y)^3}\,dx = \int \frac{a-y-y}{a^3}\,da = \int \frac{1}{a^2}\,da - 2\int \frac{y}{a^3}\,da = -\frac{1}{a} + \frac{y}{a^2} + K$$

$$= -\frac{1}{x+y} + \frac{y}{(x+y)^2} + K$$

$$\int_0^1 \left( \int_0^1 \frac{x-y}{(x+y)^3}\,dx \right) dy = \int_0^1 \left[ -\frac{1}{x+y} + \frac{y}{(x+y)^2} + K \right]_0^1 dy = \int_0^1 \left( -\frac{1}{1+y} + \frac{y}{(1+y)^2} \right) dy$$

$$\int \frac{y}{(1+y)^2}\,dy = \int \frac{b-1}{b^2}\,db = \int \left( \frac{1}{b} - \frac{1}{b^2} \right) db = \ln|b| + \frac{1}{b} + K =$$

$$\ln|y+1| + \frac{1}{y+1} + K$$

$$\therefore \int_0^1 \left( \frac{-1}{1+y} + \frac{y}{(1+y)^2} \right) dy = \left[ -\ln|1+y| + \ln|1+y| + \frac{1}{y+1} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

Portanto, a função não é integrável porque não é contínua num dado lugar geométrico do domínio $x = -y$

5) a) $$\int_0^1 \left( \int_{x^2}^{\sqrt{x}} xy\,dy \right) dx = \int_0^1 \left[ x\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[ \frac{x|x|}{2} - \frac{xx^4}{2} \right] dx$$

$$= \int_0^1 \left( \frac{x^2}{2} - \frac{x^5}{2} \right) dx = \left[ \frac{x^3}{6} - \frac{x^6}{12} \right]_0^1 = \frac{1}{12}$$

(8)

c) $\iint_D e^{x/y} \cdot dx \cdot dy \qquad D = \{(x,y) \mid 1 \le y \le 2,\ y \le x \le y^3\}$

$$\int_1^2 \int_y^{y^3} e^{x/y} \cdot dx \cdot dy = \int_1^2 \left[ y \cdot e^{x/y} + k \right]_y^{y^3} \cdot dy = \int_1^2 \left( y \cdot e^{y^2} - y \cdot e \right) \cdot dy$$

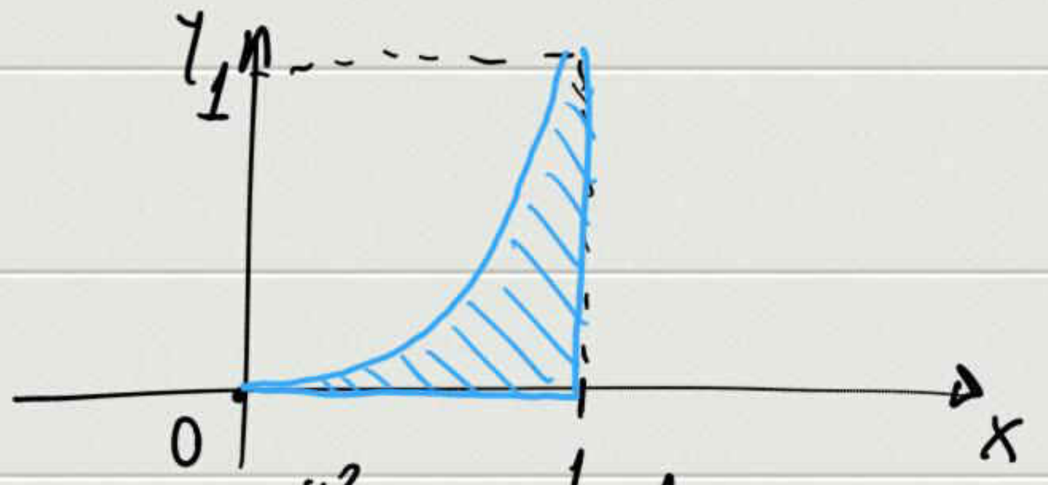
$$\int y e^{y^2} \cdot dy \ ; \quad \boxed{u = y^2} \Leftrightarrow \frac{du}{dy} = 2y \Leftrightarrow \boxed{dy = \frac{du}{2y}}$$

$$\int y \cdot e^{y^2} \cdot dy = \int y \cdot e^u \cdot \frac{du}{2y} = \int \frac{e^u \, du}{2} = \frac{e^u}{2} + k = \frac{e^{y^2}}{2} + k$$

$$\therefore \int_1^2 \left( y \cdot e^{y^2} - y \cdot e \right) dy = \left[ \frac{e^{y^2}}{2} - \frac{y^2}{2} \cdot e + k \right]_1^2 = \frac{e^4}{2} - \frac{e}{2} - \frac{4}{2} e + \frac{1}{2} e$$

$$\therefore \boxed{\iint_D e^{x/y} \cdot dx \cdot dy = \frac{1}{2}\left( e^4 - 4e \right)}$$

d) $\iint_D x \cdot \cos y \cdot dx \cdot dy \qquad D:$




$$\int_0^1 \int_0^{x^2} x \cdot \cos y \cdot dy \cdot dx = \int_0^1 \left[ x \cdot \operatorname{sen} y + k \right]_0^{x^2} \cdot dx = \int_0^1 x \cdot \operatorname{sen} x^2 \cdot dx$$

$$\boxed{u = \operatorname{sen} x^2} \Leftrightarrow \frac{du}{dx} = \cos x^2 \cdot 2x \Leftrightarrow \boxed{dx = \frac{du}{2x \cdot \sqrt{1-u^2}}}$$

$$\int x \cdot \operatorname{sen} x^2 \cdot dx = \int x \cdot u \cdot \frac{du}{2x \cdot \sqrt{1-u^2}} = \frac{1}{2} \int \frac{u}{\sqrt{1-u^2}} \cdot du \ ; \quad \boxed{z = 1-u^2} \Leftrightarrow \frac{dz}{du} = -2u$$

$$\Leftrightarrow \boxed{du = \frac{dz}{-2u}} \quad \therefore \boxed{\int \frac{u}{\sqrt{1-u^2}} \cdot du} = \int \frac{u}{z^{1/2}} \cdot \frac{dz}{-2u} = -\frac{1}{2} \cdot 2 \, z^{1/2} + k =$$

$$-\sqrt{1-u^2} + k = -\sqrt{1 - \operatorname{sen}^2 x^2} + k = -\cos x^2 + k$$

$$\therefore \boxed{\int x \cdot \operatorname{sen} x^2 \cdot dx = -\frac{1}{2} \cos x^2 + k} \qquad \therefore \int_0^1 x \operatorname{sen} x^2 dx = \left[ -\frac{1}{2} \cos x^2 + k \right]_0^1 \Leftrightarrow$$

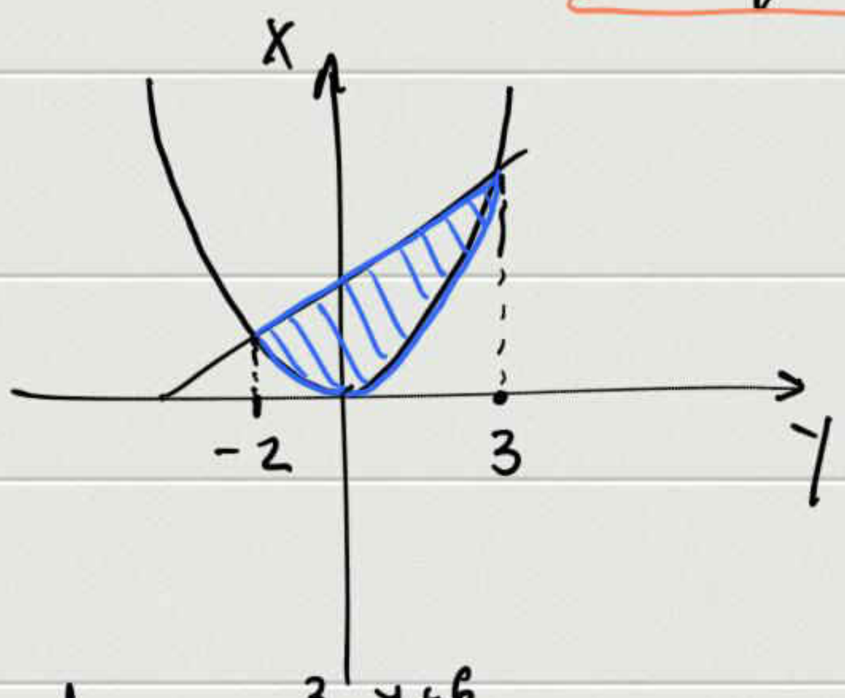
$$\boxed{\iint_D x \cdot \cos y \cdot dx \, dy = \frac{1}{2}\left( 1 - \cos 1 \right)}$$

(9)

e) $\iint_D 4y^3 . dx . dy$ $\quad y = x - 6 \quad ; \quad y^2 = x$

$x = y + 6$

D:




$y^2 = y + 6$

$y^2 - y - y - 2 = 0$

$(y+2)(y-2) - (y+2) =$

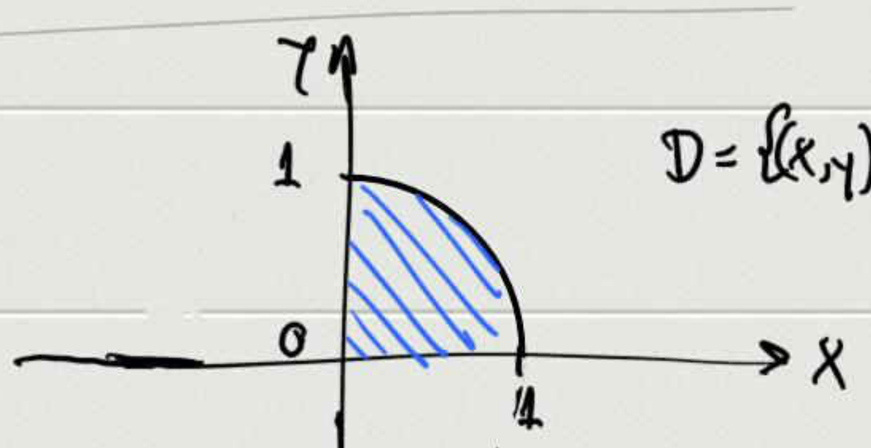
$(y+2)(y-2-1) = 0$

$$\iint_D 4y^3 . dx . dy = \int_{-2}^{3} \int_{y^2}^{y+6} 4y^3 . dx . dy = \int_{-2}^{3} [4y^3 . x + k]_{y^2}^{y+6} . dy$$

$$= 4 \int_{-2}^{3} (y^4 + 6y^3 - y^5) . dy = 4 \left[ \frac{1}{5}y^5 + \frac{6}{4}y^4 - \frac{1}{6}y^6 + k \right]_{-2}^{3}$$

$$= 4 \left( \frac{3^5 + 2^5}{5} + \frac{6}{4}(3^4 - 2^4) - \frac{1}{6}(3^6 - 2^6) \right)$$

f) $\iint_D xy . dx dy$

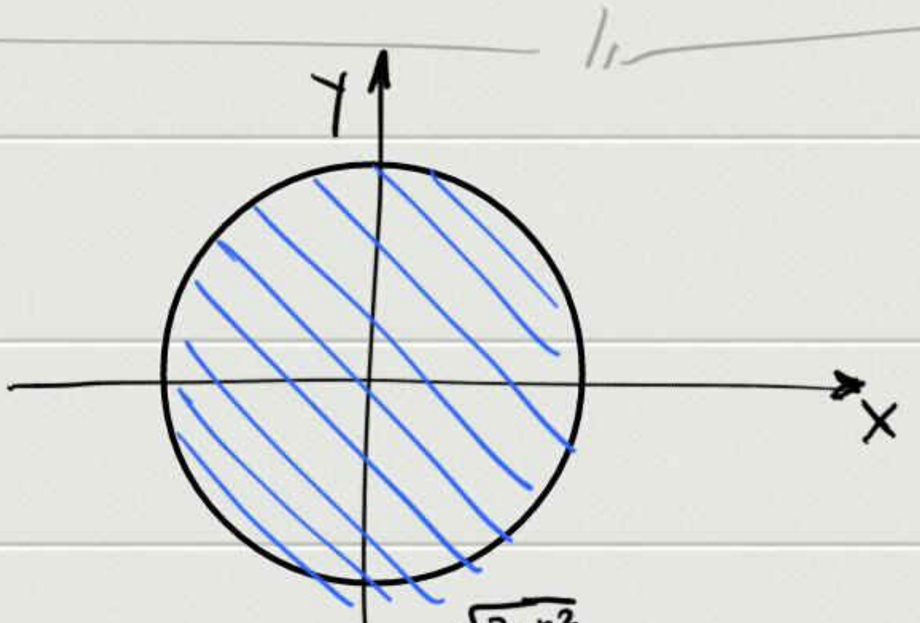



$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \sqrt{1-x^2}$ e $0 \le x \le 1\}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy . dx dy = \int_0^1 \left[ \frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} . dx = \int_0^1 x \frac{|1-x^2|}{2} dx =$$

$$\int_0^1 x \frac{(1-x^2)}{2} . dx = \int_0^1 \frac{(x - x^3)}{2} . dx = \frac{1}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8}$$

g) $\iint_D (x^2 \operatorname{tg} x + y^3 + 4) . dx . dy$




$D = \{(x,y) \mid x^2 + y^2 \le 2\}$

$-\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$ e $-\sqrt{2} \le x \le \sqrt{2}$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (x^2 . \operatorname{tg} x + y^3 + 4) . dy . dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left[ \frac{1}{4}y^4 + y(x^2 . \operatorname{tg} x + 4) \right]_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} . dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left( \frac{1}{4}(2-x^2)^2 - \frac{1}{4}(2-x^2)^2 + \sqrt{2-x^2}(x^2 . \operatorname{tg} x + 4) - (-\sqrt{2-x^2})(x^2 . \operatorname{tg} x + 4) \right) dx = \int_{-\sqrt{2}}^{\sqrt{2}} 2\sqrt{2-x^2}(x^2 . \operatorname{tg} x + 4) \, dx$$

$$= \int_{-\sqrt{2}}^{1} 2\sqrt{2-x^2} . x^2 \operatorname{tg} x . dx + \int_{-1}^{1} 8\sqrt{2-x^2} . dx \ \ldots$$

(10)

$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\cdot x^2 \cdot \operatorname{tg} x \cdot dx$ → essa integral não sai da maneira convencional, e fique claro que ela não dá zero, se fosse assim, $\int_{-\sqrt{2}}^{\sqrt{2}} 8\sqrt{2-x^2}\cdot dx$ dava a resposta, mas não dá, só usando umas fatorações e uns métodos russos absurdos pra resolver esse exercício do jeito que eu o comecei, nem vale apena mostrar. Comece de novo quando for assim, mudando o jeito de descrever o lugar geométrico → age com consciência, a sacanagem está no termo $x^2 \cdot \operatorname{tg} x$

$$-\sqrt{2} \le y \le \sqrt{2} \quad \text{e} \quad -\sqrt{2-y^2} \le x \le \sqrt{2-y^2}$$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} (x^2 \cdot \operatorname{tg} x + y^3 + 4)\cdot dx\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left(2\sqrt{2-y^2}\cdot y^3 + 8\sqrt{2-y^2}\right)\cdot dy$$

$$\int_{-\sqrt{2}}^{\sqrt{2}} 2\sqrt{2}\sqrt{1-\frac{y^2}{2}}\cdot y^3 \cdot dy + 8\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2}\sqrt{1-\frac{y^2}{2}}\cdot dy$$

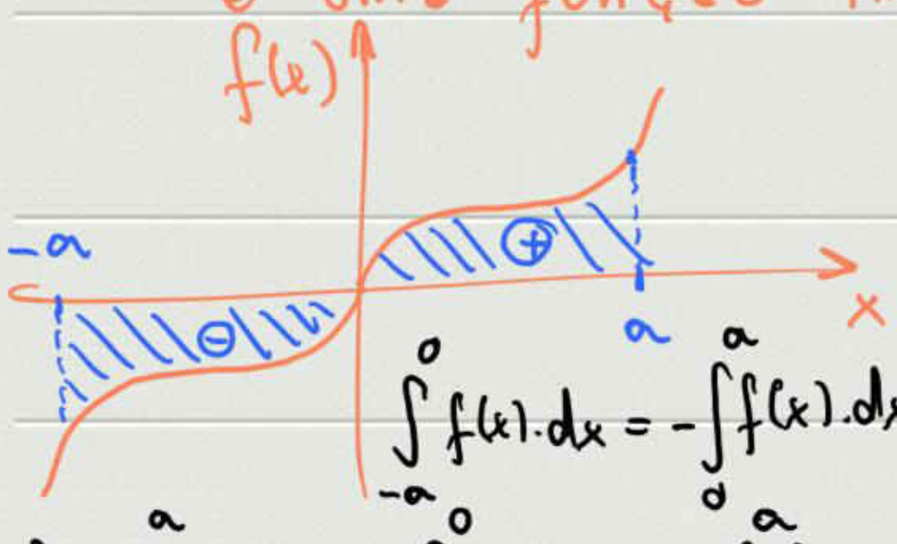
$$\operatorname{sen} t = \frac{y}{\sqrt{2}} \Leftrightarrow \cos t \cdot dt = \frac{dy}{\sqrt{2}}$$

$$2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \operatorname{sen}^3 t \cdot \sqrt{2}\cdot \cos t \cdot dt + 8\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \sqrt{2}\cos t \cdot dt$$

$$4\cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sen}^3 t \cdot \cos^2 t \cdot dt + 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \cdot dt =$$

$$0 + 16\left(\left[\frac{\operatorname{sen} t \cdot \cos t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot dt\right) = 16\cdot\frac{\pi}{2} = 8\pi$$

$\int_{-a}^{a} x^2 \cdot \operatorname{tg} x \cdot dx = 0 \mid a \in \mathbb{R}$ é uma função ímpar

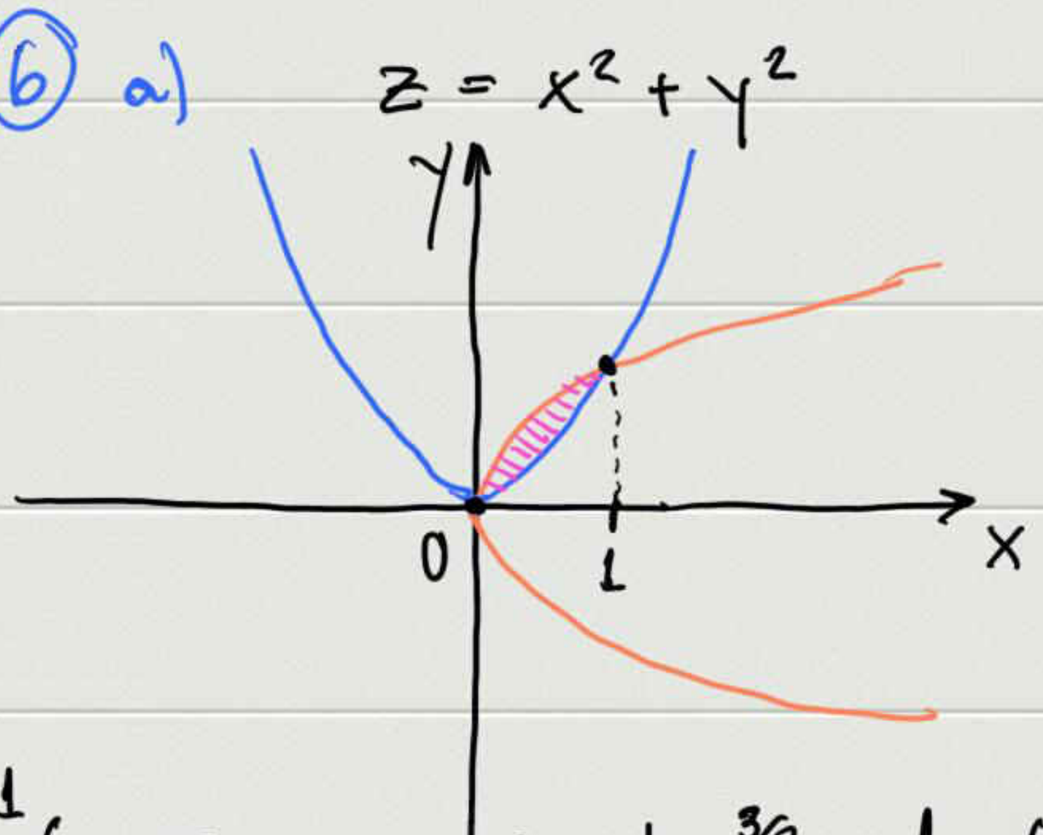



$$\int_{-a}^{0} f(x)\cdot dx = -\int_{0}^{a} f(x)\cdot dx$$

$$\therefore \int_{0}^{a} f(x)\cdot dx + \int_{-a}^{0} f(x)\cdot dx = \int_{-a}^{a} f(x)\cdot dx = 0$$

O bizu: fazer $x^2 \operatorname{tg} x$ sumir o quanto antes, pois $\sqrt{2-x^2}\cdot x^2 \operatorname{tg} x$ não é ímpar nem par.

6) a)




$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ y = \pm\sqrt{x} \end{cases}$$

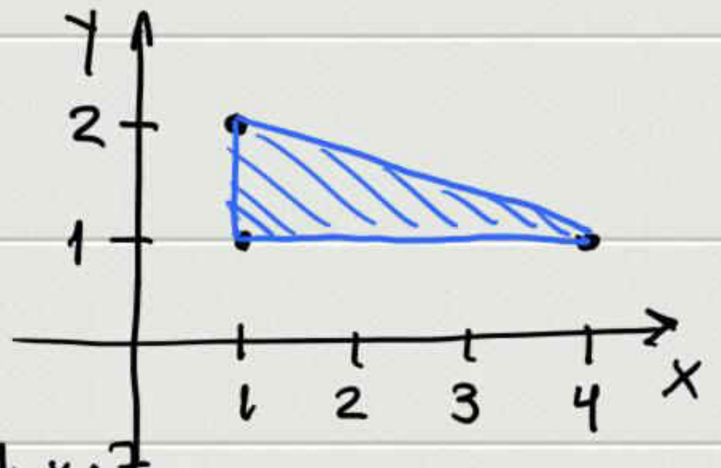
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$$

$$V = \int_0^1 \left[ x^2 y + \frac{1}{3} y^3 \right]_{x^2}^{\sqrt{x}} dx =$$

$$\int_0^1 \left( x^2\sqrt{x} - x^2 x^2 + \frac{1}{3} x^{3/2} - \frac{1}{3} x^6 \right) dx = \left[ \frac{2}{7} x^{7/2} - \frac{1}{5} x^5 + \frac{2}{5}\frac{1}{3} x^{5/2} - \frac{1}{7}\frac{1}{3} x^7 \right]_0^1$$

$$= \frac{2}{7} - \frac{1}{5} + \frac{2}{15} - \frac{1}{21} = \frac{30 - 21 + 14 - 5}{7 \cdot 3 \cdot 5} = \frac{18}{7 \cdot 3 \cdot 5} = \frac{6}{35}$$

b) $z = x \cdot y$




$$\begin{cases} 1 \le y \le -\frac{1}{3}x + \frac{7}{3} \\ 1 \le x \le 4 \end{cases} \quad \left| \begin{array}{l} -\frac{1}{3}x + b = 1 \\ -\frac{4}{3} + b = 1 \end{array} \right| \quad b = \frac{7}{3}$$

$$V = \int_1^4 \int_1^{-\frac{1}{3}x + \frac{7}{3}} xy\, dy\, dx = \int_1^4 \left[ \frac{xy^2}{2} \right]_1^{-\frac{1}{3}x + \frac{7}{3}} dx = \frac{1}{2} \int_1^4 \left( x\left(-\frac{1}{3}x + \frac{7}{3}\right)^2 - x \right) dx$$

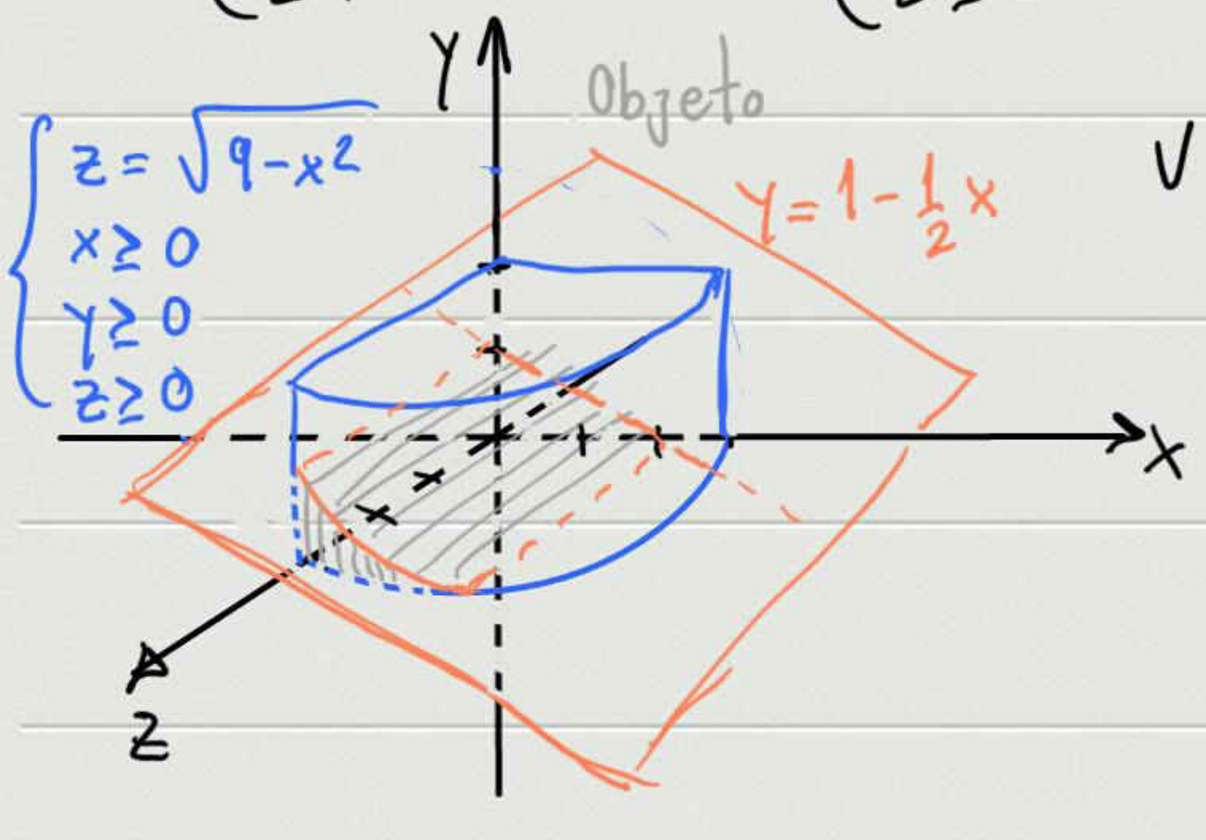
$$V = \boxed{\frac{1}{18}} \int_1^4 \left( x\left(x^2 - 14x + 49\right) - \boxed{9}x \right) dx = \frac{1}{18} \int_1^4 \left( x^3 - 14x^2 + 40x \right) dx$$

$$V = \frac{1}{18}\left[ \frac{x^4}{4} - \frac{14}{3}x^3 + \frac{40}{2}x^2 \right]_1^4 = \frac{1}{18}\left( \left( \frac{256 - 1}{4} \right) - \frac{14}{3}(64 - 1) + \frac{40}{2}(16 - 1) \right)$$

$$\boxed{V = \frac{31}{8}}$$

c)

$$\begin{cases} x^2 + z^2 = 9 \\ x + 2y = 2 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases} \Leftrightarrow \begin{cases} z^2 = 9 - x^2 \\ y = 1 - \frac{1}{2}x \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases} \Leftrightarrow \begin{cases} z = \pm\sqrt{9 - x^2} \\ y = 1 - \frac{1}{2}x \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases}$$




$$V = \int_0^2 \int_0^{\sqrt{9-x^2}} \left(1 - \frac{1}{2}x\right) dz\, dx$$

$$V = \int_0^2 \left[ z\left(1 - \frac{1}{2}x\right) \right]_0^{\sqrt{9-x^2}} dx$$

$$V = \int_0^2 \sqrt{9 - x^2}\left(1 - \frac{1}{2}x\right) dx$$

(12)

$$V = \int_0^2 \sqrt{9-x^2}\,.dx - \frac{1}{2}\int_0^2 x.\sqrt{9-x^2}.dx = \int_0^{\arcsen 2/3} 9.\cos^2\theta.d\theta - \frac{1}{2}\int_9^5 \frac{u^{1/2}}{-2}.du$$

$$= 9.\int_0^{\arcsen 2/3} \frac{2\cos^2\theta - 1 + 1}{2}.d\theta + \frac{1}{4}.\frac{2}{3}\left[u^{3/2}\right]_9^5 = \frac{9}{2}.\int_0^{\arcsen 2/3} \cos 2\theta.d\theta + \frac{9}{2}\int_0^{\arcsen 2/3} 1.d\theta +$$

$$\frac{1}{6}(5\sqrt{5} - 27) = \frac{9}{4}.\left[\sen 2\theta\right]_0^{\arcsen 2/3} + \frac{9}{2}\arcsen\frac{2}{3} + \frac{1}{6}(5\sqrt{5}-27)$$

$$= \frac{9}{4}\left[2.\sen\theta.\cos\theta\right]_0^{\arcsen 2/3} + \frac{9}{2}\arcsen\frac{2}{3} + \frac{1}{6}(5\sqrt{5}-27) = \sqrt{5} + \frac{9}{2}\arcsen\frac{2}{3} + \frac{1}{6}\ldots$$

$$\therefore \boxed{V = \frac{1}{6}(11\sqrt{5}-27) + \frac{9}{2}\arcsen\left(\frac{2}{3}\right)}$$

Outra maneira de interpretar

$$V = \int_0^2\int_0^{1-\frac{1}{2}x} \sqrt{9-x^2}.dy\,dx = \int_0^2 \left[y.\sqrt{9-x^2}\right]_0^{1-\frac{1}{2}x}.dx = \int_0^2 \sqrt{9-x^2}\,(1-\tfrac{1}{2}x).dx$$

d)

$$\begin{cases} x+y+z=1 \\ x\geq 0 \\ y\geq 0 \\ z\geq 0 \end{cases} \Leftrightarrow \begin{cases} z = 1-x-y \\ x\geq 0 \\ y\geq 0 \\ z\geq 0 \end{cases} \Leftrightarrow \begin{cases} z = 1-(x+y) \\ x\geq 0 \\ y\geq 0 \\ z\geq 0 \end{cases} \Leftrightarrow \begin{cases} z = 1-(x+y) \\ 1-(x+y)\geq 0 \\ x\geq 0 \\ y\geq 0 \end{cases} \Leftrightarrow$$

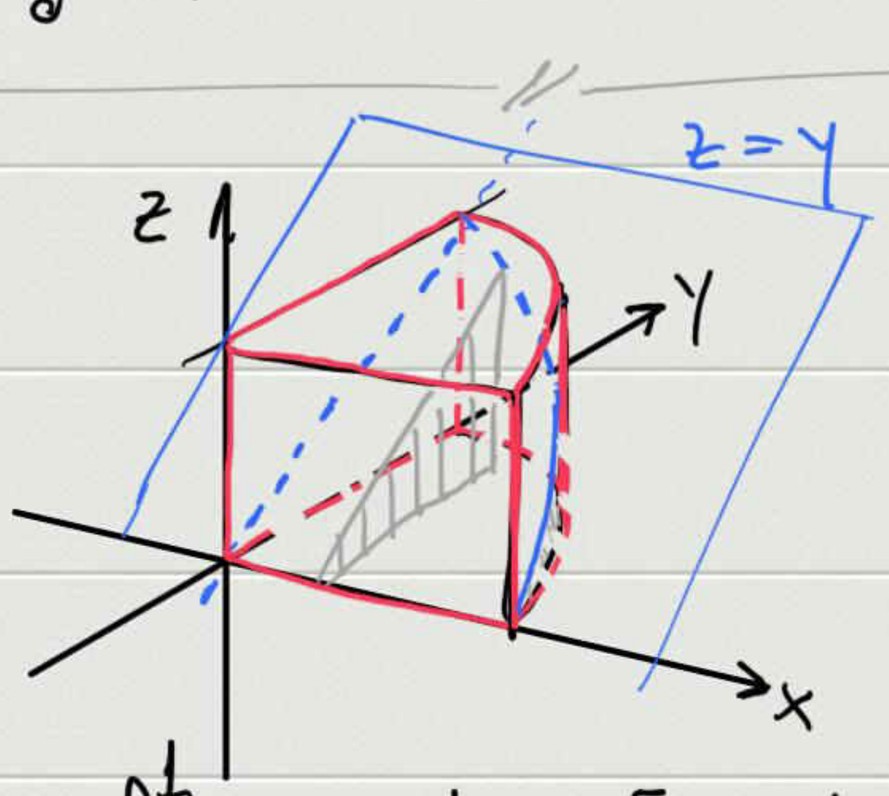
$$\begin{cases} z = 1-(x+y) \\ x \leq 1-y \\ x\geq 0 \\ y\geq 0 \end{cases} \quad \therefore D = \{(x,y)\in\mathbb{R}^2 \mid 0\leq x\leq 1-y \text{ e } 0\leq y\leq 1\}$$

$$V = \int_0^1\int_0^{1-y}(1-(x+y)).dx.dy = \int_0^1 \left[x - \frac{x^2}{2} - xy\right]_0^{1-y}.dy$$

$$V = \int_0^1 \left(1-y - \frac{(1-y)^2}{2} - y(1-y)\right).dy = \int_0^1 \left(1-y-\frac{1-2y+y^2}{2} - y + y^2\right).dy$$

$$V = \int_0^1 \left(\frac{1}{2} - y + \frac{y^2}{2}\right).dy = \left[\frac{1}{2}y - \frac{y^2}{2} + \frac{y^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

e) Objeto: uma espécie de pirâmide cilíndrica




$$V = \int_0^1\int_0^{\sqrt{1-x^2}} y\ .dy.dx$$

$$V = \int_0^1 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}}.dx = \int_0^1 \left[\frac{1-x^2}{2}\right].dx$$

$$V = \int_0^1 \frac{(1-x^2)}{2}.dx = \frac{1}{2}\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2}\left(1-\frac{1}{3}\right) = \frac{1}{3}$$

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f) $S:\begin{cases} x^2+y^2=a^2 \\ y^2+z^2=a^2 \\ a>0 \end{cases} \Leftrightarrow \begin{cases} x=\pm\sqrt{a^2-y^2} \\ z=\pm\sqrt{a^2-y^2} \\ a>0 \end{cases}$ D: se $x,z\in\mathbb{R}$, então: $-a\le y\le a$

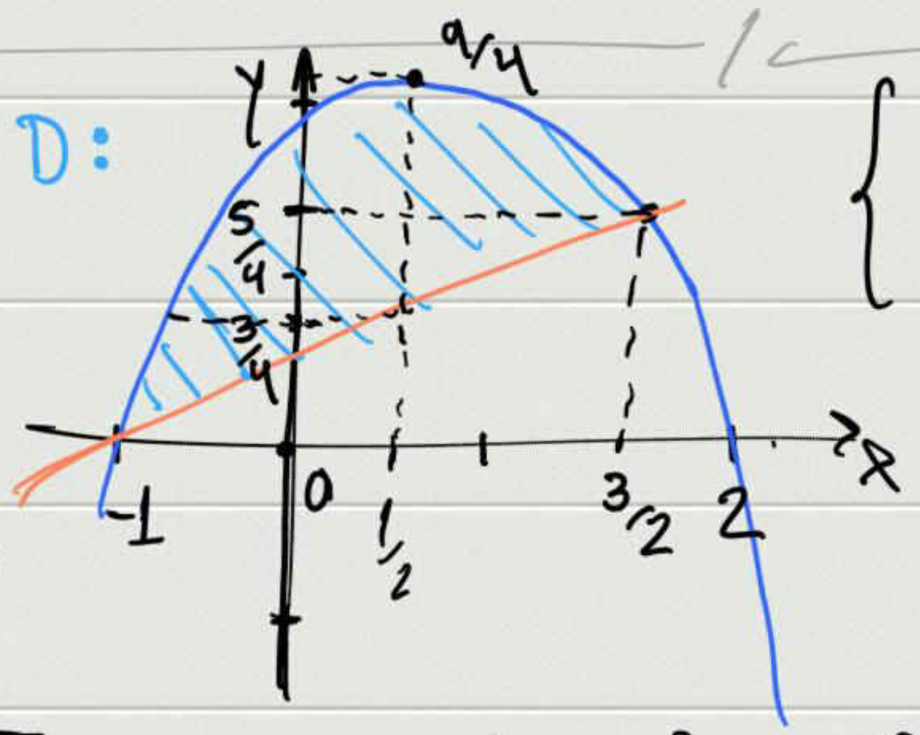
$$V=\iiint_D dx\,dy\,dz = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dz\,dx\,dy \Leftrightarrow$$

$$V=\int_{-a}^{a} 4(a^2-y^2)dy = 4a^2[y]_{-a}^{a} - 4\cdot\frac{1}{3}[y^3]_{-a}^{a} = 4a^2\,2a - \frac{4}{3}(2a^3)$$

$$V = 8a^3\left(1-\frac{1}{3}\right) = \frac{16}{3}a^3$$

//

7) $\iint_D f(x,y)\,dx\,dy$ D:




$$\begin{cases} y=-x^2+x+2 \\ x-2y+1=0 \end{cases} \Leftrightarrow$$

$x-2(-x^2+x+2)+1=0$

$x+2x^2-2x-4+1=0$

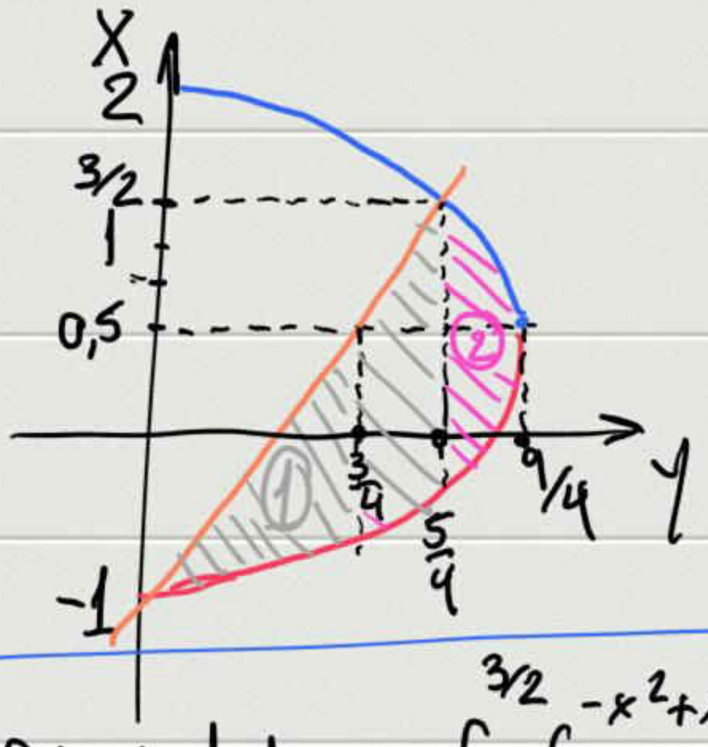
$2x^2-x-3=0$

$x=\frac{1\pm\sqrt{1+4.3.2}}{2.2}=\frac{1\pm5}{4}=\begin{cases} x_1=3/2 \\ x_2=-1 \end{cases} \Leftrightarrow \begin{cases} y_1=5/4 \\ y_2=0 \end{cases}$

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{3/2}\int_{\frac{x+1}{2}}^{-x^2+x+2} f(x,y)\,dy\,dx = \cdots$$

$y=-x^2+x+2 \Leftrightarrow y=-x^2+x-\frac{1}{4}+\frac{9}{4} \Leftrightarrow \frac{9}{4}-y=(x-1/2)^2 \Leftrightarrow$

$|x-1/2|=\sqrt{\frac{9}{4}-y} \Leftrightarrow \begin{cases} x=\sqrt{\frac{9}{4}-y}+1/2, & \text{se } x\ge 1/2 \\ x=-\sqrt{\frac{9}{4}-y}+1/2, & \text{se } x<1/2 \end{cases}$



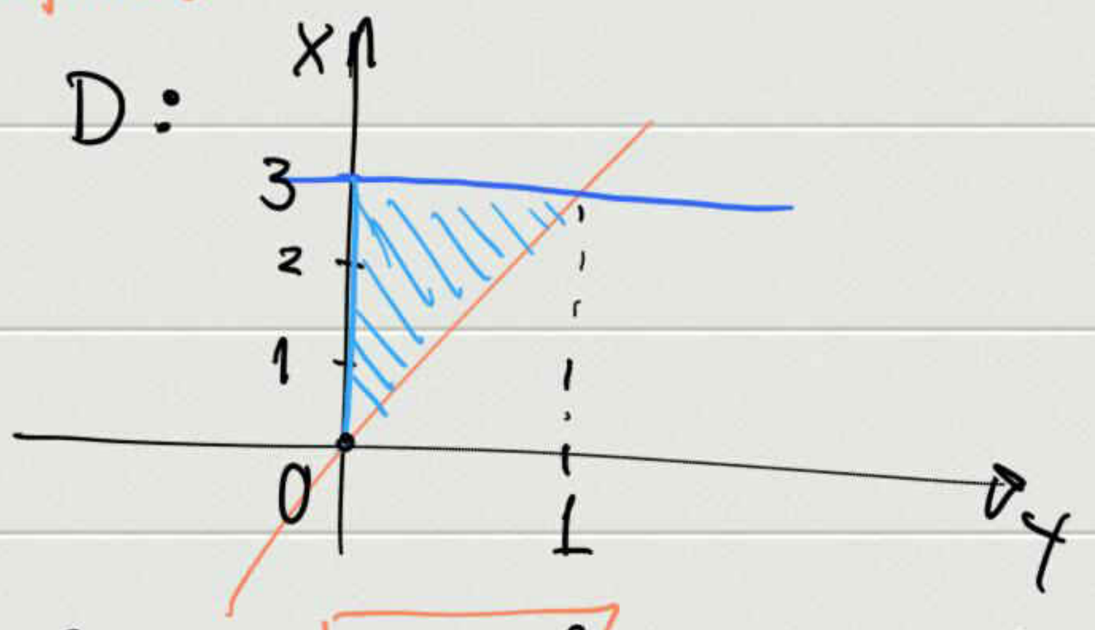

$$= \cdots \int_{0}^{5/4}\int_{-\sqrt{\frac{9}{4}-y}+1/2}^{2y-1} f(x,y)\,dx\,dy + \int_{5/4}^{9/4}\int_{-\sqrt{\frac{9}{4}-y}+1/2}^{\sqrt{\frac{9}{4}-y}+1/2} f(x,y)\,dx\,dy$$

$$\therefore \iint_D f(x,y)\,dx\,dy = \int_{-1}^{3/2}\int_{\frac{x+1}{2}}^{-x^2+x+2} f(x,y)\,dy\,dx = \int_{0}^{5/4}\int_{-\sqrt{\frac{9}{4}-y}+1/2}^{2y-1} f(x,y)\,dx\,dy + \int_{5/4}^{9/4}\int_{-\sqrt{\frac{9}{4}-y}+1/2}^{\sqrt{\frac{9}{4}-y}+1/2} f(x,y)\,dx\,dy$$

(14)

(8) a) $\int_0^1 \int_{3y}^{3} e^{x^2} \, dx \, dy$ → não existe a indefinida

D:

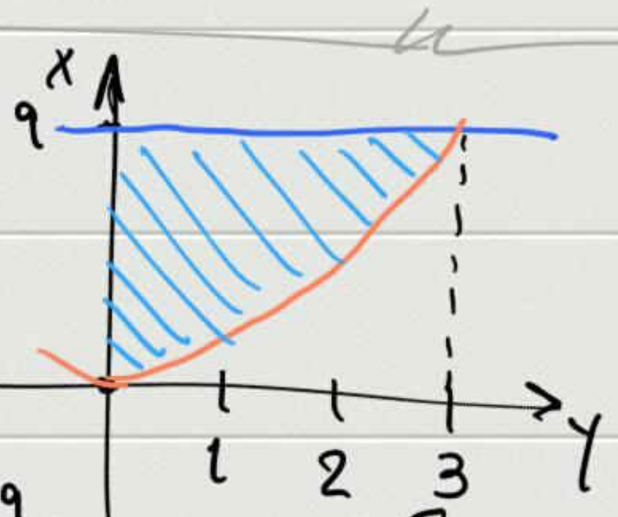



$$\int_0^1 \int_{3y}^{3} e^{x^2} \, dx \, dy = \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2} \, dy \, dx$$

$$= \int_0^3 \left[ y \cdot e^{x^2} \right]_0^{\frac{1}{3}x} \, dx = \int_0^3 \frac{1}{3} \cdot x \cdot e^{x^2} \, dx \; ; \quad \boxed{u = e^{x^2}} \Leftrightarrow \frac{du}{dx} = e^{x^2} \cdot 2x \Leftrightarrow$$

$$\boxed{dx = \frac{du}{e^{x^2} \cdot 2x}} \quad \therefore \int_0^3 \frac{1}{3} \cdot x \cdot e^{x^2} \cdot \frac{du}{e^{x^2} \cdot 2x} = \frac{1}{6} \int_0^3 du = \frac{1}{6}(e^9 - 1)$$

b) $\int_0^3 \int_{y^2}^{9} y \cdot \cos(x^2) \, dx \, dy$ D:




$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \sqrt{x} \text{ e } 0 \le x \le 9\}$$

$$\iint_D y \cdot \cos x^2 \, dx \, dy = \int_0^9 \int_0^{\sqrt{x}} y \cdot \cos x^2 \, dy \, dx = \int_0^9 \left[ \frac{y^2}{2} \cdot \cos x^2 \right]_0^{\sqrt{x}} \, dx = \int_0^9 \frac{|x|}{2} \cdot \cos x^2 \, dx$$

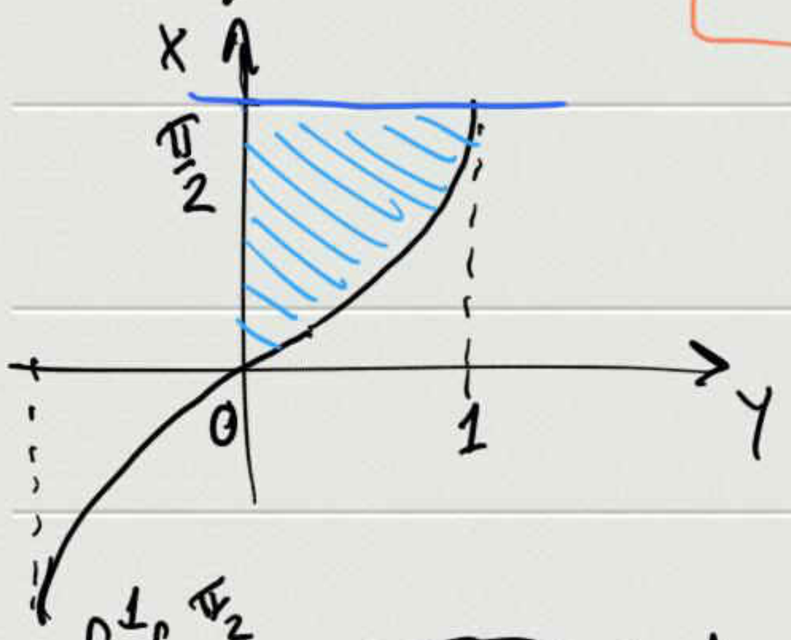
$$= \int_0^9 \frac{x}{2} \cdot \cos x^2 \, dx \; ; \quad \boxed{u = x^2} \Leftrightarrow \frac{du}{dx} = 2x \Leftrightarrow \boxed{dx = \frac{du}{2x}}$$

$$\int_0^{81} \frac{x}{2} \cdot \cos u \cdot \frac{du}{2x} = \frac{1}{4} \int_0^{81} \cos u \, du = \frac{1}{4} \left[ \operatorname{sen} u \right]_0^{81} = \frac{\operatorname{sen} 81}{4}$$

c) $\int_0^1 \int_{\operatorname{arcsen} y}^{\pi/2} \cos x \sqrt{1+\cos^2 x} \, dx \, dy \; ; \quad D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \text{ e } \operatorname{arcsen} y \le x \le \pi/2\}$

$\operatorname{arcsen} y \le x \le \pi/2 \Leftrightarrow y \le \operatorname{sen} x \le 1 \Rightarrow \boxed{y \le \operatorname{sen} x}$

Se $y = 0$ então $\boxed{0 \le x \le \pi/2}$

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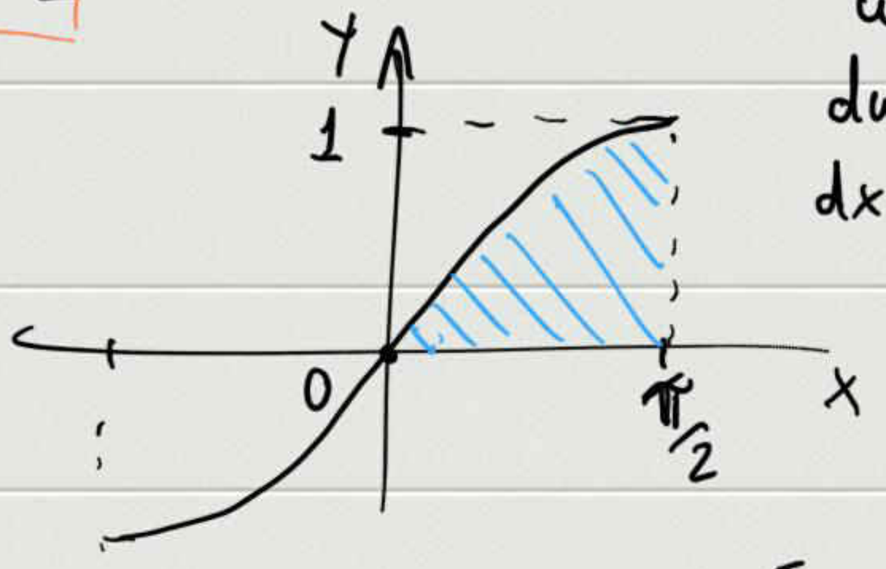


$u = 1 + \cos^2 x$

$du = 2 \cdot \cos x (-\operatorname{sen} x) \, dx$

$dx = \frac{-du}{2 \cos x \operatorname{sen} x}$

$$\int_0^1 \int_{\operatorname{arcsen} y}^{\pi/2} \cos x \sqrt{1+\cos^2 x} \, dx \, dy = \int_0^{\pi/2} \int_0^{\operatorname{sen} x} \cos x \cdot \sqrt{1+\cos^2 x} \, dy \, dx = \int_0^{\pi/2} \left[ y \cdot \cos x \sqrt{1+\cos^2 x} \right]_0^{\operatorname{sen} x} dx$$

$$= \int_0^{\pi/2} \operatorname{sen} x \cdot \cos x \sqrt{1+\cos^2 x} \, dx = \int_2^1 \left(-\tfrac{1}{2}\right) \cdot \sqrt{u} \, du = \left[ -\frac{1}{3} \cdot u \cdot \sqrt{u} \right]_2^1 = \frac{1}{3}(2\sqrt{2} - 1)$$

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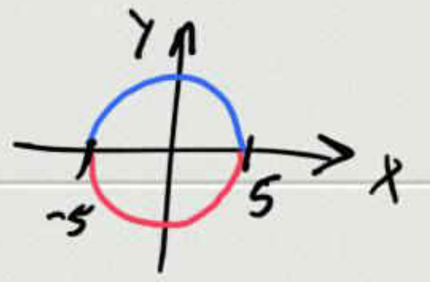
9 a) $\iint_R x\,dx\,dy$ $\quad R:\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 25\}$




$y=\pm\sqrt{25-x^2} \Rightarrow -\sqrt{25-x^2} \le y \le \sqrt{25-x^2}$ e $-5\le x\le 5$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\,dy\,dx = \int_{-5}^{5} 2\sqrt{25-x^2}\cdot x\,dx \quad ; \quad u=25-x^2 \Leftrightarrow dx=\frac{du}{-2x}$$

$$2\int_{0}^{0}\sqrt{u}\,du = 0$$

Por substituição

$$(x,y)=\varphi(r,\theta)$$

$$\begin{cases} x=r\cdot\cos\theta \\ y=r\cdot\operatorname{sen}\theta \end{cases} \Leftrightarrow x^2+y^2=r^2 \Rightarrow r^2\le 25 \Leftrightarrow -5\le r\le 5$$

$r\ge 0$, então $\boxed{0\le r\le 5}$

$$\det\begin{bmatrix}\nabla x(r,\theta)\\ \nabla y(r,\theta)\end{bmatrix} = r \rightarrow$$ sempre será isto para polares

Geometricamente obtém-se que $-5\le x\le 5 \Leftrightarrow -5\le r\cdot\cos\theta\le 5 \Leftrightarrow$

$-1\le\cos\theta\le 1 \Leftrightarrow \boxed{0+2\pi\cdot K\le\theta\le 2\pi+2\pi\cdot K \mid K\in\mathbb{Z}}$

Portanto de $\iint_B f(x,y)\,dx\,dy = \iint_{Buv} f(x(u,v),y(u,v)\cdot\det\begin{bmatrix}\nabla x(u,v)\\ \nabla y(u,v)\end{bmatrix}\,du\,dv$

Obtém-se:

$$\iint_B x\,dx\,dy = \iint_{Bre} r\cdot\cos\theta\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^5 r^2\cos\theta\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi} 125\cdot\cos\theta\,d\theta =$$

$$\frac{125}{3}\left[\operatorname{sen}\theta\right]_0^{2\pi} = 0$$

b) Dá para fazer sem substituição, porém, farei apenas com a substituição

$$\iint_R xy\,dx\,dy \qquad R=\{(x,y)\in\mathbb{R}^2 \mid 4\le x^2+y^2\le 25 \mid x\ge 0 \text{ e } y\ge 0\}$$

$(x,y)=\varphi(r,\theta)$

$$\begin{cases} x=r\cos\theta \\ y=r\operatorname{sen}\theta \end{cases} ; \boxed{\frac{\partial(x,y)}{\partial(r,\theta)}=r} ; \quad 4\le r^2\le 25 \Leftrightarrow \boxed{2\le r\le 5}$$

$$\begin{cases} x\ge 0 \\ y\ge 0 \end{cases} \Leftrightarrow \begin{cases} r\cdot\cos\theta\ge 0 \\ r\cdot\operatorname{sen}\theta\ge 0 \end{cases} \Leftrightarrow \begin{cases} \cos\theta\ge 0 \\ \operatorname{sen}\theta\ge 0 \end{cases} \Leftrightarrow \begin{cases} -\frac{\pi}{2}+2\pi\cdot K\le\theta\le\frac{\pi}{2}+2\pi\cdot K \\ 0+2\pi\cdot K\le\theta\le\pi+2\pi\cdot K \end{cases} \Rightarrow$$

$$\boxed{0+2\pi\cdot K\le\theta\le\frac{\pi}{2}+2\pi\cdot K \mid K\in\mathbb{Z}}$$

$$\therefore \iint_R xy\,dx\,dy = \iint_{Rre} r\cos\theta\cdot r\operatorname{sen}\theta\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_2^5 r^3\cdot\cos\theta\cdot\operatorname{sen}\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{4}\left[625-16\right]\operatorname{sen}\theta\cdot\cos\theta\,d\theta = \frac{609}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 2\cdot\operatorname{sen}\theta\cdot\cos\theta\,d\theta = \frac{609}{8}\int_0^{\pi/2}\operatorname{sen}(2\theta)\,d\theta =$$

$$\frac{609}{8}\cdot\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = \frac{609}{8}\left[-\frac{1}{2}(-1-1)\right] = \frac{609}{8}$$

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c) $\iint_R \frac{1}{\sqrt{x^2+y^2}} \cdot dx\,dy \quad R_{r,\theta} = D = \{(r,\theta) \in \mathbb{R}^2 \mid 1 \le r \le 1+\operatorname{sen}\theta\}$

$0 \le \theta \le \pi$

$x(r,\theta),\ y(r,\theta)$

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \operatorname{sen}\theta \end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\left[\nabla x(r,\theta)\right]\right| = \left|\begin{vmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{vmatrix}\right| = r$$

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \iint_{R_{r,\theta}} \frac{1}{\sqrt{r^2}} \cdot r \cdot dr\,d\theta = \int_0^{\pi}\int_1^{1+\operatorname{sen}\theta} \frac{1}{|r|} \cdot r \cdot dr \cdot d\theta$$

$$= \int_0^{\pi}\int_1^{1+\operatorname{sen}\theta} dr\,d\theta = \int_0^{\pi} \operatorname{sen}\theta \cdot d\theta = \left[-\cos\theta\right]_0^{\pi} = -\cos\pi + \cos 0 = 2$$

d) $\iint_R (x^2+y^2)\,dx\,dy \qquad R_{r,\theta} = D = \{(r,\theta) \in \mathbb{R}^2 \mid \theta \le r \le 2\theta \text{ e } 0 \le \theta \le 2\pi\}$

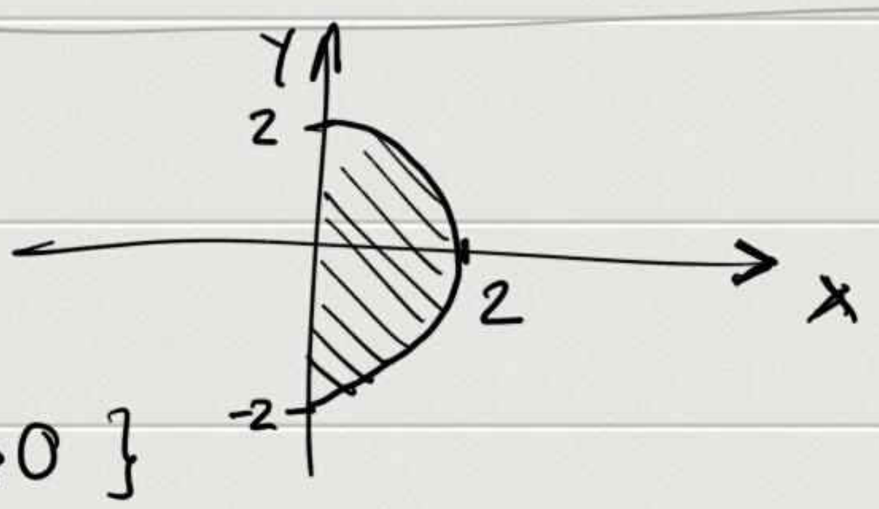
$x(r,\theta)\quad y(r,\theta)$

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \operatorname{sen}\theta \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$\iint_R (x^2+y^2) \cdot dx\,dy = \iint_{R_{r,\theta}} r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi}\int_{\theta}^{2\theta} r^3 \cdot dr \cdot d\theta = \int_0^{2\pi} \frac{15}{4}\theta^4 \cdot d\theta =$$

$$\frac{15}{20} \cdot 32\pi^5 = 24\pi^5$$

e) $\iint_D e^{-x^2-y^2}\,dx\,dy$




$D = \{(x,y) \mid x^2+y^2 = 4 \text{ e } y \ge 0\}$

$D_{r,\theta} = \{(r,\theta) \mid 0 \le r \le 2 \text{ e } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \operatorname{sen}\theta \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$\iint_D e^{-(x^2+y^2)} \cdot dx\,dy = \iint_{D_{r,\theta}} e^{-r^2} \cdot r \cdot dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r \cdot e^{-r^2} \cdot dr \cdot d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{-4} \frac{-e^u}{2} \cdot du\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^0 - e^{-4}\right) \cdot d\theta = \frac{\pi}{2}\left(1 - e^{-4}\right)$$

(17)

f) $\iint_D \sqrt{(x-1)^2+y^2}\,.dxdy \qquad D=\{(x,y) \mid x^2+y^2 \le 1 \text{ e } y \ge 0\}$

$x(r,\theta)$ e $y(r,\theta)$

$$\begin{cases} x-1 = r.\cos\theta \\ y = r.\text{sen}\theta \end{cases} \Leftrightarrow \begin{cases} x = r.\cos\theta + 1 \\ y = r\text{sen}\theta \end{cases}$$

$\hookrightarrow (x-1)^2+y^2 = r^2$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\left[\nabla_x(r,\theta)\right]\right| = \left|\begin{matrix} \cos\theta & -r\text{sen}\theta \\ \text{sen}\theta & r\cos\theta \end{matrix}\right| = |r| = r$$

$x^2+y^2 \le 1 \Leftrightarrow x^2-2x+1+y^2 \le 1-2x+1 \Leftrightarrow (x-1)^2+y^2 \le 2(1-x)$

$\Leftrightarrow r^2 \le -2(x-1) \Leftrightarrow r^2 \le -2(r\cos\theta) \Leftrightarrow r(r+2\cos\theta) \le 0$

$\Leftrightarrow 0 \le r \le -2\cos\theta \rightarrow -2\cos\theta \ge 0 \Leftrightarrow \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ | Ineq. 0

$y \ge 0 \Leftrightarrow r.\text{sen}\theta \ge 0 \Leftrightarrow 0 \le \theta \le \pi$ | Ineq. 1

De ineq. 0 e ineq. 1 : $\frac{\pi}{2} \le \theta \le \pi$

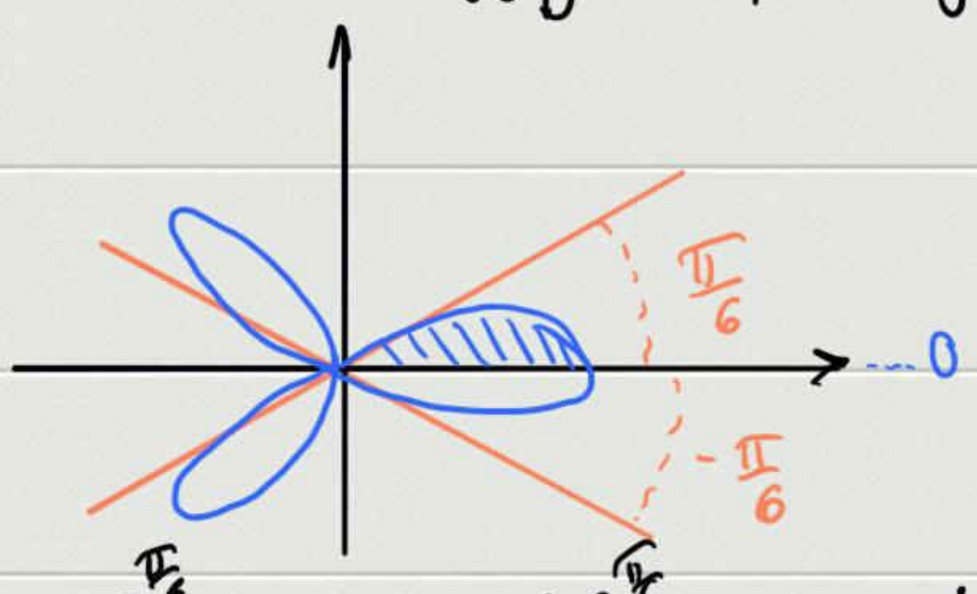
$$\iint_D \sqrt{(x-1)^2+y^2}\,.dxdy = \iint_{D_{r,\theta}} |r|.r.dr.d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-2\cos\theta} r^2.dr.d\theta =$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{3}(-2\cos\theta)^3.d\theta = \frac{-8}{3}\int_{\frac{\pi}{2}}^{\pi} \cos^3\theta.d\theta = -\frac{8}{3}\left(0-0-\left(0+\frac{2}{3}\right)\right) = \frac{16}{9}$$

$$\int \cos^3 x\,dx = \frac{1}{3}.\text{sen}x.\cos^2 x + \frac{2}{3}\int \cos x\,dx = \frac{1}{3}\text{sen}x.\cos^2 x + \frac{2}{3}\text{sen}x$$

Da fórmula de recorrência

(10) a) $A = \iint_D 1.dxdy = \iint_{D_{r,\theta}} r.dr.d\theta$




$$\therefore A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r.dr.d\theta$$

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2 3\theta}{2}.d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos 6\theta + 1}{4}.d\theta$$

$$A = \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 6\theta.d\theta + \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1.d\theta = \frac{1}{24}\left(\text{sen}\left(6.\frac{\pi}{6}\right) - \text{sen}\left(-6\frac{\pi}{6}\right)\right) + \frac{1}{4}\left(\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{12}$$

(18)

b) $A = \iint_D dx\,dy$ $\qquad r^2 = 4.\cos 2\theta \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

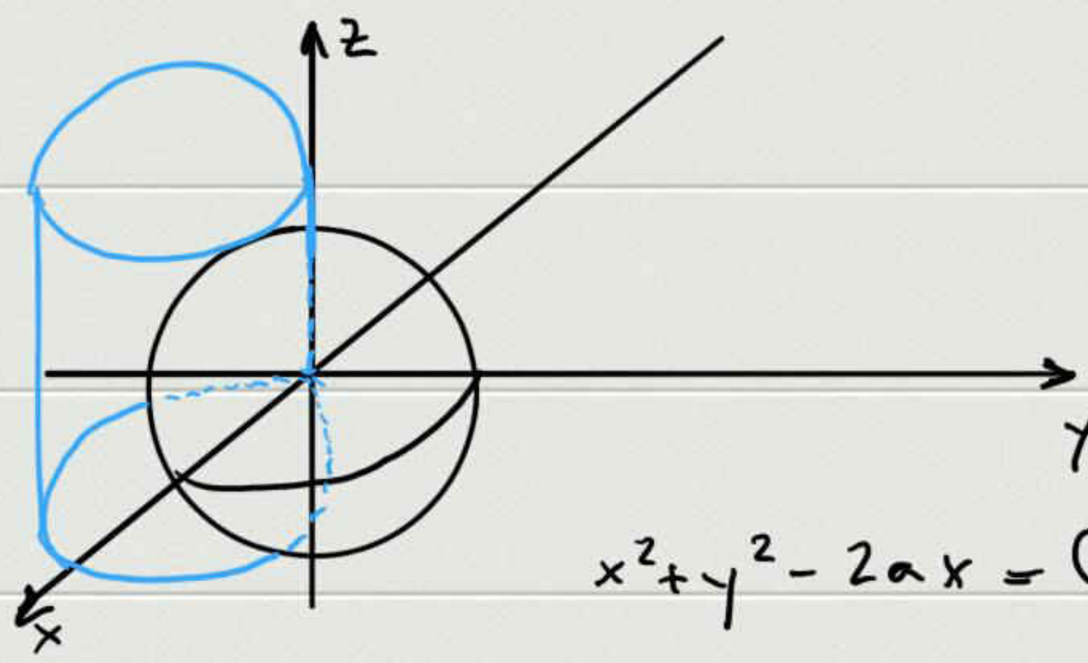
$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{4\cos 2\theta}} r \,.dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos 2\theta \,.d\theta = 2\,(1-(-1)) = 4$$

(11) $x^2+y^2+z^2 = 4a^2$ (Eq.0) e exterior $x^2+y^2 = 2ax$

$x^2+y^2-2ax = 0 \Leftrightarrow x^2-2ax+a^2+y^2 = a^2 \Leftrightarrow (x-a)^2+y^2 = a^2$ (Eq.1)

isolando-se

$$-\sqrt{4a^2-(x^2+y^2)} \le z \le \sqrt{4a^2-(x^2+y^2)}$$




$x(r,\theta)$ e $y(r,\theta)$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$x^2+y^2-2ax = 0 \Leftrightarrow r^2-2ar.\cos\theta = 0 \Leftrightarrow r(r-2a\cos\theta) = 0$

$\therefore \quad 0 \le r \le 2a\cos\theta$

$$V = \iiint_D dx\,dy\,dz = \iiint_{D(r,\theta)} r.\,dr.d\theta.dz = \int_0^{\pi}\int_0^{2a\cos\theta}\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r.\,dz.dr.d\theta =$$

$$\int_0^{\pi}\int_0^{2a\cos\theta} 2\sqrt{4a^2-r^2}.r\,dr.d\theta \overset{u=4a^2-r^2}{\Leftrightarrow} \int_0^{\pi}\int_{4a^2}^{4a^2\operatorname{sen}^2\theta} 2\,u^{1/2}\,\frac{r}{-2r}.du.d\theta = -\int_0^{\pi}\frac{2}{3}\left[\left(4a^2\operatorname{sen}^2\theta\right)^{3/2}-\left(4a^2\right)^{3/2}\right]d\theta$$

$$= -\frac{2}{3}\int_0^{\pi} 8a^3(\operatorname{sen}^3\theta-1).d\theta = -\frac{16a^3}{3}.\left(\int_0^{\pi}\operatorname{sen}^3\theta.d\theta - \int_0^{\pi} 1.d\theta\right) = -\frac{16a^3}{3}\left(-\frac{2}{3}(-1-1)-\pi\right)$$

$$\boxed{V = \frac{16a^3}{3}\left(\frac{4}{3}+\pi\right)}$$

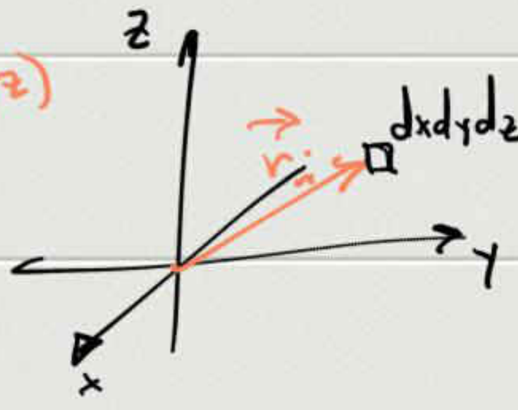
$$\int \operatorname{sen}^3 x.dx = -\frac{1}{3}\cos x.\operatorname{sen}^2 x + \frac{2}{3}(-\cos x) + k$$

(12) a) $\delta(x,y) = x^2$ $\qquad D = \{(x,y)\in\mathbb{R} \mid -1 \le x \le 1,\ 0 \le y \le 1\}$

$m = \delta.V$

$dm = \delta.dV$ → Se $\delta(x,y,z)$

$\boxed{dm = \delta.dx\,dy\,dz}$




$$M = \iiint_D dm$$

$$\vec{R} = \sum_{i=1}^{n} \frac{\vec{r_i}.m_i}{M}$$

$n \to \infty$

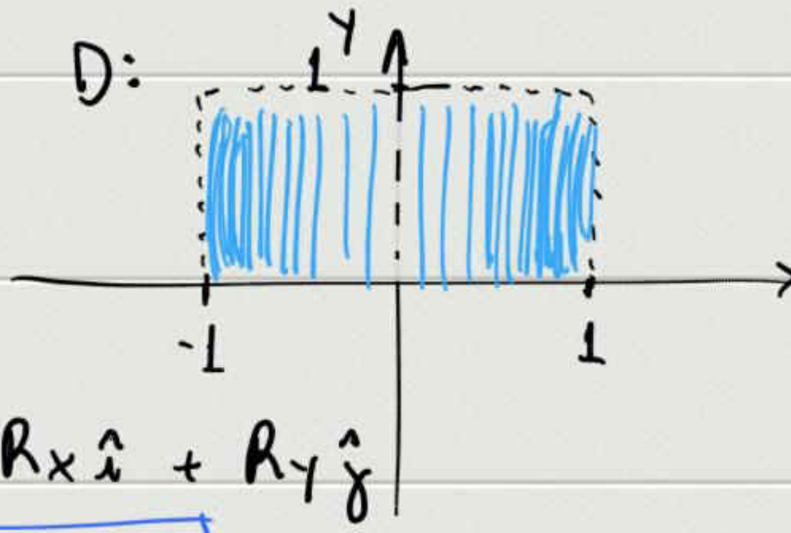
$$\vec{R} = \int \frac{\vec{r}\,dm}{M}$$

19)

$$M = \iint_D dm = \iint_D \delta(x,y)\cdot dx\,dy = \int_{-1}^{1}\int_{0}^{1} x^2\cdot dy\cdot dx = \int_{-1}^{1} x^2\cdot dx \cdot \int_{0}^{1} dy \;\Rightarrow$$

$M = 2/3$

D:




$$R_x = \frac{\iint_D x\,dm}{M} = \int_{-1}^{1}\int_{0}^{1} x\cdot x^2\cdot dy\cdot dx$$

$$R_x = \frac{3}{2}\int_{-1}^{1} x^3\,dx = 0$$

$$\vec{R} = \vec{R}(x,y) = R_x\,\hat{i} + R_y\,\hat{j}$$

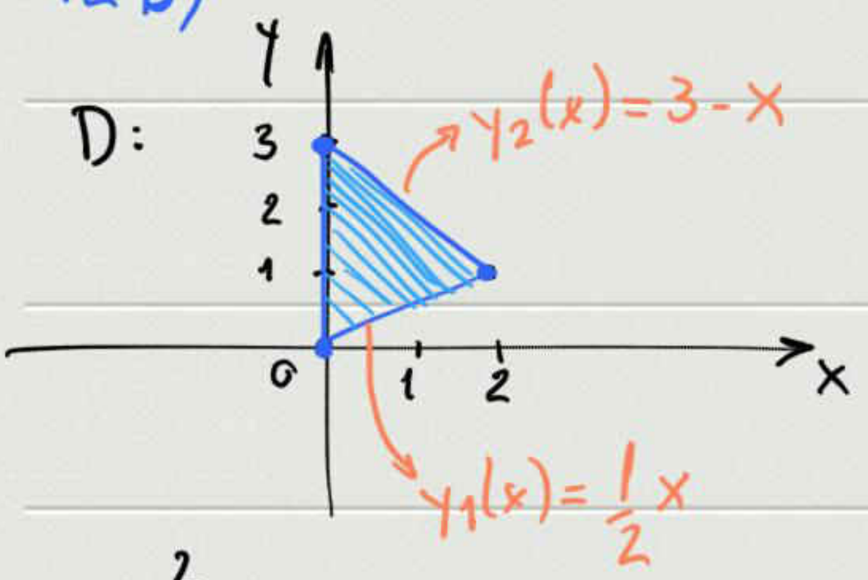
$$R_y = \frac{\iint_D y\cdot dm}{M} = \frac{3}{2}\cdot\int_{0}^{1}\int_{-1}^{1} y\cdot x^2\,dx\,dy$$

$$\therefore \vec{R}(x,y) = \left(0, \tfrac{1}{2}\right)$$

$$R_y = \frac{3}{2}\int_{0}^{1} y\cdot\frac{2}{3}\cdot dy = \frac{1}{2}(1-0) = \frac{1}{2}$$

12 b)

D:




$$M = \iint_D dm = \iint_D \delta(x,y)\cdot dx\cdot dy = \int_{0}^{2}\int_{\frac{1}{2}x}^{3-x} (x+y)\,dy\cdot dx$$

$$= \int_{0}^{2}\left(x\left(3-x-\tfrac{1}{2}x\right) + \frac{1}{2}\left((3-x)^2 - \left(\tfrac{1}{2}x\right)^2\right)\right)\cdot dx$$

$$= \int_{0}^{2}\left[3x - \frac{3}{2}x^2 + \frac{1}{2}\left((9-6x+x^2) - \frac{1}{4}x^2\right)\right]\cdot dx$$

$$= \int_{0}^{2} 36 - \frac{9x^2}{8}\cdot dx = \frac{36}{8}(2-0) - \frac{9}{8}\cdot\frac{1}{3}\left(2^3 - 0^3\right) = 9 - 3 = 6$$

$$\vec{R}(x,y) = \left(\frac{\iint_D x\cdot dm}{M}, \frac{\iint_D y\cdot dm}{M}\right) \;;\; \iint_D x\,dm = \int_{0}^{2}\int_{\frac{1}{2}x}^{3-x} x\cdot(x+y)\cdot dy\cdot dx =$$

$$\int_{0}^{2}\left[x^2y + \frac{xy^2}{2}\right]_{\frac{1}{2}x}^{3-x}\cdot dx = \int_{0}^{2}\left[x^2\left(3-x-\tfrac{1}{2}x\right) + \frac{x}{2}\left((3-x)^2 - \left(\tfrac{1}{2}x\right)^2\right)\right]\cdot dx =$$

$$\int_{0}^{2}\frac{9}{8}(4x - x^3)\,dx = \frac{9}{8}\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = \frac{9}{8}\cdot\left(8 - \frac{1}{4}\cdot 16\right) = \frac{9}{2}$$

$$\iint_D y\cdot dm = \int_{0}^{2}\int_{\frac{1}{2}x}^{3-x} y\cdot(x+y)\cdot dy\cdot dx = \int_{0}^{2}\left[\frac{xy^2}{2} + \frac{1}{3}y^3\right]_{\frac{1}{2}x}^{3-x}\cdot dx =$$

$$\int_{0}^{2}\left[\frac{x}{2}\left((3-x)^2 - \left(\tfrac{1}{2}x\right)^2\right) + \frac{1}{3}\left((3-x)^3 - \left(\tfrac{1}{2}x\right)^3\right)\right]\cdot dx =$$

$$\int_{0}^{2}\left[\frac{x}{2}\left(9 - 6x + x^2 - \frac{1}{4}x^2\right) + \frac{1}{3}\left(27 - 27x + 9x^2 - x^3 - \frac{1}{8}x^3\right)\right]\cdot dx$$

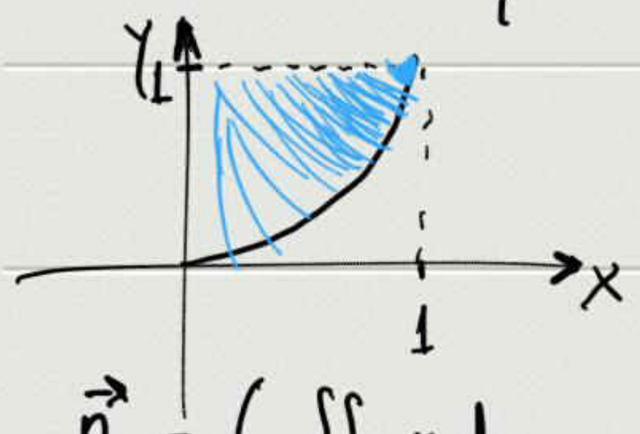
$$\int_{0}^{2}\frac{1}{24}\left(108x - 72x^2 + 9x^3 + 8\cdot 27 - 216x + 72x^2 - 9x^3\right)\cdot dx = \int_{0}^{2}\frac{1}{24}(8\cdot 27 - 108x)\cdot dx$$

$$= \frac{1}{24}\left(8\cdot 27\cdot 2 - \frac{108}{2}\cdot 2^2\right) = 9$$

$$\therefore \vec{R} = \left(\frac{3}{4}, \frac{3}{2}\right)\ u.c.$$

12 c) $D: x \geq 0 \quad y \geq 0 \quad y = x^2 \quad y = 1$

$\delta(x,y) = xy$




$$M = \iint_D dm = \int_0^1 \int_{x^2}^1 xy \, dy \, dx = \int_0^1 \frac{x}{2}\left(1^2 - (x^2)^2\right) dx$$

$$M = \int_0^1 \frac{x}{2} dx - \int_0^1 \frac{x^5}{2} dx = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$$\vec{R} = \left( \frac{\iint_D x \, dm}{M}, \frac{\iint_D y \, dm}{M} \right); \quad \iint_D x \, dm = \int_0^1 \int_{x^2}^1 x \cdot xy \, dy \, dx =$$

$$\int_0^1 \int_{x^2}^1 x^2 y \, dy \, dx = \int_0^1 \frac{x^2}{2}(1 - x^4) \, dx = \int_0^1 \frac{x^2}{2} dx - \int_0^1 \frac{x^6}{2} dx = \left(\frac{1}{3} - \frac{1}{7}\right) \cdot \frac{1}{2} = \frac{2}{21}$$

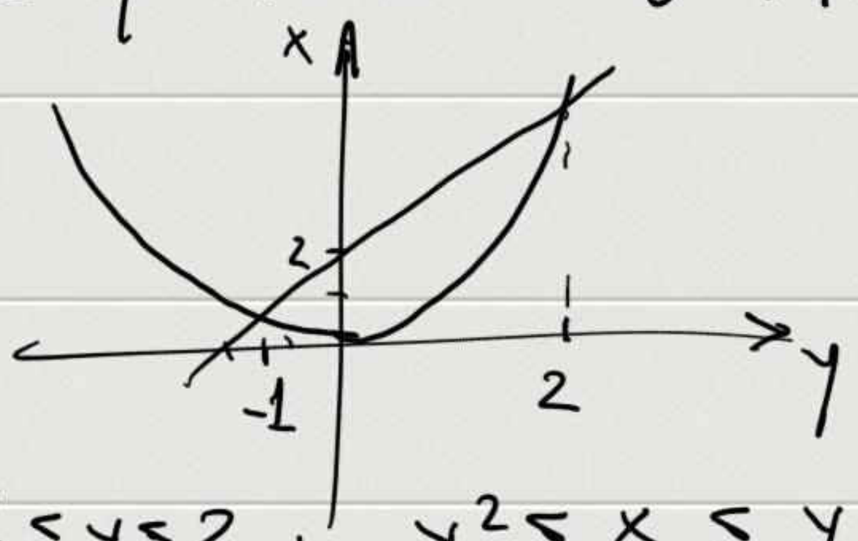
$$\iint_D y \, dx \, dy = \int_0^1 \int_{x^2}^1 y \cdot xy \, dy \, dx = \int_0^1 \frac{1}{3} x \left(1^3 - (x^2)^3\right) dx = \int_0^1 \frac{x}{3} dx - \int_0^1 \frac{x^7}{3} dx$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{1}{8}$$

$$\therefore \vec{R} = \left(\frac{2}{21} \cdot 6, \frac{1}{8} \cdot 6\right) \Leftrightarrow \boxed{\vec{R} = \left(\frac{4}{7}, \frac{3}{4}\right) \text{ u.c.}}$$

12 d) $y^2 = x \quad e \quad y = x - 2 \quad \delta(x,y) = 3$

$$\begin{cases} x = y^2 \\ x = y + 2 \end{cases}$$




$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$(y - 2)(y + 1) = 0$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 2, \; y^2 \leq x \leq y + 2\}$$

$$M = \iint_D dm = \iint_D \delta(x,y) \, dx \, dy = \int_{-1}^2 \int_{y^2}^{y+2} 3 \, dx \, dy = \int_{-1}^2 3(y + 2 - y^2) \, dy$$

$$3 \cdot \left[ -\frac{1}{3}\left(2^3 - (-1)^3\right) + \frac{1}{2}\left(2^2 - (-1)^2\right) + 2(2 - (-1)) \right] = 3\left[-3 + \frac{3}{2} + 6\right] = \frac{27}{2}$$

$$\vec{R} = \left( \frac{\iint_D x \, dm}{M}, \frac{\iint_D y \, dm}{M} \right); \quad \iint_D x \, dm = \int_{-1}^2 \int_{y^2}^{y+2} x \cdot 3 \, dx \, dy \qquad \text{u.m.}$$

$$= \int_{-1}^2 \frac{3}{2}\left((y+2)^2 - y^4\right) dy = \int_{-1}^2 \frac{3}{2}\left(4 + 4y + y^2 - y^4\right) dy =$$

$$\frac{3}{2}\left[4(2 - (-1)) + \frac{4}{2}\left(2^2 - (-1)^2\right) + \frac{1}{3}\left(2^3 - (-1)^3\right) - \frac{1}{5}\left(2^5 - (-1)^5\right)\right] =$$

$$\frac{3}{2}\left[12 + 6 + 3 - \frac{33}{5}\right] = \frac{3}{2} \cdot \frac{105 - 33}{5} = \frac{3}{2} \cdot \frac{72}{5} = \frac{108}{5}$$

21)

$$R_y = \iint_D y \cdot dm = \iint_D y \cdot 3\,dx\,dy = \int_{-1}^{2}\int_{y^2}^{y+2} 3y\,dx\,dy = \int_{-1}^{2} 3y\left[y+2-y^2\right].dy$$

$$= \int_{-1}^{2} 3\left(y^2+2y-y^3\right).dy = 3\left[\frac{1}{3}\left(2^3-(-1)^3\right)+\frac{2}{2}\left(2^2-(-1)^2\right)-\frac{1}{4}.\left(2^4-(-1)^4\right)\right]$$

$$= 3\left[3+3-\frac{15}{4}\right] = 3\left[\frac{24-15}{4}\right] = \frac{27}{4}$$

$$\vec{R} = \left(\frac{108}{5}\cdot\frac{2}{27}, \frac{27}{4}\cdot\frac{2}{27}\right) \Leftrightarrow \boxed{\vec{R} = \left(\frac{8}{5}, \frac{1}{2}\right) \text{ u.c.}}$$

e) $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \operatorname{sen} x\,,\ 0 \le x \le \pi\}$ $\delta(x,y) = y$

$$M = \iint_D dm = \int_0^{\pi}\int_0^{\operatorname{sen} x} y.\,dy.dx = \int_0^{\pi} \frac{\operatorname{sen}^2 x}{2}.dx = -\int_0^{\pi} \frac{-2\operatorname{sen}^2 x + 1 - 1}{4}.dx$$

$$= -\int_0^{\pi} \frac{\cos 2x}{4}.dx - \int_0^{\pi} -\frac{1}{4}.dx = -\frac{1}{4}\left[\int_0^{\pi} \cos 2x.dx - \int_0^{\pi} 1.dx\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2}\left(\operatorname{sen} 2\pi - \operatorname{sen} 0\right) - (\pi - 0)\right] = \frac{\pi}{4}\ u.m$$

$$\vec{R} = \left(\frac{\iint_D x.dm}{M}, \frac{\iint_D y.dm}{M}\right)\ ;\ \iint_D x.dm = \int_0^{\pi}\int_0^{\operatorname{sen} x} x.y.dy.dx = \int_0^{\pi} x.\frac{\operatorname{sen}^2 x}{2}.dx =$$

$$\int x.\operatorname{sen}^2 x.dx = x.\left(\frac{1}{2}\operatorname{sen} 2x + x\right) - \int 1.\left(\frac{1}{2}\operatorname{sen} 2x + x\right).dx =$$

$$\frac{x.\operatorname{sen} 2x}{2} + x^2 + \frac{1}{4}.\cos 2x - \frac{x^2}{2} + K = \frac{x.\operatorname{sen} 2x}{2} + \frac{x^2}{2} + \frac{1}{4}.\cos 2x + K$$

$$\frac{1}{2}.\frac{1}{2}\left[0 + \frac{\pi^2}{2} + \frac{1}{4}\left(\cos 2\pi - \cos 0\right)\right] = \frac{\pi^2}{8}$$

$$\iint_D y.dm = \int_0^{\pi}\int_0^{\operatorname{sen} x} y.y.dy.dx = \int_0^{\pi}\int_0^{\operatorname{sen} x} y^2.dy.dx = \int_0^{\pi} \frac{1}{3}\operatorname{sen}^3 x.dx =$$

$$\frac{1}{3}\int_0^{\pi} \operatorname{sen}^3 x.dx = \frac{1}{3}\left[-\frac{\cos x.\operatorname{sen}^2 x}{3} - \frac{2}{3}\cos x + K\right]_0^{\pi} = \frac{1}{3}\left[0 - \frac{2}{3}\left(\cos\pi - \cos 0\right)\right]$$

$$= \frac{4}{9}$$

$$\vec{R} = \left(\frac{\pi^2}{8}\cdot\frac{4}{\pi}, \frac{4}{9}\cdot\frac{4}{\pi}\right) \Leftrightarrow \vec{R} = \left(\frac{\pi}{2}, \frac{16}{9\pi}\right)$$

Obs: Devido à simetria do problema e à densidade, Rx só poderia estar no ponto médio do segmento de 0 à $\pi$.

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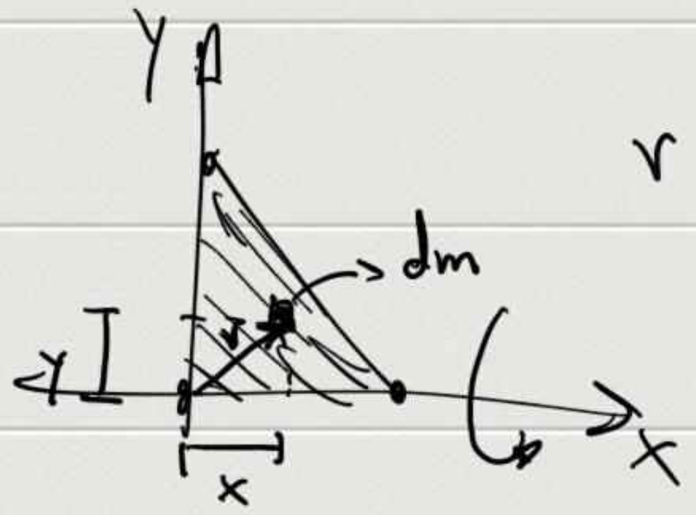
13 $dI = dm\, r^2 \iff I = \int_0^M r^2 \cdot dm \qquad dm = \delta(x,y) \cdot dx\,dy$

Então $I = \iint_D r^2 \cdot \delta(x,y) \cdot dx \cdot dy$




$r = \sqrt{x^2+y^2}$

~~Steiner~~ Não se precisa calcular o $I_{cm}$

~~$I = I_{cm} + M d^2$~~

$dI_x = dm\, y^2 = \delta(x,y) \cdot y^2\, dx\,dy \iff I_x = \iint_S \delta(x,y) \cdot y^2 \cdot dx\,dy$

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le 1,\ 0 \le x \le 1\}$

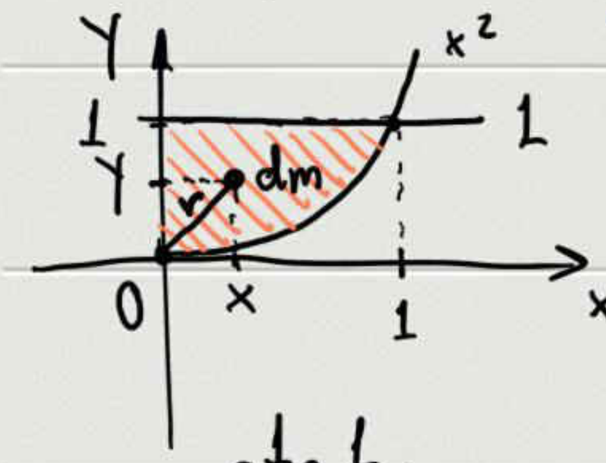
$I_x = \int_0^1 \int_{x^2}^1 xy\, y^2 \cdot dy\,dx = \int_0^1 \frac{x}{4}\left(1^4 - (x^2)^4\right) \cdot dx = \frac{1}{4} \cdot \int_0^1 (x - x^9)\,dx$

$I_x = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{10}\right) = \frac{1}{10}$ um. uc$^2$

$dI_y = dm \cdot x^2 \iff dI_y = \delta(x,y) \cdot x^2 \cdot dx\,dy \iff I_y = \iint_S \delta(x,y)\, x^2 \cdot dx\,dy$

$I_y = \int_0^1 \int_{x^2}^1 xy \cdot x^2 \cdot dy\,dx = \int_0^1 \frac{x^3}{2}\left(1^2 - (x^2)^2\right) \cdot dx = \frac{1}{2}\int_0^1 (x^3 - x^7) \cdot dx$

$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{8}\right) = \frac{1}{16}$ um. uc$^2$

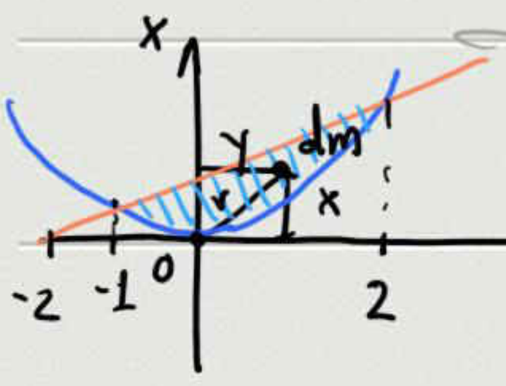



$dI_0 = dm\, r^2 \iff dI_0 = \delta(x,y) \cdot dx\,dy \cdot r^2$

$I_0 = \iint_S \delta(x,y) \cdot r^2 \cdot dx\,dy = \int_0^1 \int_{x^2}^1 xy\,(x^2+y^2) \cdot dy\,dx$

$I_0 = \int_0^1 \int_{x^2}^1 (xy \cdot x^2 + xy \cdot y^2) \cdot dy\,dx = \int_0^1 \int_{x^2}^1 xy\, x^2 \cdot dy\,dx + \int_0^1 \int_{x^2}^1 xy\, y^2 \cdot dy\,dx$

$= I_y + I_x = \frac{1}{10} + \frac{1}{16} = \frac{8+5}{80} = \frac{13}{80}$ um. uc$^2$




$\begin{cases} x = y^2 \\ x = y+2 \end{cases} \Rightarrow \begin{matrix} y^2 = y+2 \\ y^2 - y - 2 = 0 \end{matrix} \Big| \quad (y-2)(y+1) = 0$

$S = \{(x,y) \in \mathbb{R}^2 \mid y^2 \le x \le y+2,\ -1 \le y \le 2\}$

$dI_x = dm\, y^2 \iff I_x = \iint_S \delta(x,y) \cdot y^2\, dx\,dy = \int_{-1}^2 \int_{y^2}^{y+2} 3y^2 \cdot dx\,dy = 3\int_{-1}^2 y^2\,(y+2-y^2)\,dy$

$= 3\int_{-1}^2 (y^3 + 2y^2 - y^4)\,dy = 3\left[\frac{1}{4}(2^4 - (-1)^4) + \frac{2}{3}(2^3 - (-1)^3) - \frac{1}{5}(2^5 - (-1)^5)\right]$

(23)

$$I_x = 3\left(\frac{15}{4} + 6\quad\frac{33}{5}\right) = \frac{3}{20}(75 + 120 - 132) = \frac{189}{20}\ um.uc^2$$

$$dI_y = dm \cdot x^2 \Leftrightarrow I_y = \iint_S \delta(x,y).x^2.\,dxdy = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2.\,dxdy$$

$$= \int_{-1}^{2}\left[(y+2)^3 - (y^2)^3\right]dy = \int_{-1}^{2}\left(y^3+6y^2+12y+8-y^6\right).dy$$

$$= \left[\frac{1}{4}\left(2^4-(-1)^4\right) + \frac{6}{3}\left(2^3-(-1)^3\right) + \frac{12}{2}\left(2^2-(-1)^2\right) + 8(2-(-1)) - \frac{1}{7}\left(2^7-(-1)^7\right)\right]$$

$$= \frac{1}{4}.15 + 2.9 + 6.3 + 8(3) - \frac{129}{7} = \frac{105+504+504+672-516}{28}$$

$$I_y = \frac{1269}{28}\ um.uc^2$$

$$dI_o = dm\,r^2 \Leftrightarrow I_o = \iint_S \delta(x,y).r^2.\,dxdy = \int_{-1}^{2}\int_{y^2}^{y+2} 3.(x^2+y^2).dxdy =$$

$$\int_{-1}^{2}\int_{y^2}^{y+2}(3x^2+3y^2).dxdy = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2.dxdy + \int_{-1}^{2}\int_{y^2}^{y+2} 3y^2.dxdy = I_y + I_x$$

$$\therefore I_o = \frac{189}{20} + \frac{1269}{28} = \frac{7.189+5.1269}{4.35} = \frac{1323+6345}{4.35} = \frac{7668}{4.35} = \frac{1917}{35}\ um\,uc^2$$

(14) a) $D = \{(x,y) \mid (x-2y+3)^2 + (3x+4y-1)^2 \le 100\}$

$\delta(x,y) = x - 2y + 18$

Fazendo-se uma substituição dupla (x,y) para polares (r,θ)

$$\begin{cases} x-2y+3 = r.\cos\theta \\ 3x+4y-1 = r.\operatorname{sen}\theta \end{cases} \Leftrightarrow \begin{cases} 5x+5 = 2r\cos\theta + r\operatorname{sen}\theta \\ 10y-10 = r.\operatorname{sen}\theta - 3r\cos\theta \end{cases} \Rightarrow$$

$$\begin{cases} x = \dfrac{2r\cos\theta + r\operatorname{sen}\theta - 1}{5} \\ y = \dfrac{r\operatorname{sen}\theta - 3r\cos\theta}{10} + 1 \end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{bmatrix}\nabla x(r,\theta)\\ \nabla y(r,\theta)\end{bmatrix}\right| = \left|\det\begin{bmatrix}\frac{2}{5}\cos\theta + \frac{1}{5}\operatorname{sen}\theta & -\frac{2r\operatorname{sen}\theta + r\cos\theta}{5}\\ \frac{\operatorname{sen}\theta - 3\cos\theta}{10} & \frac{r\cos\theta + 3r\operatorname{sen}\theta}{10}\end{bmatrix}\right|$$

$$\left|\frac{1}{50}\left[\left(2r\cos^2\theta + 6r\cos\theta\operatorname{sen}\theta + r\cos\theta\operatorname{sen}\theta + 3r\operatorname{sen}^2\theta\right) - \left(-2r\operatorname{sen}^2\theta + 6r\operatorname{sen}\theta\cos\theta + r\cos\theta\operatorname{sen}\theta - 3r\cos^2\theta\right)\right]\right|$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\frac{1}{50}\left[5r\cos^2\theta + 5r\operatorname{sen}^2\theta\right]\right| = \left|\frac{r}{10}\right| = \frac{r}{10}, \text{ pois } r \ge 0$$

$$M = \iint_D \delta(x,y).dxdy = \iint_{D(r,\theta)} \delta(r,\theta).\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|.dr\,d\theta$$

(24)

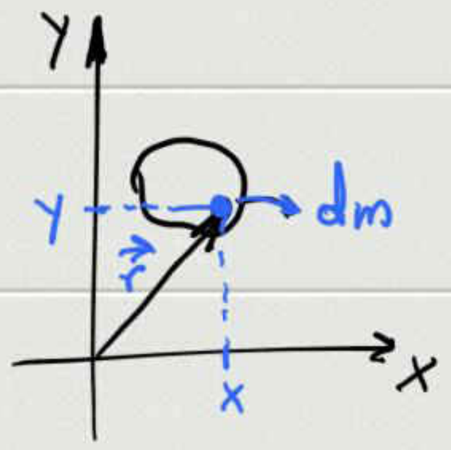
$$\delta(r,\theta) = x - 2y + 18 = x - 2y + 3 + 15 = r.\cos\theta + 15$$

restrições $\quad D_{(r,\theta)} = \{(r,\theta) \in \quad | \; 0 \le r \le 10\}$

$$M = \int_0^{2\pi}\int_0^{100} (r.\cos\theta + 15).\frac{r}{10}.dr.d\theta = \frac{1}{10}\int_0^{2\pi}\int_0^{10}(r^2.\cos\theta + 15r).dr\,d\theta =$$

$$\frac{1}{10}\int_0^{2\pi}\left(\frac{10^3}{3}.\cos\theta + \frac{15}{2}.10^2\right).d\theta = \frac{1}{10}\cdot\left[\frac{10^3}{3}.0 + \frac{15}{2}.10^2.2\pi\right] = 150\pi \text{ u.m}$$

14 b) $\quad \vec{R} = \dfrac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \Rightarrow \vec{R} = \dfrac{\iint_D \vec{r}.dm}{\iint_D dm}$




$$\vec{R} = \frac{1}{\iint_D dm}.\iint_D (x,y).dm = \frac{1}{\iint_D dm}.\iint_D (x\hat{i} + y\hat{\gamma}).dm$$

$$= \frac{1}{\iint_D dm}.\left(\iint_D x.\hat{i}.dm + \iint_D y.\hat{\gamma}.dm\right) = \frac{1}{\iint_D dm}.\left(\iint_D x.dm, \iint_D y\,dm\right)$$

$$D = \{(x,y) \in \mathbb{R} \mid x^2 + y^2 \le r^2\} \quad \delta(x,y) = (x-r)^2 + y^2$$

Alterando-se as coordenadas cartesianas

$$\begin{cases} x = \rho.\cos\theta \\ y = \rho.\operatorname{sen}\theta \end{cases} ; \; \rho^2 \le r^2 \Leftrightarrow \boxed{0 \le \rho \le r} ;$$

$$\delta(\rho,\theta) = (\rho.\cos\theta - r)^2 + (\rho.\operatorname{sen}\theta)^2 = \rho^2\cos^2\theta - 2\rho.r\cos\theta + r^2 + \rho^2.\operatorname{sen}^2\theta$$

$$\Leftrightarrow \boxed{\delta(\rho,\theta) = \rho^2 - 2\rho r\cos\theta + r^2}$$

Se não há restrições para $\theta$, então: $\boxed{0 \le \theta \le 2\pi}$

$$\boxed{\left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right| = \rho} \quad \therefore \quad M = \iint_D dm = \iint_D \delta(x,y).dx\,dy = \iint_{D\rho\theta} \delta(\rho,\theta).\rho.d\rho.d\theta$$

$$M = \int_0^{2\pi}\int_0^{r}(\rho^2 - 2\rho r\cos\theta + r^2).\rho.d\rho.d\theta = \int_0^{2\pi}\int_0^{r}(\rho^3 - 2\rho^2 r.\cos\theta + r^2.\rho).d\rho.d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^4}{4} - \frac{2}{3}r^4.\cos\theta + \frac{r^4}{2}\right).d\theta = \frac{3}{4}r^4.2\pi$$

$$\iint_D x.dm = \iint_{D\rho\theta} \rho.\cos\theta.\rho.(\rho^2 - 2\rho r\cos\theta + r^2)\,d\rho\,d\theta = \int_0^{2\pi}\int_0^{r}(\rho^4.\cos\theta - 2\rho^3 r\cos^2\theta + r^2\rho^2\cos\theta)\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^5}{5}.\cos\theta - \frac{2}{4}r^5\cos^2\theta + \frac{r^5}{3}\cos\theta\right)d\theta \; ; \; \int\cos^2\theta.d\theta = \int\frac{\cos 2\theta + 1}{2}d\theta = \frac{1}{4}\operatorname{sen}2\theta + \frac{1}{2}\theta + k$$

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$$\iint_D x\,dm = \int_0^{2\pi}\left(\frac{8}{15}r^5\cos\theta + \frac{r^5}{2}\cdot\cos^2\theta\right) = 0 + 0 + \frac{r^5}{4}\cdot 2\pi$$

$$\iint_D y\,dm = \iint_{D\rho,\theta} \rho\cdot\operatorname{sen}\theta\cdot\rho\cdot(\rho^2 - 2\rho r\cos\theta + r^2)\,d\rho\,d\theta$$

$$= \int_0^{2\pi}\int_0^{r}(\rho^4\operatorname{sen}\theta - 2\rho^3 r\cdot\operatorname{sen}\theta\cdot\cos\theta + r^2\cdot\rho^2\cdot\operatorname{sen}\theta)\,d\rho\,d\theta$$

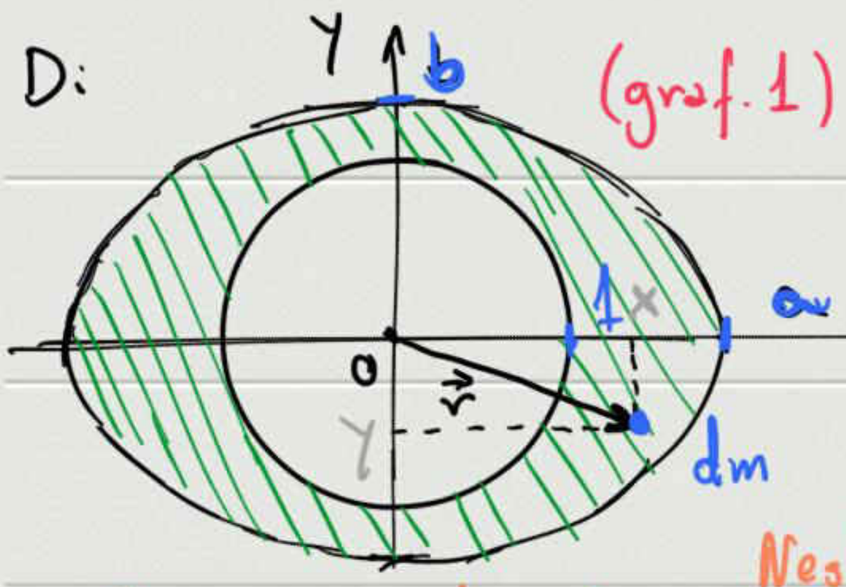
$$= \int_0^{r}\int_0^{2\pi}(\rho^4\operatorname{sen}\theta - \rho^3\cdot r\operatorname{sen}2\theta + r^2\rho^2\operatorname{sen}\theta)\,d\theta\,d\rho = \int_0^{r} 0\,d\rho = 0$$

$$\therefore \vec{R} = \frac{1}{\frac{3}{4}r^4\cdot 2\pi}\cdot\left(\frac{r^5}{4}\cdot 2\pi, 0\right) \Leftrightarrow \boxed{\vec{R} = \left(\frac{r}{3}, 0\right)\ u.c.}$$

14 c) $D = \{(x,y)\in\mathbb{R} \mid x^2+y^2 \geq 1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\}$ $a,b\in\mathbb{R}$, $a>1$ e $b>1$ $\quad \delta(x,y) = 1$

$I_o$: Momento de Inércia em relação à origem

D: (graf. 1)




$$dI_o = dm\cdot r^2 = dm(x^2+y^2)$$

$$dI_o = dm\,x^2 + dm\,y^2$$

$$I_o = \iint_D x^2\,dm + \iint_D y^2\,dm = I_y + I_x$$

Neste conjunto D, fica horrível isolar uma variável para estabelecer os limites de integração, vai precisar de 4 integrais para calcular o momento de inércia

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\operatorname{sen}\theta \end{cases}$$

$x^2+y^2 \geq 1 \Leftrightarrow \boxed{r\geq 1}$ $(r\geq 0)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1 \Leftrightarrow b^2x^2 + a^2y^2 \leq a^2b^2 \Leftrightarrow b^2r^2\cdot\cos^2\theta + a^2r^2\operatorname{sen}^2\theta \leq (ab)^2$

$$r^2\left(\underbrace{(b^2-a^2)}_{>0}\cos^2\theta + a^2\right) \leq (ab)^2 \Leftrightarrow r^2 \leq \frac{(ab)^2}{(b^2-a^2)\cos^2\theta + a^2}$$

$$\therefore \boxed{1\leq r\leq \frac{ab}{\sqrt{(b^2-a^2)\cos^2\theta + a^2}}} \quad ; \quad \boxed{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r}$$

$$\therefore D_{r,\theta} = \left\{(r,\theta)\in \mid 1\leq r\leq \frac{ab}{\sqrt{(b^2-a^2)\cos^2\theta + a^2}} \text{ e } 0\leq\theta\leq 2\pi\right\}$$

$$I_y = \iint_D x^2\,dm = \iint_{Dr,\theta} r^2\cos^2\theta\cdot r\cdot\delta(r,\theta)\,dr\,d\theta = \int_0^{2\pi}\int_1^{\frac{ab}{\sqrt{(b^2-a^2)\cos^2\theta + a^2}}} r^3\cos^2\theta\,dr\,d\theta$$

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$$x^2+y^2 \geq 1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1$$

$$\begin{cases} x = ar.\cos\theta \\ y = br.\operatorname{sen}\theta \end{cases} \qquad ab.r = \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$r^2 \leq 1$$

$$a^2r^2.\cos^2\theta + b^2\operatorname{sen}^2\theta.r^2 \geq 1$$

$$r^2 \geq \frac{1}{a^2\cos^2\theta + b^2\operatorname{sen}^2\theta}$$

$$u = \frac{x}{a} \qquad v = \frac{y}{b} \qquad u^2+v^2 \leq 1$$

$$a^2u^2 + b^2v^2 \geq 1$$

O máximo que cheguei usando umas fatorizações russas.

$$2a^4b^2\int \frac{(1+tg^2\theta)^2}{\left(\frac{a^2}{b^2}tg^2\theta+1\right)^2}.d\theta$$ , se alguém quiser arriscar

Já digo que o bizu está em como se começa, eu tentei encontrar um bom começo, mas cansei.

(27)

$$I_y = \int_0^{2\pi} \frac{1}{4} \cdot \cos^2\theta \left[ \left( \frac{ab}{((b^2-a^2)\cos^2\theta + a^2)^{1/2}} \right)^4 - 1^4 \right] \cdot d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \frac{(ab)^4}{((b^2-a^2)\cos^2\theta + a^2)^2} \cdot \cos^2\theta \cdot d\theta - \frac{1}{4} \int_0^{2\pi} \cos^2\theta \cdot d\theta$$

$$= \frac{1}{4}$$

$$\gamma = (b^2-a^2)\cos^2\theta + a^2 \quad \Rightarrow \quad \boxed{\cos^2\theta = \frac{\gamma - a^2}{(b^2-a^2)}}$$

$$d\gamma = (b^2-a^2)\, 2 \cdot \cos\theta \cdot \operatorname{sen}\theta \cdot d\theta$$

$$d\gamma = (b^2-a^2)\, \operatorname{sen} 2\theta \cdot d\theta$$

$$\boxed{(b^2-a^2)\frac{\cos 2\theta + 1}{2} + a^2}$$

$$\int \frac{\cos^2\theta}{(b^2-a^2)\cdot\cos^2\theta + a^2} \cdot d\theta$$

$$\int \frac{1}{(b^2-a^2) + a^2 \sec^2\theta} \cdot d\theta \qquad \sec^2\theta = \operatorname{tg}^2\theta + 1$$

$$\int \frac{1}{b^2 + a^2(\sec^2\theta - 1)} \cdot d\theta \qquad \gamma = k^2 + \operatorname{tg}^2\theta$$

$$\int \frac{1}{b^2 + a^2\operatorname{tg}^2\theta} \cdot d\theta = \frac{1}{a^2}\int \frac{1}{\left(\frac{b}{a}\right)^2 + \operatorname{tg}^2\theta} \cdot d\theta \qquad d\gamma = 2\operatorname{tg}\theta \sec^2\theta \cdot d\theta$$

$$\int \frac{1}{k^2+\gamma^2} \cdot \frac{1}{1+\gamma^2} \cdot d\gamma \qquad \int \frac{1}{k^2+\operatorname{tg}^2\theta} \cdot d\theta$$

$$\frac{1}{k^2} \int \frac{1}{1+\frac{\gamma^2}{k^2}} \cdot \frac{1}{1+\gamma^2} \cdot d\gamma \qquad \gamma = \operatorname{tg}\theta$$

$$d\gamma = \sec^2\theta \cdot d\theta$$

$$d\gamma = (1+\operatorname{tg}^2\theta)\, d\theta$$

$$d\theta = \frac{d\gamma}{1+\operatorname{tg}^2\theta}$$

$$\gamma = \frac{b\operatorname{tg}\theta}{a}$$

$$d\gamma = \frac{b}{a}\sec^2\theta\, d\theta$$

$$d\theta = \frac{a}{b}\frac{d\gamma}{1+\operatorname{tg}^2\theta}$$

$$u = k^2 + \gamma^2$$

$$du = 2\gamma \cdot d\gamma$$

$$= \frac{}{b^2\cos^2\theta + a^2\operatorname{sen}^2\theta}$$

$$\left(\frac{1}{b^2 + a^2\operatorname{tg}^2\theta}\right)' = \cdot$$

$$= \frac{a^2\operatorname{tg}\theta \cdot \sec^2\theta}{(b^2 + a^2\operatorname{tg}^2\theta)^2} \cdot d\theta$$

$$u = \operatorname{sen}\theta$$

$$du = -\cos\theta \cdot d\theta$$

(15) a) $\int_0^1 \int_a^z \int_0^y xyz \, dx\,dy\,dz = \int_0^1 \int_a^z \frac{y^3}{2} \cdot z \, dy\,dz = \int_a^1 \frac{1}{6} \cdot z^4 \cdot z \, dz$

$= \frac{1}{48}$

b) $\int_a^\pi \int_a^2 \int_0^{\sqrt{4-z^2}} z \cdot \operatorname{sen} y \cdot dx\,dz\,dy = \int_a^\pi \int_0^2 \sqrt{4-z^2} \cdot z \cdot \operatorname{sen} y \, dz\,dy$

$u = 4 - z^2 \Leftrightarrow du = -2z\,dz \Leftrightarrow dz = \frac{du}{-2z}$

$\int_0^\pi \int_a^2 \sqrt{4-z^2} \cdot z \cdot \operatorname{sen} y \, dz\,dy = \int_a^\pi \int_4^0 \frac{u^{1/2}}{-2} \cdot \operatorname{sen} y \, du\,dy = \int_a^\pi \frac{2}{3} \cdot \frac{(0^{3/2} - (4)^{3/2})}{-2} \cdot \operatorname{sen} y \, dy$

$= \frac{8}{3} \cdot (\cos\pi - \cos 0) = -\frac{16}{3}$

(16) a) $\iiint_D yz \, dx\,dy\,dz \qquad D = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le 1, \ 0 \le y \le 2z, \ 0 \le x \le z+2\}$

$\iiint_D yz \, dx\,dy\,dz = \int_a^1 \int_0^{2z} \int_0^{z+2} y \cdot z \, dx\,dy\,dz = \int_a^1 \int_a^{2z} (z+2) yz \, dy\,dz = \int_a^1 (z+2) z \frac{4z^2}{2} \, dz$

$\int_a^1 (2z^4 + 4z^3) \, dz = \frac{2}{5} + 1 = \frac{7}{5}$

b) $\iiint_D y \, dx\,dy\,dz \qquad D = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le x+2y, \ 0 \le y \le x^2, \ 0 \le x \le 1\}$

$\iiint_D y \, dx\,dy\,dz = \int_a^1 \int_0^{x^2} \int_a^{x+2y} y \, dz\,dy\,dx = \int_a^1 \int_a^{x^2} (x+2y) \cdot y \, dy\,dx =$

$\int_0^1 \int_0^{x^2} (xy + 2y^2) \, dy\,dx = \int_a^1 \left( \frac{x}{2}((x^2)^2 - 0^2) + \frac{2}{3}(x^2)^3 \right) dx = \int_0^1 \left( \frac{x^5}{2} + \frac{2}{3}x^6 \right) dx$

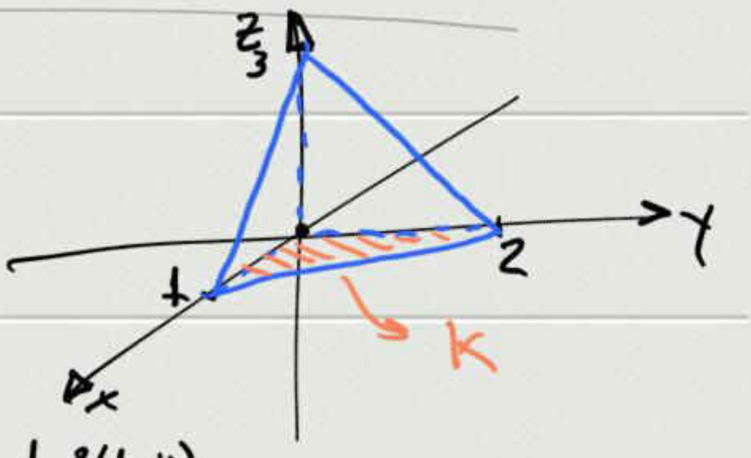
$\frac{1}{12} + \frac{2}{21} = \frac{7 + 2 \cdot 4}{84} = \frac{15}{84}$

c) $E = \iiint_D xy \, dx\,dy\,dz$

Equação segmentária do plano $\frac{1}{(x-x_0)} + \frac{1}{2}(y-y_0) + \frac{1}{3}(z-z_0) = 0$

$6x + 3y + 2z - 6 = 0 \Leftrightarrow 0 \le z \le \frac{6 - 6x - 3y}{2}$

definindo-se a região K: $0 \le y \le 2 - 2x, \quad 0 \le x \le 1$




$E = \int_0^1 \int_0^{2-2x} \int_0^{\frac{6-6x-3y}{2}} xy \, dz\,dy\,dx = \int_a^1 \int_0^{2(1-x)} \frac{6-6x-3y}{2} xy \, dy\,dx = \frac{1}{2} \cdot \int_a^1 \int_0^{2(1-x)} \left[ 3(1-x)xy - \frac{3}{2}xy^2 \right] dy\,dx =$

$\int_a^1 \left[ 3(1-x) \cdot x \frac{2^2}{2}(1-x)^2 - \frac{3}{2}x \frac{2^3}{3}(1-x)^3 \right] dx = \int_0^1 (6-4)x(1-x)^3 \, dx =$

$2 \int_a^1 (x - 3x^2 + 3x^3 - x^4) \, dx = 2\left[ \frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5} \right] = 2\left[ \frac{1}{4} - \frac{1}{5} \right] = \frac{1}{10}$

(29)

d) $\iiint_D z\,dx\,dy\,dz$, $D=\{(x,y,z)\in\mathbb{R}^3 \mid x\geq 0, y\geq 0, z\geq 0,$

$z=1-y \quad x=1-z\}$ $\therefore D=\{(x,y,z)\in\mathbb{R}^3 \mid 0\leq x\leq 1-z, 0\leq z\leq 1-y,$

$0\leq y\leq 1\}$ $\therefore \iiint_D z\,dx\,dy\,dz = \int_0^1\int_0^{1-y}\int_0^{1-z} z\,dx\,dz\,dy = \int_0^1\int_0^{1-y}(1-z).z\,dz\,dy$

$= \int_0^1\int_0^{1-y}(z-z^2)\,dz\,dy = \int_0^1\left(\frac{(1-y)^2}{2}-\frac{1}{3}(1-y)^3\right).dy = \int_0^1\left(\frac{1-2y+y^2}{2}-\frac{(1-3y+3y^2-y^3)}{3}\right).dy$

$= \int_0^1\left(\frac{1}{6}-\frac{y^2}{2}+\frac{y^3}{3}\right).dy = \frac{1}{6}-\frac{1}{6}+\frac{1}{4.3} = \frac{1}{12}$

e) $\iiint_D x\,dx\,dy\,dz \quad D=\{(x,y,z)\in\mathbb{R}^3 \mid 4y^2+4z^2\leq x\leq 4\}$

melhor trabalhar com coordenadas cilíndricas

$\begin{cases} x = x \\ y = r.\cos\theta \\ z = r.\operatorname{sen}\theta \end{cases}$ → eixo axial $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,x)}\right| = \left|\det\begin{bmatrix}\nabla x(r,\theta,x)\\ \nabla y(r,\theta,x)\\ \nabla z(r,\theta,x)\end{bmatrix}\right| = \left|\det\begin{bmatrix}0 & 0 & 1\\ \cos\theta & -r.\operatorname{sen}\theta & 0\\ \operatorname{sen}\theta & r.\cos\theta & 0\end{bmatrix}\right|$

$= |r\cos^2\theta + r.\operatorname{sen}^2\theta| = |r| = r$

$4y^2+4z^2 = 4.r^2 \leq x \leq 4 \Rightarrow 4r^2\leq 4 \Leftrightarrow 0\leq r\leq 1$

$\int_0^{2\pi}\int_0^1\int_{4r^2}^4 x.r.dx.dr.d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(4^2-(4r^2)^2)\,r\,dr.d\theta = \int_0^{2\pi}\int_0^1 (8-8r^4).r.dr.d\theta =$

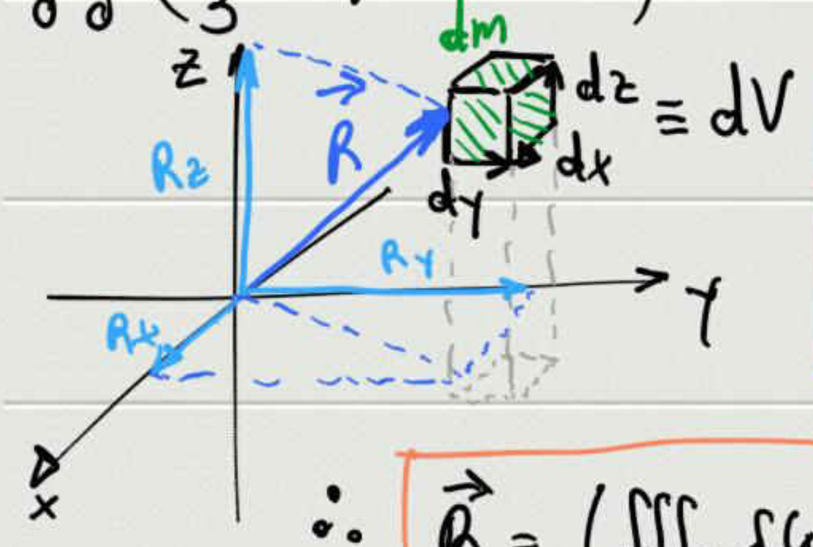
$\int_0^{2\pi}\int_0^1 (8r-8r^5).dr.d\theta = \int_0^{2\pi} 8.\left(\frac{1}{2}-\frac{1}{6}\right).d\theta = 2\pi.8.\frac{1}{3} = \frac{16\pi}{3}$

(17) $Q=[0,a]\times[0,a]\times[0,a] \Leftrightarrow 0\leq x\leq a, 0\leq y\leq a, 0\leq z\leq b$ $\quad dm=\delta.dV$

$\delta(x,y,z) = x^2+y^2+z^2$

$M = \iiint_Q dm = \iiint_Q \delta(x,y,z).dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2).dx\,dy\,dz =$

$\int_0^a\int_0^a\left(\frac{a^3}{3}+(y^2+z^2).a\right)dy\,dz = \int_0^a\left(\frac{a^4}{3}+\frac{a^4}{3}+z^2a^2\right)dz = \frac{2}{3}a^5+\frac{a^3}{3}.a^2 = a^5$ u.m.




$\vec{R} = \frac{\sum_{i=1}^n dm_i.\vec{r}_i}{\sum_{i=1}^n dm_i}$ $\quad R_x = \frac{\sum_{i=1}^n dm_i.x_i}{\sum_{i=1}^n dm_i}$, $R_y = \frac{\sum_{i=1}^n dm_i\,y_i}{\sum_{i=1}^n dm_i}$ ↷ $R_{z\cdots}$ $z_i$

$\therefore \vec{R} = \left(\frac{\iiint_Q \delta(x,y,z).x.dx\,dy\,dz}{M}, \frac{\iiint_Q \delta(x,y,z)\,y.dx\,dy\,dz}{M}, \frac{\iiint_Q \delta(x,y,z)\,z\,dx\,dy\,dz}{M}\right)$

(30)

$$R_x = \frac{1}{M}\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2).x\,dxdydz = \frac{1}{M}\int_0^a\int_0^a\int_0^a (x^3+xy^2+xz^2)\,dxdydz =$$

$$\frac{1}{M}\int_0^a\int_0^a \left(\frac{a^4}{4} + \frac{a^2}{2}\cdot y^2 + \frac{a^2}{2}z^2\right)dy\,dz = \frac{1}{a^5}\int_0^a \left(\frac{a^4}{4}\cdot a + \frac{a^5}{6} + \frac{a^3}{2}z^2\right)dz = \frac{1}{a^5}\left(\frac{a^6}{4}+\frac{a^6}{6}+\frac{a^6}{6}\right)$$

$$= a\left(\frac{3+2+2}{12}\right) = \frac{7}{12}a \text{ u.c.}$$

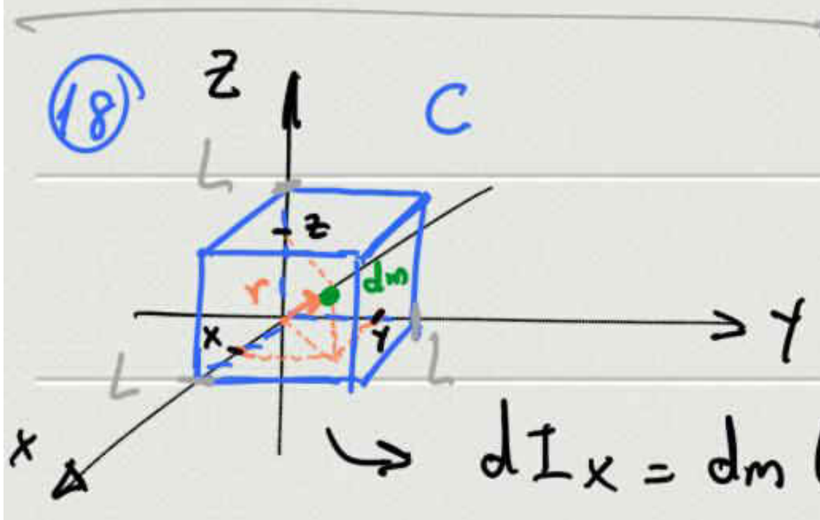
$$R_y = \frac{1}{M}\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)y\,dxdydz = \frac{1}{a^5}\int_0^a\int_0^a\int_0^a (y^3+yx^2+yz^2).dxdydz = \frac{7}{12}a \text{ u.c.}$$

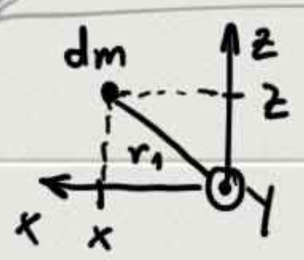
pela simetria da integral

$$R_z = \frac{1}{M}\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)z\,dxdydz = \frac{1}{a^5}\int_0^a\int_0^a\int_0^a (z^3+zx^2+zy^2).dxdydz = \frac{7}{12}a \text{ u.c.}$$

$$\therefore \vec{R} = \frac{7}{12}a(1,1,1) \text{ u.c.}$$

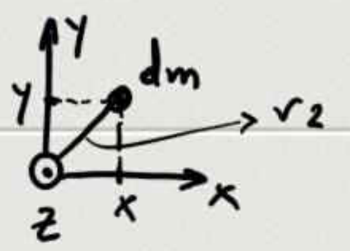
(18)




$dI_y = dm \cdot r_1^2$

$dI_y = dm\,(x^2+z^2)$




$dI_z = dm\,r_2^2$

$dI_z = dm\,(x^2+y^2)$

$\hookrightarrow dI_x = dm\,(z^2+y^2)$

$$I_x = \iiint_C (z^2+y^2).dm = \iiint_C (z^2+y^2).K.dxdydz = \int_0^L\int_0^L\int_0^L (z^2+y^2).K.dxdydz =$$

$$K\int_0^L\int_0^L L.(z^2+y^2).dy\,dz = K.L\int_0^L \left(L\,z^2 + \frac{L^3}{3}\right)dz = KL.\left(\frac{L^4}{3}+\frac{L^4}{3}\right) = \frac{2}{3}KL^5 \text{ u.m. u.c.}^2$$

$$I_y = \iiint_C (z^2+x^2)\,dm = \int_0^L\int_0^L\int_0^L (z^2+x^2)K\,dxdydz = \frac{2}{3}KL^5 \text{ u.m. uc}^2$$

$$I_z = \iiint_C (z^2+x^2)\,dm = \int_0^L\int_0^L\int_0^L (y^2+x^2)K.dxdydz = \frac{2}{3}KL^5 \text{ u.m. uc}^2$$

simétricas

(19) a) $\iiint_E (x^2+y^2)\,dxdydz \qquad E = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le x^2+y^2\le 4\ ,\ -1\le z\le 2\}$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \\ z = z \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r\ ,\quad 0\le r^2\le 4 \Leftrightarrow 0\le r\le 2$$

r sempre maior que zero.

Se não há mais restrições $\Rightarrow 0\le\theta\le 2\pi$

$$\iiint_E (x^2+y^2)\,dxdydz = \iiint_{E\,r,\theta,z} r^2.r.\,drd\theta dz = \int_{-1}^{2}\int_0^{2\pi}\int_0^2 r^3.dr.d\theta.dz =$$

$$\int_{-1}^{2}\int_0^{2\pi} \frac{2^4}{4}.d\theta.dz = 2\pi.4.(2-(-1)) = 24\pi$$

31

b) $\iiint_E y\,dx\,dy\,dz \qquad E=\{(x,y,z)\in\mathbb{R}^3 \mid 1\le x^2+y^2\le 4 \text{ e } 0\le z\le x+2\}$

$$\begin{cases} x=r.\cos\theta \\ y=r.\operatorname{sen}\theta \\ z=z \end{cases} \qquad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \;,\quad 1\le r^2\le 4 \Leftrightarrow 1\le r\le 2 \;,\quad 0\le z\le r.\cos\theta+2$$
$$\text{e}\quad 0\le\theta\le 2\pi$$

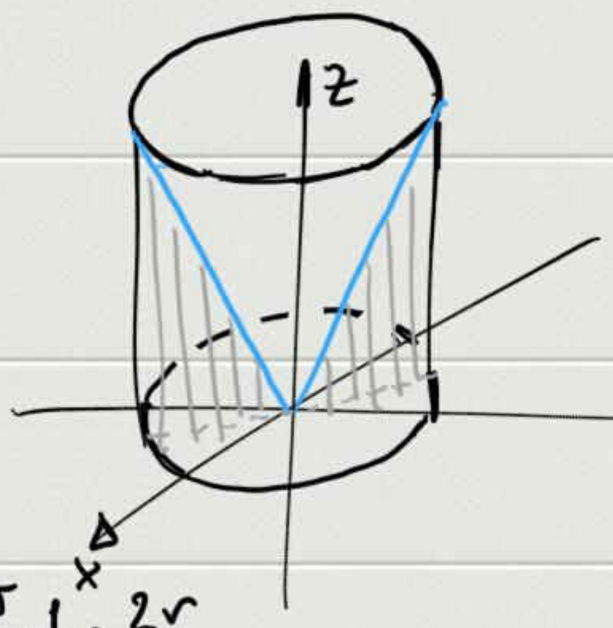
$$\iiint_E y\,dx\,dy\,dz = \iiint_{E(r,\theta,z)} r.\operatorname{sen}\theta\,.\,r\,.\,dr.d\theta\,dz = \int_0^{2\pi}\int_1^2\int_0^{r.\cos\theta+2} r^2.\operatorname{sen}\theta\,dz.dr.d\theta =$$

$$\int_0^{2\pi}\int_1^2 (r.\cos\theta+2).r^2.\operatorname{sen}\theta.dr.d\theta = \int_0^{2\pi}\int_1^2 (r^3.\cos\theta.\operatorname{sen}\theta+2r^2.\operatorname{sen}\theta).dr\,d\theta =$$

$$\int_0^{2\pi}\left(\frac{1}{4}(2^4-1^4).\frac{\operatorname{sen}2\theta}{2}+2.\frac{1}{3}.(2^3-1^3).\operatorname{sen}\theta\right).d\theta = \int_0^{2\pi}\frac{15}{8}.\underbrace{\operatorname{sen}2\theta}_{=0}.d\theta+\int_0^{2\pi}\frac{14}{3}.\underbrace{\operatorname{sen}\theta}_{=0}.d\theta =$$

$$\frac{15}{8}\left(\frac{\cos 4\pi-\cos 0}{2}\right)+\frac{14}{3}(\cos 2\pi-\cos 0) = 0$$

c) $\iiint_E x^2.dx\,dy\,dz \qquad E=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le x^2+y^2\le 1 \text{ e } 0\le z\le 4x^2+4y^2\}$




$$\begin{cases} x=r.\cos\theta \\ y=r.\operatorname{sen}\theta \\ z=z \end{cases} ; \left|\frac{\partial(x,y,z)}{\partial(r\theta,z)}\right| = r ; \quad 0\le r^2\le 1 \Leftrightarrow 0\le r\le 1$$

$$0\le z^2\le 4r^2 \Leftrightarrow 0\le z\le 2r \qquad 0\le\theta\le 2\pi$$
$$\text{e}\quad z\ge 0$$

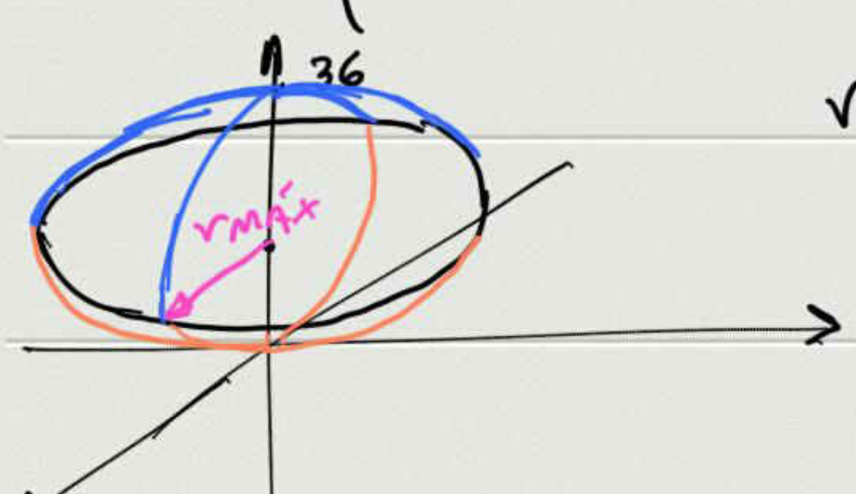
$$\iiint_E x^2.dx\,dy\,dz = \iiint_{E r,\theta,z} (r.\cos\theta)^2.\,r.\,dr\,d\theta\,dz =$$

$$\int_0^{2\pi}\int_0^1\int_0^{2r} r^3.\cos^2\theta.\,dz.dr.d\theta = \int_0^{2\pi}\int_0^1 2r^4.\cos^2\theta.dr.d\theta = \int_0^{2\pi}\frac{2}{5}.\cos^2\theta.d\theta =$$

$$\frac{2}{5}\left(\left[\frac{1}{2}.\operatorname{sen}\theta.\cos\theta\right]_0^{2\pi}+\frac{1}{2}\int_0^{2\pi}1.d\theta\right) = \frac{2}{5}(0+\pi) = \frac{2}{5}\pi$$

20 $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le z\le 36-3x^2-3y^2\}$

$$D_{r,\theta} \Rightarrow \begin{cases} x=r.\cos\theta \\ y=r.\operatorname{sen}\theta \\ z=z \end{cases} \qquad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \;;$$




$r=\sqrt{x^2+y^2}$, em $r_{máx} \Rightarrow \quad x^2+y^2 = 36-3x^2-3y^2$

$\Leftrightarrow \quad 4(x^2+y^2)=36 \Leftrightarrow \quad 4r^2=36 \Leftrightarrow r_{máx}=3$

$\therefore \quad 0\le r\le 3 \quad \text{e} \quad 0\le\theta\le 2\pi$

$$r^2\le z\le 36-3r^2$$

$$\iiint_D dx\,dy\,dz = \iiint_{D r,\theta,z} r.\,dr.\,d\theta\,dz$$

(32)

$$= \int_0^{2\pi}\int_a^3\int_{r^2}^{36-3r^2} r\,.\,dz\,.dr\,.d\theta = \int_0^{2\pi}\int_a^3 r.\left[36-3r^2-(r^2)\right].dr\,d\theta = \int_0^{2\pi}\int_0^3 (36r-4r^3).dr.d\theta =$$

$$\int_0^{2\pi}\left(\frac{36}{2}3^2-\frac{4}{4}3^4\right).d\theta = 2\pi\left(81\,(2-1)\right) = 162\pi \text{ u.c}^3$$

(21) $S = \{(x,y,z)\in\mathbb{R}^3 \mid 4x^2+4y^2 \le z \le a \mid a\in\mathbb{R}_+^*\}$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \\ z = z \end{cases} ; \quad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \; ; \quad \delta(x,y,z) = K$$

$4r^2 \le z \le a$

$x^2+y^2 = r$ ; limite de $r \Leftrightarrow 4r^2 \le a \Leftrightarrow 0 \le r \le \frac{\sqrt{a}}{2}$

$0 \le \theta \le 2\pi$

$$M = \iiint_S dm = \iiint_S \delta(x,y,z)\,dx\,dy\,dz = \iiint_{S_{r,\theta,z}} \delta(x,y,z).r.dr.d\theta\,dz =$$

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K.r.dz.dr.d\theta = \int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} K.(a-4r^2).r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} K\,(ar-4r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} K.\left(\frac{a}{2}.\frac{a}{4}-\frac{4}{4}.\frac{a^2}{16}\right)d\theta = 2\pi.K.\left(\frac{a^2}{8}-\frac{a^2}{16}\right) = K\pi\frac{a^2}{8} \text{ u.m.}$$

$$\vec{R} = \left(\frac{\iiint_S dm.x}{M}, \frac{\iiint_S .dm\,y}{M}, \frac{\iiint_S dm\,z}{M}\right)$$

$$\iiint_S dm\,x = \iiint_S x.K.dx\,dy\,dz = \iiint_{S_{r\theta z}} r\cos\theta.K.r\,dr\,d\theta\,dz =$$

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K.r^2.\cos\theta.dz.dr.d\theta = K\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}(a-4r^2).r^2.\cos\theta.dr.d\theta =$$

$$K\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}(a.r^2-4r^4).\cos\theta.dr.d\theta = K.\int_0^{2\pi}\left(\frac{a}{3}\left(\frac{\sqrt{a}}{2}\right)^3-\frac{4}{5}.\left(\frac{\sqrt{a}}{2}\right)^5\right).\cos\theta\,d\theta = 0$$

$$\iiint_S .dm\,y = \iiint_S y.K.dx\,dy\,dz = \iiint_{S_{r,\theta,z}} K.r.\operatorname{sen}\theta.r.dr.d\theta\,dz =$$

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K.r^2.\operatorname{sen}\theta.dz\,dr\,d\theta = \int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a}\int_0^{2\pi} K.r^2.\operatorname{sen}\theta.d\theta.dz.dr = \int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K.r^2.\left[-\cos\theta\right]_0^{2\pi}dz\,dr$$

$$= 0$$

$$\iiint_S dm\,z = \iiint_S z.K.dx\,dy\,dz = \iiint_{S_{r,\theta,z}} z.K.r.dr\,d\theta\,dz = \int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} z.Kr.dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} \frac{a^2-(4r^2)^2}{2}.K.r.dr.d\theta = \int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}} \frac{(a^2-16r^4).K.r.dr\,d\theta}{2} =$$

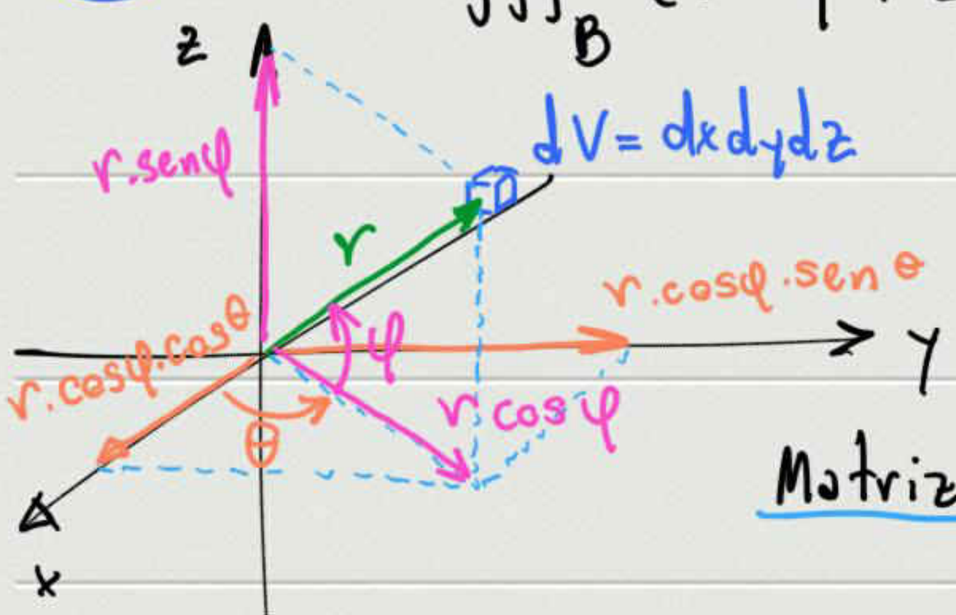
$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\left(\frac{a^2.Kr}{2}-8K.r^5\right).dr.d\theta = \int_0^{2\pi}\left(\frac{a^2K}{2}.\frac{1}{2}\left(\frac{\sqrt{a}}{2}\right)^2-\frac{8K}{6}\left(\frac{\sqrt{a}}{2}\right)^6\right).d\theta = \frac{2}{3}\frac{a^3K}{16}.2\pi$$

$$\therefore \frac{a^3K.\pi}{12}$$

(33)

$$R_y = \frac{\frac{a^3 \cdot \pi \cdot \pi}{12}}{\frac{\pi \cdot a^2}{8}} = \frac{2}{3}a \quad \therefore \boxed{\vec{R} = \left(0, 0, \frac{2}{3}a\right) \mu.c.}$$

(22) a) $\iiint_B (x^2+y^2+z^2)\,dxdydz \qquad B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$




parametrização

$$\begin{cases} x(r,\theta,\varphi): x = r.\cos\varphi.\cos\theta \\ y(r,\theta,\varphi): y = r.\cos\varphi.\operatorname{sen}\theta \\ z(r,\theta,\varphi): z = r.\operatorname{sen}\varphi \end{cases}$$

Matriz Jacobiana : $J = \begin{bmatrix} \nabla x(r,\theta,\varphi) \\ \nabla y(r,\theta,\varphi) \\ \nabla z(r,\theta,\varphi) \end{bmatrix} \Leftrightarrow$

$$J = \begin{bmatrix} \cos\varphi.\cos\theta & r.\cos\varphi(-\operatorname{sen}\theta) & r.(-\operatorname{sen}\varphi).\cos\theta \\ \cos\varphi\operatorname{sen}\theta & r.\cos\varphi\cos\theta & r.(-\operatorname{sen}\varphi).\operatorname{sen}\theta \\ \operatorname{sen}\varphi & 0 & r.\cos\varphi \end{bmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = |\det J| = |r^2.\cos^3\varphi.\cos^2\theta + r^2.\cos\varphi.\operatorname{sen}^2\varphi.\operatorname{sen}^2\theta + r^2.\cos\varphi.\operatorname{sen}^2\varphi.\cos^2\theta$$

$$+ r^2.\cos^3\varphi.\operatorname{sen}^2\theta| = |r^2.\cos^3\varphi + r^2\cos\varphi.\operatorname{sen}^2\varphi| = |r^2.\cos\varphi|.|\cos^2\varphi + \operatorname{sen}^2\varphi| =$$

$r^2.|\cos\varphi|$, sendo $\varphi$ um ângulo entre $-\frac{\pi}{2}$ e $\frac{\pi}{2}$ $|\cos\varphi| = \cos\varphi$

$\forall\ \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\therefore \boxed{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2.\cos\varphi}$ Coordenadas esféricas

$$\therefore B_{r,\theta,\varphi} = \{r \in \mathbb{R}_+, \theta \in [0,2\pi], \varphi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \mid r^2 \leq 1\} \Leftrightarrow 0 \leq r \leq 1$$

$$\therefore \iiint_B (x^2+y^2+z^2)\,dxdydz = \iiint_{B\,r\theta\varphi} r^2.r^2.\cos\varphi.\,dr.d\theta\,d\varphi =$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^4.\cos\varphi.\,drd\varphi.d\theta = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}.\cos\varphi.\,d\varphi.d\theta = \int_0^{2\pi} \frac{1}{5}\left(\operatorname{sen}\tfrac{\pi}{2} - \operatorname{son}\left(-\tfrac{\pi}{2}\right)\right).d\theta =$$

$$\frac{2}{5}.2\pi = \frac{4\pi}{5}$$

b) $\iiint_E y^2\,dxdydz \qquad E = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, x^2+y^2+z^2 \leq 1\}$

Condições

$$\begin{cases} x = r.\cos\varphi.\cos\theta \\ y = r.\cos\varphi.\operatorname{sen}\theta \\ z = r.\operatorname{sen}\varphi \end{cases} \qquad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2.\cos\varphi$$

$r \geq 0$ e $\cos\varphi \geq 0$

$\therefore \boxed{0 \leq r \leq 1}$

$$\begin{matrix} Eq.1 \\ Eq.2 \end{matrix}\begin{cases} r.\cos\varphi.\cos\theta \geq 0 \\ r.\cos\varphi.\operatorname{sen}\theta \geq 0 \end{cases} \Leftrightarrow \quad Eq.1 \times Eq.2$$

$r^2.\cos^2\varphi.\operatorname{sen}\theta.\cos\theta \geq 0 \Leftrightarrow \operatorname{sen}\theta.\cos\theta \geq 0 \Leftrightarrow 2\operatorname{sen}\theta.\cos\theta \geq 0 \Leftrightarrow \operatorname{son} 2\theta \geq 0 \Leftrightarrow$

$0 \leq 2\theta \leq \pi \Leftrightarrow \boxed{0 \leq \theta \leq \frac{\pi}{2}}$, naturalmente $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

(34)

$$\therefore \iiint_E y^2 \cdot dx\,dy\,dz = \iiint_{E r,\theta,\varphi} (r\cdot\cos\varphi\cdot\operatorname{sen}\theta)^2 \cdot r^2\cdot\cos\varphi \cdot dr\,d\theta\,d\varphi =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 r^4\cdot\cos^3\varphi\cdot\operatorname{sen}^2\theta\cdot dr\,d\theta\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{5}\cdot\cos^3\varphi\cdot\operatorname{sen}^2\theta\,d\theta\,d\varphi \rightarrow \text{fórmula de recorrência}$$

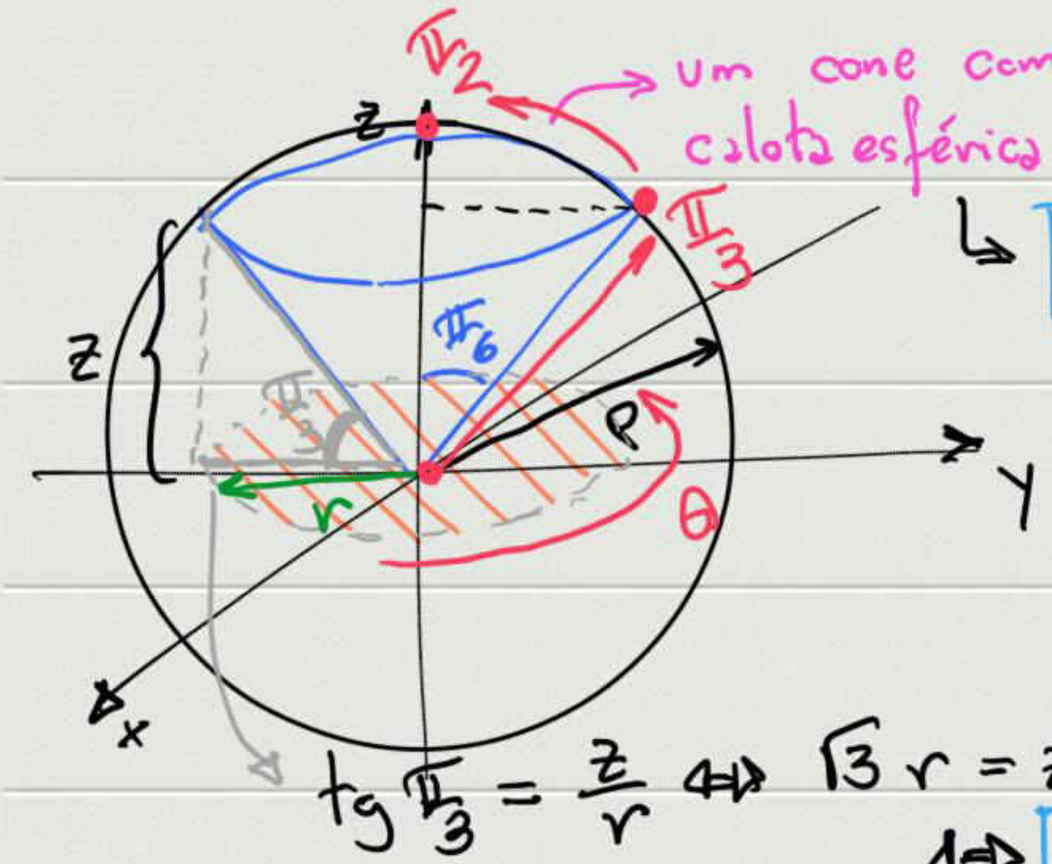
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cdot\cos^3\varphi\cdot\left(\left[-\frac{1}{2}\cdot\cos x\cdot\operatorname{sen}x\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\cdot d\theta\right)\cdot d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot\cos^3\varphi\cdot d\varphi = \rightarrow$$

$$\frac{\pi}{20}\cdot\left(\left[\frac{\operatorname{sen}\varphi\cdot\cos^2\varphi}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\cdot d\varphi\right) = \frac{\pi}{20}\left(0 + \frac{2}{3}\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\left(-\frac{\pi}{2}\right)\right)\right) = \frac{\pi}{20}\cdot\frac{4}{3}$$

$= \frac{\pi}{15}$ falar primeiro quadrante implica 2 octantes, se o enunciado falar primeiro octante, então $z \geq 0 \Leftrightarrow r\cdot\operatorname{sen}\varphi \geq 0 \Leftrightarrow 0 \leq \varphi \leq \frac{\pi}{2}$ que consequentemente é metade da resposta anterior $\left(\frac{\pi}{30}\right)$, eu posso estar errado talvez, se alguém puder confirmar para mim.

c) $\iiint_E \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \qquad E = \{(x,y,z) \in \mathbb{R}^3 \mid \dots$

Segue o seguinte modelamento matemático




(casquinha de sorvete)

$$\text{tg}\,\frac{\pi}{3} = \frac{z}{r} \Leftrightarrow \sqrt{3}\,r = z$$

$$\Leftrightarrow \boxed{z = \sqrt{3}\cdot\sqrt{x^2+y^2}}$$

$$\hookrightarrow \boxed{x^2+y^2+z^2 = \rho}$$

qualquer ponto do círculo em laranja pode ser representado por $r = \sqrt{x^2+y^2}$, um eixo do plano xy. Faz-se uma operação análoga a uma reta para determinar a inclinação da geratriz do cone.

sem a substituição para coordenadas esféricas fica até difícil descrever algebricamente o lugar geométrico. Em coordenadas esféricas é muito simples, é apenas a circunferência cujo raio varia de zero à $\rho$, $\theta$ de zero à $2\pi$ e $\varphi$, $\frac{\pi}{3}$ à $\frac{\pi}{2}$ segundo meu modelamento matemático

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{\rho} r\cdot r^2\cdot\cos\varphi\cdot dr\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\rho^4}{4}\cdot\cos\varphi\cdot d\varphi\,d\theta = \int_0^{2\pi} \frac{\rho^4}{4}\cdot\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{3}\right)\cdot d\theta$$

$$= 2\pi\,\frac{\rho^4}{4}\cdot\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{4}\rho^4\left(2-\sqrt{3}\right)$$

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23) $M = \iiint_H dm$ , $H = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0 \ , \ x^2+y^2+z^2 \leq a \mid a \in \mathbb{R}, a>0\}$ , $\delta(x,y,z) = K.\sqrt{x^2+y^2+z^2}$

$$\begin{cases} x = r.\cos\varphi.\cos\theta \\ y = r.\cos\varphi.\operatorname{sen}\theta \\ z = r.\operatorname{sen}\varphi \end{cases} ; \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2.\cos\varphi \ ; \ r^2 \leq a \Leftrightarrow 0 \leq r \leq a$$

$z \geq 0 \Leftrightarrow r.\operatorname{sen}\varphi \geq 0 \Leftrightarrow$

$0 \leq \theta \leq 2\pi$ $\qquad$ $0 \leq \varphi \leq \frac{\pi}{2}$

$$M = \iiint_H K.\delta(x,y,z).dxdydz = \iiint_{H\,r,\theta,\varphi} K.r.r^2\cos\varphi.dr.d\theta.d\varphi =$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} K.r^3.\cos\varphi.dr.d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} K.\frac{a^4}{4}.\cos\varphi.d\varphi d\theta = \int_0^{2\pi} K.\frac{a^4}{4}.\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}0\right).d\theta$$

$$= 2\pi.\frac{Ka^4}{4} = \frac{\pi.Ka^4}{2}$$

24) $E = \left\{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1 \mid a,b,c \in \mathbb{R}_+^*\right\}$

$$V_E = \iiint_E dV = \iiint_E dxdydz \ ; \ \begin{cases} \frac{x}{a} = r.\cos\varphi.\cos\theta \\ \frac{y}{b} = r.\cos\varphi.\operatorname{sen}\theta \\ \frac{z}{c} = r.\operatorname{sen}\varphi \end{cases} \Leftrightarrow \begin{cases} x = a.r.\cos\varphi.\cos\theta \\ y = b.r.\cos\varphi.\operatorname{sen}\theta \\ z = c.r.\operatorname{sen}\varphi \end{cases}$$

percebe-se que (pelo fato de a, b e c serem constantes) ao se fazer as derivadas parciais, a, b e c estarão multiplicando cada gradiente:

$$\det J = \det\begin{bmatrix} \nabla x(r,\theta,\varphi) \\ \nabla y(r,\theta,\varphi) \\ \nabla z(r,\theta,\varphi) \end{bmatrix} = \det\begin{bmatrix} a.\nabla\frac{x}{a}(r,\theta,\varphi) \\ b.\nabla\frac{y}{b}(r,\theta,\varphi) \\ c.\nabla\frac{z}{c}(r,\theta,\varphi) \end{bmatrix} =$$

$$a.\det\begin{bmatrix} \nabla\frac{x}{a}(r,\theta,\varphi) \\ b\nabla\frac{y}{b}(r,\theta,\varphi) \\ c\nabla\frac{z}{c}(r,\theta,\varphi) \end{bmatrix} = a.b.c\det\begin{bmatrix} \nabla\frac{x}{a}(r,\theta,\varphi) \\ \nabla\frac{y}{b}(r,\theta,\varphi) \\ \nabla\frac{z}{c}(r,\theta,\varphi) \end{bmatrix} = a.b.c.\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = a.b.c.r^2.\cos\varphi$$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1 \Leftrightarrow r^2 \leq 1 \Leftrightarrow 0 \leq r \leq 1$ , $0 \leq \theta \leq 2\pi$ , $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$$\iiint_E dedydz = \iiint_{E\,r,\theta,\varphi} a.b.c\,r^2\cos\varphi.\ drd\theta d\varphi = abc\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{1} r^2.\cos\varphi.dr.d\varphi d\theta =$$

$$abc\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}.\cos\varphi.d\varphi.d\theta = abc\int_0^{4\pi}\frac{1}{3}.\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\left(\frac{-\pi}{2}\right)\right) = abc.2\pi.\frac{2}{3}$$

$$= \frac{4\pi}{3}.abc$$

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(25) $\iiint_E x\,dx\,dy\,dz \qquad E=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{4}+\frac{y^2}{9}+z^2\le 1,\ x\ge 0\}$

$$\begin{cases} x = 2r.\cos\varphi.\cos\theta \\ y = 3r.\cos\varphi.\operatorname{sen}\theta \\ z = r.\operatorname{sen}\varphi \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = 6.r^2\cos\varphi \quad , \ x\ge 0 \Leftrightarrow 2r.\cos\varphi.\cos\theta \ge 0 \Leftrightarrow$$

$\Leftrightarrow -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ ; $r^2\ge 1 \Leftrightarrow 0\le r\le 1$ ; $-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$\iiint_E x\,dx\,dy\,dz = \iiint_{E_{r,\theta,\varphi}} 2r.\cos\varphi.\cos\theta.6r^2.\cos\varphi\,dr\,d\theta\,d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 12r^3.\cos^2\varphi.\cos\theta\,dr\,d\varphi\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3.\cos^2\varphi.\cos\theta\,d\varphi\,d\theta = \longrightarrow \text{recorrência}$$

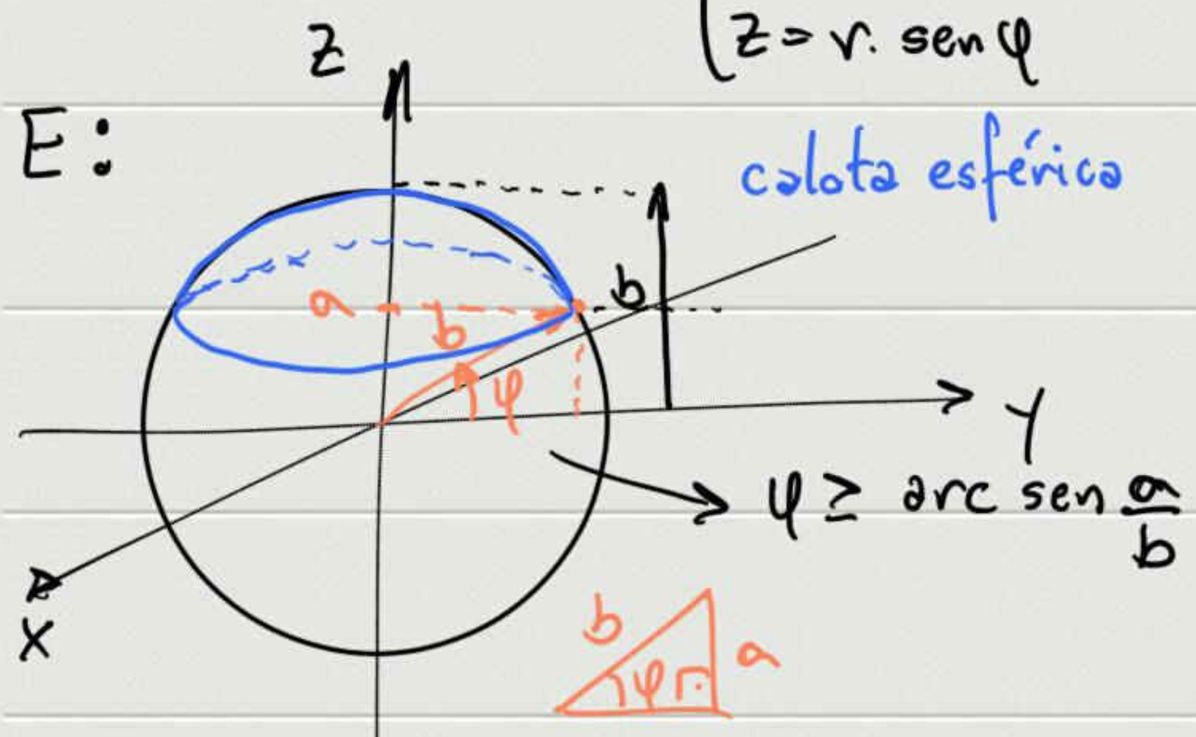
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3.\cos\theta\left(\left[\frac{\operatorname{sen}\varphi.\cos\varphi}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.\,d\varphi\right)d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3.\cos\theta\left(0+\frac{1}{2}\pi\right)d\theta = \frac{3\pi}{2}\left(\operatorname{sen}\frac{\pi}{2}-\operatorname{sen}\left(-\frac{\pi}{2}\right)\right)$$

$= 3\pi$

(26) $E=\{(x,y,z)\in\mathbb{R}^3 \mid \underbrace{x^2+y^2+z^2\le b^2}_{\text{esfera}} \ , \ z\ge a>0 \mid a,b\in\mathbb{R}\ ,\ a<b\}$

$\delta(x,y,z)=z$ ;
$$\begin{cases} x = r.\cos\varphi.\cos\theta \\ y = r.\cos\varphi\operatorname{sen}\theta \\ z = r.\operatorname{sen}\varphi \end{cases} ; \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2.\cos\varphi \ ; \quad r.\ge\frac{a}{\operatorname{sen}\varphi}$$

E:




calota esférica

verifica-se que:

$\frac{a}{\operatorname{sen}\varphi}\le r\le b$ ; $\operatorname{arc\,sen}\frac{a}{b}\le\varphi\le\frac{\pi}{2}$

$0\le\theta\le 2\pi$

$\varphi\ge\operatorname{arc\,sen}\frac{a}{b}$

$\cos 2\varphi = 1-2\operatorname{sen}^2\varphi$

$$m = \iiint_E \delta(x,y,z).\,dx\,dy\,dz = \iiint_E \delta(r,\theta,\varphi).r^2.\cos\varphi.\,dr.d\theta.d\varphi =$$

$$\int_0^{2\pi}\int_{\operatorname{arcsen}\frac{a}{b}}^{\frac{\pi}{2}}\int_{\frac{a}{\operatorname{sen}\varphi}}^{b} r.\operatorname{sen}\varphi.r^2.\cos\varphi\,dr\,d\varphi\,d\theta = \int_0^{2\pi} d\theta.\int_{\operatorname{arcsen}\frac{a}{b}}^{\frac{\pi}{2}}\frac{\operatorname{sen}2\varphi}{2}.\frac{1}{4}\left(b^4-\frac{a^4}{\operatorname{sen}^4\varphi}\right).d\varphi =$$

$$2\pi.\frac{1}{8}\left(\int_{\operatorname{arc\,sen}\frac{a}{b}}^{\pi/2} b^4.\operatorname{sen}2\varphi\,.d\varphi - \int_{\operatorname{arc\,sen}\frac{a}{b}}^{\frac{\pi}{2}} 2\operatorname{sen}\varphi.\cos\varphi.\frac{a^4}{\operatorname{sen}^4\varphi}\,d\varphi\right) = \frac{\pi}{4}\left(b^4\left[\left(\frac{-1}{2}\right)\cos 2\varphi\right]_{\operatorname{arc\,sen}\frac{a}{b}}^{\pi/2}\right.$$

$$\left. -2\int_{\operatorname{arc\,sen}\frac{a}{b}}^{\frac{\pi}{2}} a^4\cdot\frac{\cos\varphi}{\operatorname{sen}^3\varphi}.d\varphi\right) = \frac{\pi}{4}\left(\frac{b^4}{2}\left(1+1-2\frac{a^2}{b^2}\right)-2a^4\int_{a/b}^1\frac{1}{t^3}.dt\right)$$

$t=\operatorname{sen}\varphi$ ; $dt=\cos\varphi.d\varphi$ ; $t\Big|_{\frac{a}{b}}^{1}$

$$= \frac{\pi}{4}\left(b^4\left(\frac{b^2-a^2}{b^2}\right)-2a^4\frac{1}{-2}.\left(1-\frac{b^2}{a^2}\right)\right)$$

$$= \frac{\pi}{4}\left[b^2(b^2-a^2)-a^4\left(\frac{b^2-a^2}{a^2}\right)\right] = \frac{\pi}{4}(b^2-a^2)^2 \ u.m.$$

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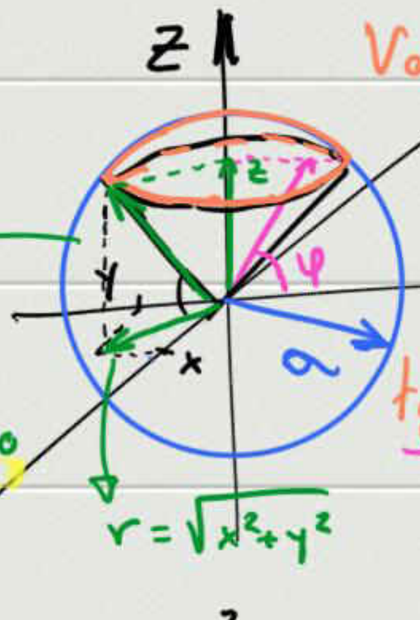
27) a)




Volume da casquinha de sorvete com sorvete

$$\begin{cases} x = \rho\cos\varphi\cdot\cos\theta \\ y = \rho\cdot\cos\varphi\cdot\operatorname{sen}\theta \\ z = \rho\cdot\operatorname{sen}\varphi \end{cases} \qquad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = \rho^2\cdot\cos\varphi$$

$tg\varphi_0 = \frac{z}{r}$

$z = r\cdot tg\varphi_0$

$z = \sqrt{x^2+y^2}\cdot tg\varphi_0$

equação do cone

$tg\varphi_0\sqrt{x^2+y^2} = \sqrt{x^2+y^2}$

$tg\varphi_0 = 1 \Leftrightarrow \boxed{\varphi_0 = \pi/4}$

Determina

$\therefore x^2+y^2+z^2 \le a^2 \Leftrightarrow \rho^2 \le a^2 \mid a\in\mathbb{R}_+^* \Leftrightarrow \underline{0\le\rho\le a}$

$\underline{\frac{\pi}{4}\le\varphi\le\frac{\pi}{2}}$

$\underline{0\le\theta\le 2\pi}$

$$\iiint_C dV = \iiint_C dx\,dy\,dz = \iiint_{C\,\rho\theta\varphi} \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| \cdot d\rho\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^a \rho^2\cdot\cos\varphi\cdot d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi \cdot \int_0^a \rho^2 =$$

$$2\pi\cdot\left(\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{4}\right)\cdot\frac{1}{3}a^3 = 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{3}a^3 = \frac{a^3\pi}{3}(2-\sqrt{2})\ u.v.$$

b) $z \ge \sqrt{x^2+y^2}$ , $x^2+y^2+z^2 \le 2az \mid a\in\mathbb{R}_+^*$ ; $\delta(x,y,z) = x^2+y^2$

esféricas

$$\begin{cases} x = r\cdot\cos\varphi\cdot\cos\theta \\ y = r\cdot\cos\varphi\cdot\operatorname{sen}\theta \\ z = r\cdot\operatorname{sen}\varphi \end{cases} ; \quad \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = \underline{r^2\cdot\cos\varphi} ; \quad r^2 \le 2a\cdot r\cdot\operatorname{sen}\varphi \Leftrightarrow$$

$r(r-2a\operatorname{sen}\varphi)\le 0 \Leftrightarrow \underline{0\le r\le 2a\operatorname{sen}\varphi}$

$r\cdot\operatorname{sen}\varphi \ge \sqrt{r^2\cdot\cos^2\varphi} \Leftrightarrow r\cdot\operatorname{sen}\varphi \ge r\cdot\cos\varphi \Leftrightarrow \underline{\frac{\pi}{4}\le\varphi\le\frac{\pi}{2}}$ , $\underline{0\le\theta\le 2\pi}$ ;

$$\iiint_R dm = \iiint_R \delta(x,y,z)\cdot dx\,dy\,dz = \iiint_R (x^2+y^2)\,dx\,dy\,dz = \iiint_{R\,r,\theta,\varphi} r^2\cos^2\varphi\cdot r^2\cos\varphi\,dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2a\operatorname{sen}\varphi} r^4\cdot\cos^3\varphi\,dr\,d\varphi\,d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{5}(2a\operatorname{sen}\varphi)^5\cdot\cos^3\varphi\,d\varphi\,d\theta = \frac{32a^5}{5}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{sen}^5\varphi\cdot\cos^3\varphi\,d\varphi\,d\theta$$

$$= \frac{32}{5}a^5\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{sen}^5\varphi(1-\operatorname{sen}^2\varphi)\cdot\cos\varphi\,d\varphi\,d\theta \ ; \quad u = \operatorname{sen}\varphi \Leftrightarrow du = \cos\varphi\,d\varphi \Leftrightarrow d\varphi = \frac{du}{\cos\varphi} \Rightarrow$$

$$\varphi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = u\Big|_{\frac{\sqrt{2}}{2}}^{1} \ ; \text{logo} = \frac{32}{5}a^5\int_0^{2\pi}\int_{\frac{\sqrt{2}}{2}}^{1} u^5(1-u^2)\cdot\cos\varphi\cdot\frac{du}{\cos\varphi}\,d\theta = \frac{32}{5}a^5\int_0^{2\pi}\int_{\frac{\sqrt{2}}{2}}^{1}(u^5-u^7)\,du\,d\theta =$$

$$\frac{32}{5}a^5\int_0^{2\pi}\frac{1}{6}\left(1^6-\left(\frac{1}{\sqrt{2}}\right)^6\right)-\frac{1}{8}\left(1^8-\left(\frac{1}{\sqrt{2}}\right)^8\right)d\theta = 2\pi\cdot\frac{32}{5}a^5\cdot\left[\frac{1}{6}\cdot\frac{8-1}{8}-\frac{1}{8}\cdot\frac{16-1}{16}\right] =$$

$$2\pi\cdot\frac{32}{5}a^5\left(\frac{1}{6}\cdot\frac{7}{8}-\frac{1}{8}\cdot\frac{15}{16}\right) = 2\pi\cdot\frac{32}{5}a^5\cdot\frac{11}{64\cdot 6} = \frac{11\cdot\pi}{30}a^5$$

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coordenadas cilíndricas

(28) a) $z(x^2+y^2)=2$, $z=0$, $x^2+y^2=1$, $x^2+y^2=2$, $x\geq 0$

$y\leq 0$ → para ser integrável, deve se ter um intervalo fechado.

$$R=\{(x,y,z)\in\mathbb{R}^3 \mid 0\leq z\leq \frac{2}{x^2+y^2};\ 1\leq x^2+y^2\leq 2,\ x\geq 0,\ y\leq 0\}$$

$\delta(x,y,z)=1$

$$x(r,\theta,z):\ y(r,\theta,z):\ z(r,\theta,z):\ \begin{cases} x=r.\cos\theta \\ y=r.\operatorname{sen}\theta \\ z=z \end{cases} \Leftrightarrow J=\begin{bmatrix}\nabla x(r,\theta,z)\\ \nabla y(r,\theta,z)\\ \nabla z(r,\theta,z)\end{bmatrix}=\begin{bmatrix}\cos\theta & r(-\operatorname{sen}\theta) & 0\\ \operatorname{sen}\theta & r.\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|=|\det J|=|r.\cos^2\theta+r.\operatorname{sen}^2\theta|=|r|=r$$

Restrições (determinação do lugar geométrico em coordenadas cilíndricas)

$x\geq 0 \Leftrightarrow r.\cos\theta\geq 0 \Leftrightarrow -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ ; $y\leq 0 \Leftrightarrow r.\operatorname{sen}\theta\leq 0 \Leftrightarrow -\pi\leq\theta\leq 0$

$\therefore\ -\frac{\pi}{2}\leq\theta\leq 0$ ; $1\leq x^2+y^2\leq 2 \Leftrightarrow 1\leq r^2\leq 2 \Leftrightarrow 1\leq r\leq\sqrt{2}$

$$R_{r,\theta,z}=\{r\in\mathbb{R}_+,\ \theta\in[0,2\pi],\ z\in\mathbb{R} \mid 1\leq r\leq\sqrt{2},\ 0\leq z\leq\frac{2}{r^2},\ -\frac{\pi}{2}\leq\theta\leq 0\}$$

$$M=\iiint_R dm=\iiint_R \delta(x,y,z).dx\,dy\,dz=\iiint_{R_{r,\theta,z}} 1.r.dr\,d\theta\,dz=$$

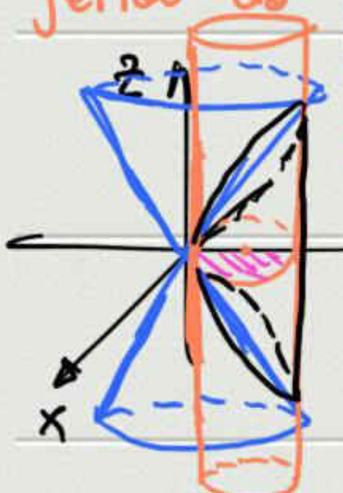
$$\int_{-\frac{\pi}{2}}^{0}\int_{1}^{\sqrt{2}}\int_{0}^{2/r^2} r.\,dz.dr.d\theta=\int_{-\frac{\pi}{2}}^{0}\int_{1}^{\sqrt{2}} r.\frac{2}{r^2}.dr.d\theta=\int_{-\frac{\pi}{2}}^{0} 2.(\ln|\sqrt{2}|-\ln|1|).d\theta=$$

$$=(0-(-\tfrac{\pi}{2}))\cdot 2\ln\sqrt{2}=\frac{\pi}{2}.\ln 2\ u.m.$$

b) $z^2=x^2+y^2$, $x^2+y^2-2y=0$, $x\geq 0$ $\delta(x,y,z)=1$

coordenadas cilíndricas

jeitão da figura




$x^2+y^2-2y+1=1$

$z=\pm\sqrt{x^2+y^2}$, $r^2$

$$\begin{cases} x=r.\cos\theta \\ y=r.\operatorname{sen}\theta \\ z=z \end{cases};\ \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|=r \quad \text{→ eixo axial}$$

$x^2+y^2-2y\leq 0 \Leftrightarrow r^2-2r.\operatorname{sen}\theta\leq 0 \Leftrightarrow r(r-2\operatorname{sen}\theta)\leq 0$

$\Leftrightarrow\ 0\leq r\leq 2\operatorname{sen}\theta$ ; $z^2\leq x^2+y^2 \Leftrightarrow -r\leq z\leq r$ ; $x\geq 0 \Leftrightarrow$

$r.\cos\theta\geq 0 \Leftrightarrow -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2} \Rightarrow 0\leq\theta\leq\frac{\pi}{2}$ → a região K está no primeiro quadrante

$$\iiint_R dm=\iiint_R 1.dx\,dy\,dz=\iiint_{R_{r,\theta,z}} 1.\ r.dr\,d\theta\,dz=\int_0^{\frac{\pi}{2}}\int_0^{2\operatorname{sen}\theta}\int_0^{r} r.dz.dr.d\theta=$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\operatorname{sen}\theta} 2\,r.r\,dr.d\theta=\int_0^{\frac{\pi}{2}} 2.\frac{1}{3}(2\operatorname{sen}\theta)^3.d\theta=\frac{16}{3}\int_0^{\frac{\pi}{2}}\operatorname{sen}^3\theta.d\theta$$

fórmula de recorrência

$$=\frac{16}{3}\cdot\left(\left[-\frac{\cos\theta.\operatorname{sen}^2\theta}{3}\right]_0^{\frac{\pi}{2}}+\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}}\operatorname{sen}\theta.d\theta\right)=\frac{32}{9}\cdot 1=\frac{32}{9}\ u.m.$$

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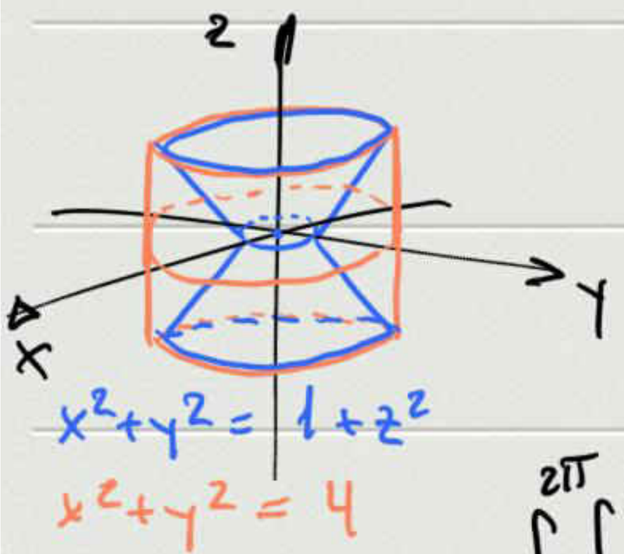
c) $x^2+y^2 = 1+z^2$ ; $x^2+y^2 = 4$ ; $\delta(x,y,z) = |z|$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \quad ; \quad x^2+y^2 \le 4 \Leftrightarrow r^2 \le 4 \Leftrightarrow \underline{0 \le r \le 2}$$

$$; \quad x^2+y^2 \le 1+z^2 \Leftrightarrow -\sqrt{x^2+y^2-1} \le z \le \sqrt{x^2+y^2-1} \Leftrightarrow$$

$$\underline{-\sqrt{r^2-1} \le z \le \sqrt{r^2-1}} \quad ; \quad \underline{0 \le \theta \le 2\pi}$$




$$\iiint_R dm = \iiint_R |z| . dx dy dz = \iiint_{R\,r,\theta,z} |z| . r . dr d\theta dz =$$

$$\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{r^2-1}}^{\sqrt{r^2-1}} |z| . r . dz . dr . d\theta = \int_0^{2\pi}\int_0^{2}\left(\int_0^{\sqrt{r^2-1}} z\, dz + \int_{-\sqrt{r^2-1}}^{0} (-z)\, dz\right) . r . dr . d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\left[\left(\frac{\sqrt{r^2-1}}{2}\right)^2 . r . - \left(0 - \left(\frac{-\sqrt{r^2-1}}{2}\right)^2\right)\right] . r . dr . d\theta = \int_0^{2\pi}\int_0^{2} |r^2-1| . r . dr . d\theta =$$

$$\int_0^{2\pi}\left(\int_0^{1} -(r^2-1) . r . dr + \int_1^{2} (r^2-1) . r . dr\right) . d\theta = \int_0^{2\pi}\left(\int_1^{2} (r^3-r) . dr - \int_0^{1} (r^3-r) . dr\right) . d\theta =$$

$$\int_0^{2\pi}\left[\frac{1}{4}(2^4-1^4) - \frac{1}{2}(2^2-1^2) - \left(\frac{1}{4}-\frac{1}{2}\right)\right] . d\theta = \int_0^{2\pi} \frac{9}{4} . d\theta = \frac{9}{2}\pi$$

d) $x^2+y^2 = 1+z^2$ , $x^2+y^2 = z^2$ $\quad |z| \le a$ $\quad \delta(x,y,z) = |z|$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\ell)}\right| = r \quad ; \quad x^2+y^2 \le 1+z^2 \ \text{ e } \ x^2+y^2 \ge z^2$$

$$z^2 \le x^2+y^2 \le 1+z^2 \Leftrightarrow z^2 \le r^2 \le 1+z^2$$

$$\Leftrightarrow \underline{z \le r \le \sqrt{1+z^2}} \quad , \quad |z| \le a \Leftrightarrow \underline{-a \le z \le a} \quad , \quad \underline{0 \le \theta \le 2\pi}$$

$$M = \iiint_R dm = \iiint_R \delta(x,y,z) . dx dy dz = \iiint_{R\,r,\theta,z} |z| . r . dr . d\theta . dz =$$

$$\int_{-a}^{a}\int_{a}^{2\pi}\int_{z}^{\sqrt{1+z^2}} |z| . r\, dr d\theta dz = \int_{-a}^{a}\int_{0}^{2\pi} |z| \frac{\left((\sqrt{1+z^2})^2 - z^2\right)}{2} . d\theta . dz = \frac{2\pi}{2} . \int_{-a}^{a} |z| . \left(|1+z^2| - z^2\right) dz$$

$$= \pi \cdot \left[\int_{-a}^{0} (-z) . (1) . dz + \int_{0}^{a} z . (1) . dz\right] = \pi\left(-\frac{(0^2-(-a)^2}{2} + \frac{a^2}{2}\right) = \underline{a^2\pi \text{ u.m.}}$$

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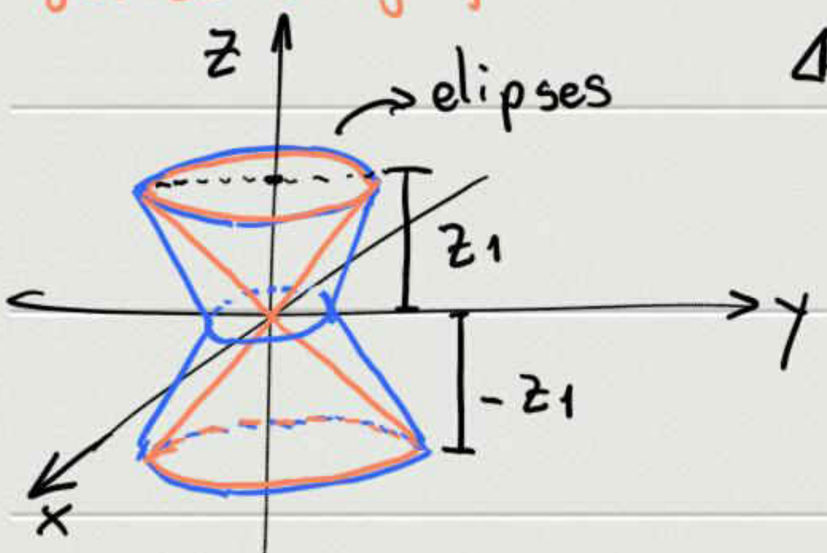
e) $\frac{x^2}{4}+\frac{y^2}{9}=1+z^2$ ; $\frac{x^2}{4}+\frac{y^2}{9}=2z^2$ $\quad \delta(x,y,z)=1$

$$\begin{cases} x = 2.r\cos\theta \\ y = 3.r.\text{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = 3.2.r \ ;$$

$$2z^2 \le \frac{x^2}{4}+\frac{y^2}{9} \le 1+z^2 \Leftrightarrow 2z^2 \le r^2 \le 1+z^2$$

Jeitão do gráfico




$$\Leftrightarrow \sqrt{2}.z \le r \le \sqrt{1+z^2}$$

$$z_1 \Rightarrow 1+z^2 = 2z^2 \Leftrightarrow z = \pm 1$$

$$\therefore -1 \le z \le 1 \ ; \quad 0 \le \theta \le 2\pi$$

$$\therefore M = \iiint_R dm = \iiint_R \delta(x,y,z).dx\,dy\,dz = \iiint_R \frac{1}{r,\theta,z}.3.2.r.dr\,d\theta\,dz =$$

$$\int_0^{2\pi}\int_{-1}^{1}\int_{\sqrt{2}z}^{\sqrt{1+z^2}} 6r.dr.dz\,d\theta = \int_0^{2\pi}\int_{-1}^{1}\frac{6}{2}\left(\left(\sqrt{1+z^2}\right)^2-\left(\sqrt{2}z\right)^2\right).dz\,d\theta =$$

$$3\cdot\int_0^{2\pi}\int_{-1}^{1}(1-z^2).dz\,d\theta = 3\cdot 2\pi.\left[(1-(-1))-\frac{1}{3}\left(1^3-(-1)^3\right)\right] = 6\pi\left[2-\frac{2}{3}\right] = 8\pi \text{ u.m.}$$

f) $x^2+y^2=1+z^2$ $\quad x^2+y^2=2+2z^2$ , $|z|\le a$ $\quad \delta(x,y,z)=z^2$

$$\begin{cases} x = r.\cos\theta \\ y = r.\text{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \ ; \quad 1+z^2 \le r^2 \le 2(1+z^2) \Leftrightarrow$$

$$\sqrt{1+z^2} \le r \le \sqrt{2}\sqrt{1+z^2}$$

$$-a \le z \le a \ , \quad 0 \le \theta \le 2\pi$$

$$M = \int_{-a}^{a}\int_0^{2\pi}\int_{\sqrt{1+z^2}}^{\sqrt{2}\sqrt{1+z^2}} z^2.r.dr.d\theta.dz = \int_{-a}^{a}\int_0^{2\pi} z^2\frac{1}{2}\left(2(1+z^2)-(1+z^2)\right).d\theta.dz =$$

$$\int_{-a}^{a}\pi.z^2(1+z^2).dz = \pi.\left[\frac{1}{3}\left(a^3-(-a)^3\right)+\frac{1}{5}\left(a^5-(-a)^5\right)\right] = \pi\left(\frac{2}{3}a^3+\frac{2}{5}a^5\right) =$$

$$2\pi\left(\frac{a^5}{5}+\frac{a^3}{3}\right) \text{ u.m.}$$

29) $\iiint_R (x+y+z)(x+y-z)\,dx\,dy\,dz$

$$R=\{(x,y,z)\in\mathbb{R}^3 \mid 1\le x+y+z\le 2,\ 0\le x+y-z\le 2,\ 1\le x-y-z\le 2\}$$

Percebe-se que a região limitada por estes planos irá gerar um paralelepípedo.

Fazendo-se uma parametrização para facilitar as contas

$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \Leftrightarrow \begin{cases} u+w = 2x \\ v-w = 2y \\ u-v = 2z \end{cases} \Rightarrow \begin{cases} x = \frac{u+w}{2} \\ y = \frac{v-w}{2} \\ z = \frac{u-v}{2} \end{cases}$$

$u(x,y,z)$ $\quad X(u,w,z)$
$v$ $\quad y$
$w$ $\quad z$
$\vdots$ $\quad \vdots$

para o Jacobiano →

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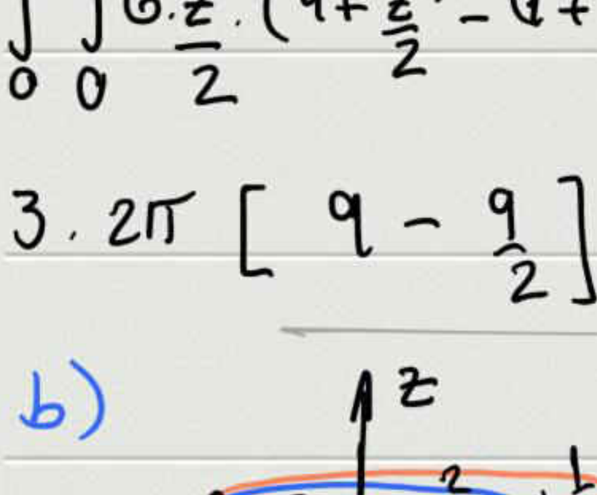
$$J = \begin{bmatrix} \nabla x(u,v,w) \\ \nabla y(u,v,w) \\ \nabla z(u,v,w) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = |\det J| = \left|\left(\frac{1}{2}\right)^3(-2)\right| = \frac{1}{4}$$

$$\iiint_R (x+y+z)(x+y-z)\,dx\,dy\,dz = \iiint_{R\,uvw} u \cdot v \cdot \frac{1}{4}\,du\,dv\,dw = \int_1^2\int_0^2\int_1^2 \frac{u.v}{4}\,du\,dv\,dw =$$

$$\int_1^2\int_0^2 \frac{v}{4} \cdot \frac{1}{2} \cdot (2^2 - 1^2)\,dv\,dw = \frac{3}{8}\int_1^2 \frac{1}{2}\,2^2\,dw = \frac{3}{4}$$

30 a)




$$\frac{x^2}{9} + \frac{y^2}{4} = 1 + z^2$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 4 + \frac{1}{2}z^2$$

$$\begin{cases} x = 3.r.\cos\theta \\ y = 2.r.\operatorname{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = 3.2.r$$

$$\sqrt{1+z^2} \le r \le \sqrt{4 + \frac{z^2}{2}}\,; \quad 0 \le \theta \le 2\pi$$

$z_1 \Leftrightarrow 1 + z^2 = 4 + \frac{1}{2}z^2 \Leftrightarrow \frac{1}{2}z^2 = 3 \Leftrightarrow -\sqrt{6} \le z \le \sqrt{6}$ e

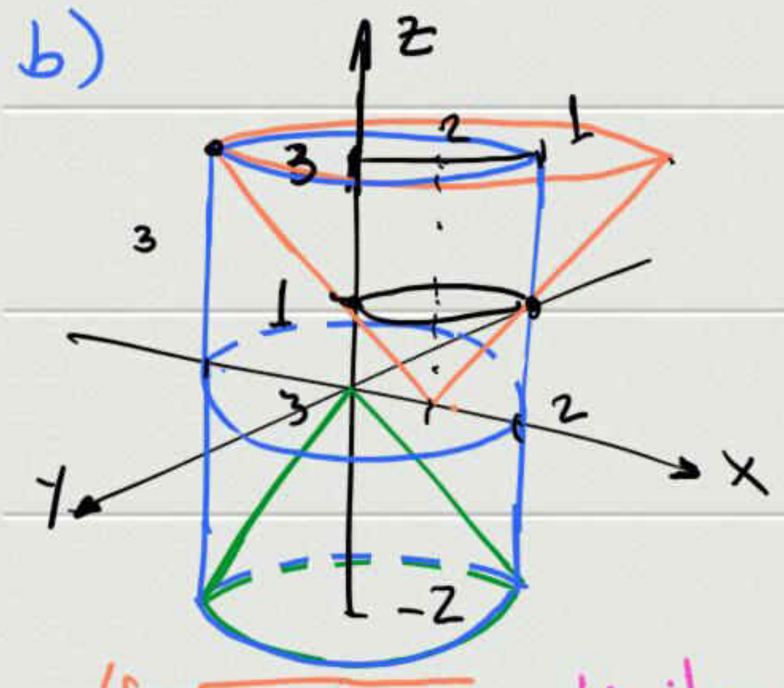
$z \ge 0 \therefore 0 \le z \le \sqrt{6}$

$$\iiint_R z\,dx\,dy\,dz = \iiint_{R\,r,\theta,z} z \cdot 6.r \cdot dr\,d\theta\,dz = \int_0^{2\pi}\int_0^{\sqrt{6}}\int_{\sqrt{1+z^2}}^{\sqrt{4+\frac{z^2}{2}}} 6.z.r.dr.d\theta.dz =$$

$$\int_0^{2\pi}\int_0^{\sqrt{6}} 6.\frac{z}{2}.\left(4 + \frac{z^2}{2} - (1+z^2)\right).dz.d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{6}} \left(3z - \frac{1}{2}z^3\right).dz = 3\int_0^{2\pi}\left[\frac{3}{2}.6 - \frac{1}{2}.\frac{1}{4}.6^2\right].d\theta$$

$$3.2\pi\left[9 - \frac{9}{2}\right] = 27\pi$$

b) cilíndrica




$$z = \sqrt{(x-1)^2 + y^2}\,; \quad z = -\sqrt{x^2+y^2}\,; \quad x^2+y^2 = 4$$

$$\begin{cases} x = r.\cos\theta \\ y = r.\operatorname{sen}\theta \\ z = z \end{cases}\,; \quad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r\,; \quad \sqrt{(x-1)^2+y^2} = \sqrt{x^2-2x+1+y^2}$$

percebe-se que é mais fácil parametrizar $z(r,\theta)$ do que $r(z,\theta)$

$$= \sqrt{r^2 - 2r.\cos\theta + 1} \therefore -r \le z \le \sqrt{r^2 - 2r.\cos\theta + 1}$$

$$0 \le r \le 2\,; \quad 0 \le \theta \le 2\pi$$

gráf: $\sqrt{(x-1)^2+y^2}$ → limite sup. de z

gráf: $-\sqrt{x^2+y^2}$ → limite inf. de z

gráf: $x^2+y^2 = 4$ → limita o plano xy

$$\iiint_A z\,dx\,dy\,dz = \iiint_{A\,r,\theta,z} z.r.dr\,d\theta\,dz =$$

$$\int_0^{2\pi}\int_0^2\int_{-r}^{\sqrt{r^2-2r\cos\theta+1}} z.r\,dz.dr.d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{(r^2-2r\cos+1) - (-r)^2}{2}\right] r.dr.d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{-2r^2\cos\theta + r}{2}.dr.d\theta = \int_0^{2\pi}\left(\frac{1}{4}.(2)^2 - \frac{1}{3}(2)^3.\cos\theta\right).d\theta = 2\pi$$

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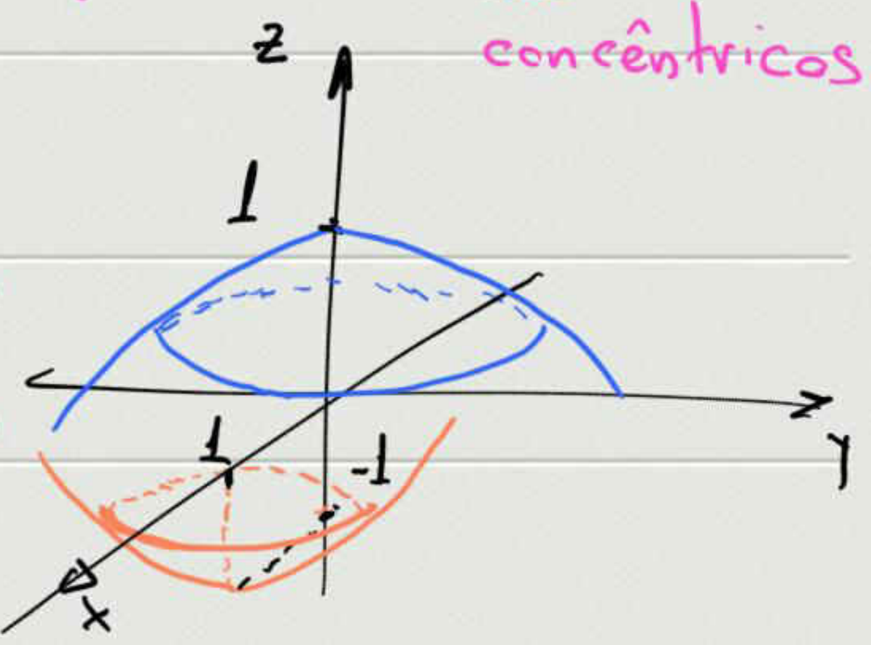
(31) $z = 1 - x^2 - y^2$ , $z = -1 + (x-1)^2 + y^2$

$z = 1 - (x^2+y^2)$ , $z = -1 + (x-1)^2 + y^2$

Neste tipo de exercício, precisa-se determinar o cilindro que engloba os paraboloides (laranja azul, semelhante ao do exercício anterior

parabolóides não-concêntricos




$-1 + x^2 - 2x + 1 + y^2 = r^2 - 2r\cdot\cos\theta$

$1 - x^2 - y^2 = -1 + (x-1)^2 + y^2 \Leftrightarrow 1 - x^2 - y^2 = -1 + x^2 - 2x + 1 + y^2 \Leftrightarrow$

$2x^2 - 2x + 2y^2 = 1 \Leftrightarrow 2x^2 - 2x + \frac{1}{2} + 2y^2 = 1 + \frac{1}{2} \Leftrightarrow 2\left(\left(x-\frac{1}{2}\right)^2 + y^2\right) = \frac{3}{2}$

$\Leftrightarrow \boxed{\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{3}{4}}$ $\quad \begin{cases} x - \frac{1}{2} = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \;|\; 0 \le r \le \frac{\sqrt{3}}{2}$

$-1 + (x-1)^2 + y^2 \le z \le 1 - (x^2+y^2) \Leftrightarrow -1 + \left(r\cos\theta - \frac{1}{2}\right)^2 + r\cdot\sin^2\theta \le z \le 1 - \left(\left(r\cos\theta + \frac{1}{2}\right)^2 + r^2\sin^2\theta\right)$

$\Leftrightarrow -\frac{3}{4} + r^2 - r\cos\theta \le z \le \frac{3}{4} - r^2 - r\cos\theta$ ; $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{-\frac{3}{4}+r^2-r\cos\theta}^{\frac{3}{4}-r^2-r\cos\theta} r\cdot dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{3}{4} - r^2 - r\cdot\cos\theta - \left(-\frac{3}{4} + r^2 - r\cos\theta\right)\right) r\cdot dr =$$

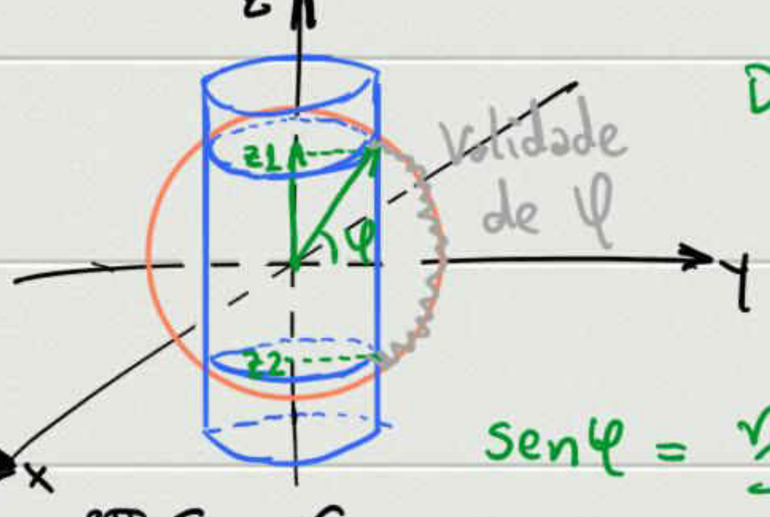
$$2\pi \cdot \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{6}{4} - 2r^2\right) r\,dr = 2\pi \cdot \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{3}{2}r - 2r^3\right) dr = 2\pi\left[\frac{3}{2}\cdot\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 - 2\cdot\frac{1}{4}\cdot\left(\frac{\sqrt{3}}{2}\right)^4\right] =$$

$$2\pi \cdot \frac{3}{4}\cdot\frac{1}{4}\left(3 - 2\cdot\frac{3}{4}\right) = \frac{9}{16}\pi \text{ u.v.}$$

(32) $x^2+y^2+z^2 \le r^2$ ; $x^2+y^2 \ge \frac{r^2}{2}$ ; $\rho \in \mathbb{R}_+$, $\theta \in [0, 2\pi]$; $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\begin{cases} x = \rho\cdot\cos\varphi\cdot\cos\theta \\ y = \rho\cdot\cos\varphi\cdot\sin\theta \\ z = \rho\cdot\sin\varphi \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\cdot\cos\varphi \; ; \quad \rho^2 \le r^2 \text{ e } \rho^2\cdot\cos^2\varphi \ge \frac{r^2}{2} \Leftrightarrow$$

$$\rho^2 \ge \frac{r^2}{2\cos^2\varphi} \quad \therefore \quad \frac{r}{\sqrt{2}\cdot\cos\varphi} \le \rho \le r \; ; \quad 0 \le \theta \le 2\pi$$




Determinando $\varphi$: $x^2+y^2+z^2 = r^2$ (Eq.1) ; $x^2+y^2 = \frac{r^2}{2}$ (Eq.2)

Eq.2 → Eq.1 (nível da curva de intersecção)

$\frac{r^2}{2} + z^2 = r^2 \Leftrightarrow z^2 = \frac{r^2}{2} \Leftrightarrow \boxed{z = \pm\frac{r}{\sqrt{2}}}$

$\sin\varphi = \frac{r/\sqrt{2}}{r} = \frac{1}{\sqrt{2}} \Leftrightarrow \varphi = \frac{\pi}{4}$ $\quad \therefore \quad -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$

$$V = \int_0^{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\frac{r}{\sqrt{2}\cdot\cos\varphi}}^{r} \rho^2\cdot\cos\varphi\cdot d\rho\cdot d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3}\left(r^3 - \left(\frac{r}{\sqrt{2}\cos\varphi}\right)^3\right)\cdot\cos\varphi\cdot d\varphi =$$

$$\frac{2\pi}{3}r^3 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos\varphi - \frac{1}{2\sqrt{2}\cos^2\varphi}\right)\cdot d\varphi = \frac{2\pi}{3}r^3\left[\sin\varphi - \frac{1}{2\sqrt{2}}\operatorname{tg}\varphi\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2\pi}{3}r^3\left[\frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2\pi r^3}{3\sqrt{2}} \text{ u.v.}$$

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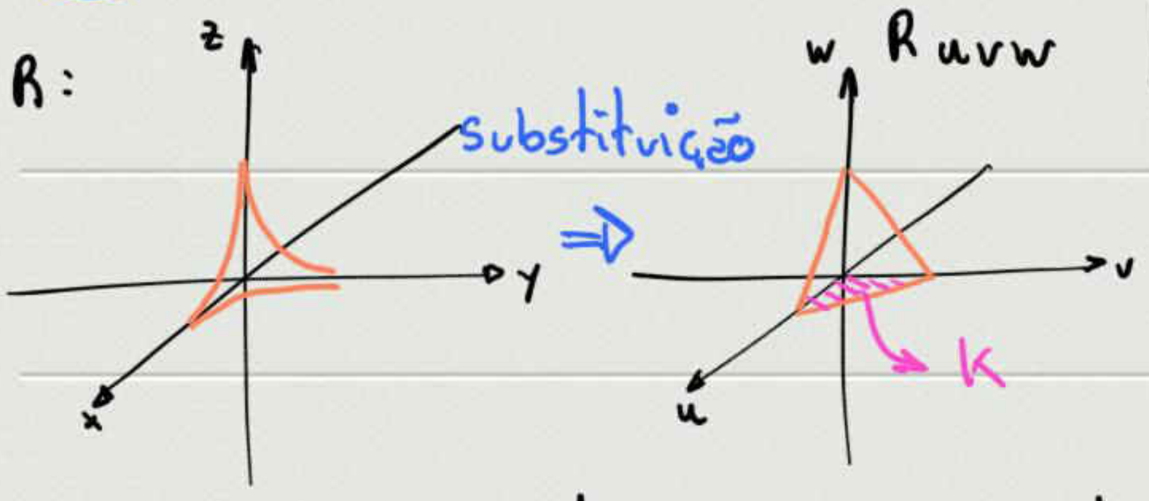
(33) $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ ; $R=\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{x}+\sqrt{y}+\sqrt{z}\le 1,\ x,y,z\in\mathbb{R}_+^*\}$

R:




$$\begin{cases} x=u^2 \\ y=v^2 \\ z=w^2 \end{cases} \Leftrightarrow J=\begin{bmatrix} \nabla x(u,v,w) \\ \nabla y(u,v,w) \\ \nabla z(u,v,w) \end{bmatrix}=\begin{bmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{bmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=|\det J|=|8uvw| \text{ ; se } \begin{matrix} u\ge 0 \\ v\ge 0 \\ w\ge 0 \end{matrix}$$

$$=8uvw$$

$|u|+|v|+|w|\le 1 \Leftrightarrow u+v+w\le 1 \Leftrightarrow w\le 1-v-u \Rightarrow \underline{0\le w\le 1-v-u}$

determinando K: $w=0 \Rightarrow u+v\le 1 \Leftrightarrow v\le 1-u \Rightarrow \underline{0\le v\le 1-u}$

determinando u: $w=0,\ v=0 \quad u\le 1 \Rightarrow \underline{0\le u\le 1}$

$$V=\iiint_R dxdydz=\iiint_{R_{uvw}} 8uvw\,dudvdw=\int_0^1\int_0^{1-u}\int_0^{1-v-u} 8uvw\,.dw.dv.du=$$

$$\int_0^1\int_0^{1-u} 8uv\left(\frac{1-v-u}{2}\right)^2.dv.du=\int_0^1\int_0^{1-u} 4uv\left(1-2(u+v).1+(u+v)^2\right).dvdu=$$

$$4\int_0^1\int_0^{1-u} uv\left(1-2u-2v+u^2+2uv+v^2\right).dvdu=4\int_0^1 uv\left(1-2u+u^2-2(1-u)v+v^2\right).dvdu=$$

$$4\int_0^1\int_0^{1-u}\left[u(1-u)^2.v-2u(1-u)v^2+uv^3\right]dvdu=4\int_0^1\left[u(1-u)^2\frac{(1-u)^2}{2}-2u(1-u)\frac{(1-u)^3}{3}\right.$$

$$\left.+u\frac{(1-u)^4}{4}\right].du=4\int_0^1\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right]u.(1-u)^4.du=4.\frac{1}{12}\int_0^1 u(1-4u+6u^2-4u^3+u^4).du=$$

$$\frac{1}{3}\cdot\int_0^1(u-4u^2+6u^3-4u^4+u^5).du=\frac{1}{3}\left[\frac{1}{2}-\frac{4}{3}+\frac{6}{4}-\frac{4}{5}+\frac{1}{6}\right]=\frac{1}{3}\left[2-4\left(\frac{8}{15}\right)+\frac{1}{6}\right]=$$

$$\frac{1}{3}\left[2-\frac{32}{15}+\frac{1}{6}\right]=\frac{1}{3}\,\frac{60-64+5}{30}=\frac{1}{90}\ u.v.$$

---

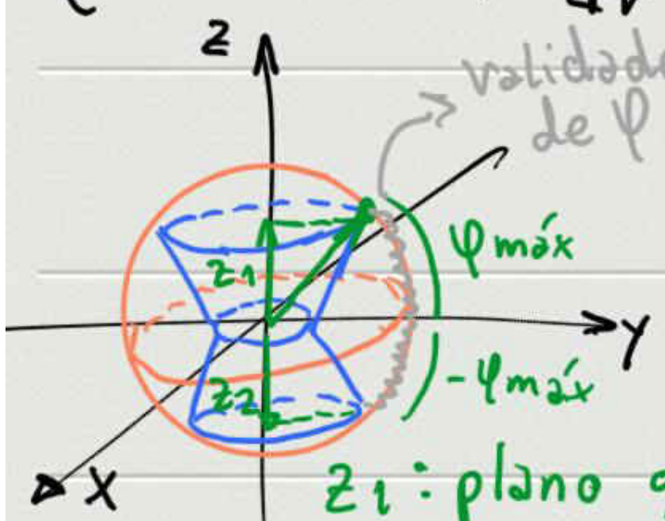
(34) $x^2+y^2+z^2\le r^2$ ; $x^2+y^2\ge\frac{r^2}{2}+z^2$ $\quad \delta(x,y,z)=x^2+y^2$

Coordenadas esféricas

$$\begin{cases} x=\rho.\cos\varphi.\cos\theta \\ y=\rho.\cos\varphi.\operatorname{sen}\theta \\ z=\rho.\operatorname{sen}\varphi \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right|=\rho^2.\cos\varphi \text{ ; } \rho^2\le r^2,\ \rho^2\cos^2\varphi\ge\frac{r^2}{2}+\rho^2\operatorname{sen}^2\varphi$$

para $\varphi\in\left]-\frac{\pi}{4},\frac{\pi}{4}\right[$

$$\Leftrightarrow \rho^2(\cos^2\varphi-\operatorname{sen}^2\varphi)\ge\frac{r^2}{2} \Leftrightarrow \rho^2\ge\frac{r^2}{2\cos 2\varphi} \Leftrightarrow$$




$$\frac{r}{\sqrt{2\cos 2\varphi}}\le\rho\le r \text{ ; obtendo } \varphi\text{:}$$

$z_1$: plano que contém a curva que representa a intersecção

$$\begin{cases} x^2+y^2=\frac{r^2}{2}+z^2 \\ x^2+y^2+z^2=r^2 \end{cases} \Leftrightarrow 2z^2=\frac{r^2}{2} \Leftrightarrow z=\pm\frac{r}{2}$$

$$\Rightarrow \operatorname{sen}\varphi=\frac{r/2}{r} \Leftrightarrow \varphi_{máx}=\frac{\pi}{6}$$

$$\therefore -\frac{\pi}{6}\le\varphi\le\frac{\pi}{6} \text{ e } 0\le\theta\le 2\pi$$

$$m=\iiint_E \delta(x,y,z).dxdydz=\iiint_E \underset{E}{\delta}(\rho,\theta,\varphi).\rho^2\cos\varphi.d\rho.d\theta d\varphi$$

(44)

$$m = \int_0^{2\pi}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\frac{r}{\sqrt{2\cos 2\varphi}}}^{r} \rho^4 \cdot \cos^3\varphi \cdot d\rho \cdot d\varphi \, d\theta = \int_0^{2\pi} d\theta \cdot \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{5}\cdot\left(r^5 - \left(\frac{r}{\sqrt{2\cos 2\varphi}}\right)^5\right)\cos^3\varphi \cdot d\varphi$$

$$\frac{2\pi}{5} r^5 \left[\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^3\varphi \, d\varphi - \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^3\varphi}{(2\cos 2\varphi)^{5/2}} d\varphi\right]$$ ; Atenção para A: $t = \operatorname{sen}\varphi$, $dt = \cos\varphi \cdot d\varphi$

$$A = \int \frac{(1-\operatorname{sen}^2\varphi)\cdot\cos\varphi}{\left(2(1-2\operatorname{sen}^2\varphi)^{\frac{5}{2}}\right)} \cdot d\varphi = \int \frac{1-t^2}{4\sqrt{2}(1-2t^2)^{5/2}} dt$$ ; A é uma integral elíptica, sem chances.

Melhor coordenadas cilíndricas

$$\begin{cases} x = \rho\cdot\cos\theta \\ y = \rho\cdot\operatorname{sen}\theta \\ z = z \end{cases} \Leftrightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)}\right| = \rho$$

$x^2+y^2+z^2 \le r^2 \Leftrightarrow \rho^2+z^2 \le r^2 \Leftrightarrow \rho^2 \le r^2 - z^2$

$x^2+y^2 \ge \frac{r^2}{2}+z^2 \Leftrightarrow \rho^2 \ge \frac{r^2}{2}+z^2$

$\therefore \sqrt{\frac{r^2}{2}+z^2} \le \rho \le \sqrt{r^2-z^2}$ ; $-\frac{r}{2} \le z \le \frac{r}{2}$

e $0 \le \theta \le 2\pi$

$$M = \iiint_{E_c} \delta(\rho,\theta,z)\cdot\rho\cdot d\rho\cdot d\theta\cdot dz = \int_0^{2\pi}\int_{-\frac{r}{2}}^{\frac{r}{2}}\int_{\sqrt{\frac{r^2}{2}+z^2}}^{\sqrt{r^2-z^2}} \rho^2\cdot\rho\cdot d\rho\cdot d\theta\cdot dz = \int_0^{2\pi} d\theta \cdot \int_{-\frac{r}{2}}^{\frac{r}{2}}\int_{\sqrt{\frac{r^2}{2}+z^2}}^{\sqrt{r^2-z^2}} \rho^3 \cdot d\rho\, dz$$

$$= 2\pi \cdot \int_{-\frac{r}{2}}^{\frac{r}{2}} \frac{1}{4}\left(\left(r^2-z^2\right)^2 - \left(\frac{r^2}{2}+z^2\right)^2\right) dz = \frac{\pi}{2}\cdot\int_{-\frac{r}{2}}^{\frac{r}{2}} \left(r^4 - 2r^2z^2+z^4 - \left(\frac{r^4}{4}+2\frac{r^2}{2}\cdot z^2+z^4\right)\right) dz$$

$$= \frac{\pi}{2}\cdot\int_{-\frac{r}{2}}^{\frac{r}{2}} \left(\frac{3}{4}r^4 - 3r^2z^2\right) dz = 3\frac{\pi}{2}r^2\left[\frac{r^2}{4}\cdot\left(\frac{r}{2}-\left(\frac{-r}{2}\right)\right) - \frac{1}{3}\left(\left(\frac{r}{2}\right)^3-\left(\frac{-r}{2}\right)^3\right)\right] =$$

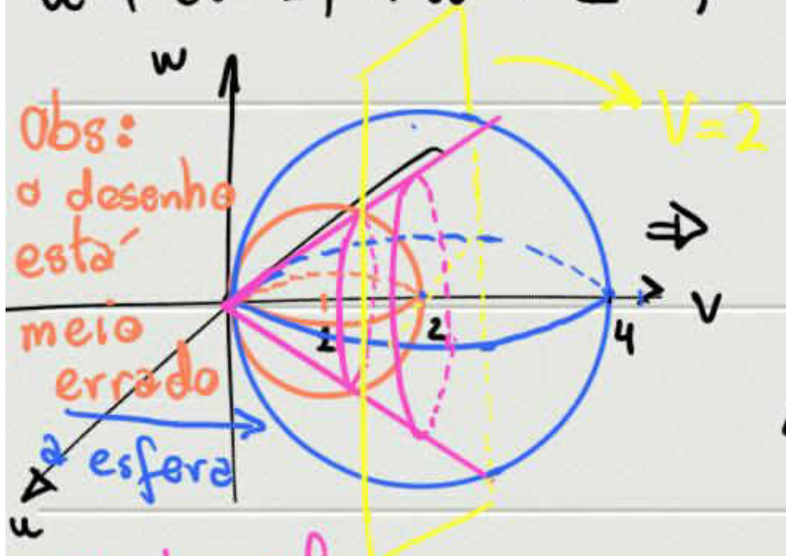
$$\frac{3\pi}{2}r^2\left[\frac{r^3}{4} - \frac{1}{3}\cdot\frac{r^3}{4}\right] = \frac{\pi}{4}r^5 \text{ u.m.}$$

$1-(v-1)^2 = (1-v+1)(1+v-1)$

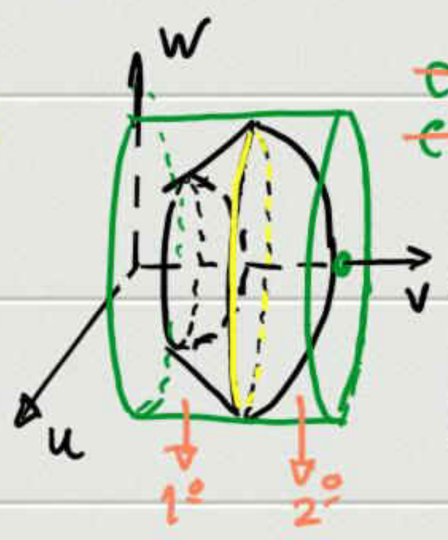
(35) $u^2+v^2+w^2 = 4v$, $u^2+v^2+w^2 = 2v$, $v \ge \sqrt{u^2+w^2}$ ; $\delta(u,v,w) = u^2+w^2$

$u^2+v^2-4v+4+w^2 = 4 \Leftrightarrow u^2+(v-2)^2+w^2 = 4$ ; $u^2+v^2-2v+1+w^2 = 1 \Leftrightarrow$

$u^2+(v-1)^2+w^2 = 1$ ; $v = \sqrt{u^2+w^2}$ superfície cônica




Obs: o desenho está meio errado




coordenadas cilíndricas são melhores neste caso seria mais simples por esféricas

$$\begin{cases} u = r\cdot\cos\theta \\ w = r\cdot\operatorname{sen}\theta \\ v = v \end{cases} \Leftrightarrow \left|\frac{\partial(u,w,v)}{\partial(r,\theta,v)}\right| = r$$

limite inferior $u^2+(v-1)^2+w^2 \ge 1 \Rightarrow v-1 \ge \sqrt{1-(u^2+w^2)} \Leftrightarrow v \ge 1+\sqrt{1-r^2}$

limite superior $u^2+(v-2)^2+w^2 \le 4 \Rightarrow v-2 \le \sqrt{4-(u^2+w^2)} \Leftrightarrow v \le 2+\sqrt{4-r^2}$

$v \ge r \Rightarrow r \le v$ ; pelo fato de se ter três superfícies se interseccionando, é melhor, para contabilizar todas as superfícies, realizar duas integrais definidas onde o plano amarelo $v=2$ faz esta divisão

(46)

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\cdot\cos\varphi \quad ; \quad dm = \delta(r,\theta,\varphi)\cdot r^2\cos\varphi\cdot dr\,d\varphi\,d\theta$$

$$m = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{\sqrt{2\cos\varphi\cos\theta}} r\cdot\cos\theta\cdot\cos\varphi\cdot r^2\cos\varphi\,dr\,d\varphi\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{\sqrt{2\cos\varphi\cos\theta}} r^3\cdot\cos^2\varphi\cdot\cos\theta\cdot dr\,d\varphi\,d\theta =$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{4}\left(4\cos^2\theta\cdot\cos^2\varphi - 1\right)\cdot\cos^2\varphi\cdot\cos\theta\cdot d\varphi\,d\theta = \frac{1}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(4\cos^4\varphi\cdot\cos^3\theta - \cos^2\varphi\cdot\cos\theta\right)d\varphi\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^4\varphi\cdot\cos^3\theta\cdot d\theta\,d\varphi - \frac{1}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2\varphi\cdot\cos\theta\cdot d\theta\,d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^3\theta\,d\theta\cdot\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^4\varphi\cdot d\varphi - \frac{1}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos\theta\cdot d\theta\cdot$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2\varphi\cdot d\varphi \quad ; \quad \int\cos^n x\cdot dx = \frac{\sin x\cdot\cos^{n-1}x}{n} + \frac{n-1}{n}\int\cos^{n-2}x\cdot dx$$

Fórmula de Recorrência

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos x\cdot dx = \sqrt{3} \quad ; \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2 x\cdot dx = \left[\frac{\sin x\cdot\cos x}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} dx = \frac{\sqrt{3}}{4} + \frac{\pi}{3} \quad ; \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^3 x\cdot dx =$$

$$\left[\frac{\sin x\cdot\cos^2 x}{3}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \frac{2}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos x\cdot dx = \frac{2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{4}}{3} + \frac{2}{3}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{12} = \frac{\sqrt{3}}{6}$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^4 x\cdot dx = \left[\frac{\sin x\cos^3 x}{4}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2 x\cdot dx = 2\cdot\frac{\frac{\sqrt{3}}{2}\cdot\frac{1}{8}}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{3} = \frac{\sqrt{3}}{32} + \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

substituindo os valores

$$\frac{\sqrt{3}}{6}\cdot\left(\frac{9\sqrt{3}}{32} + \frac{\pi}{3}\right) - \frac{1}{4}\left(\sqrt{3}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)\right)$$

$$\frac{3.3}{64} + \frac{\sqrt{3}\pi}{6} - \frac{\sqrt{3}\pi}{12} - \frac{3}{16.4} = \frac{\sqrt{3}\pi}{12} - \frac{3}{64} \text{ m.m.} \neq \frac{9}{8}\pi$$

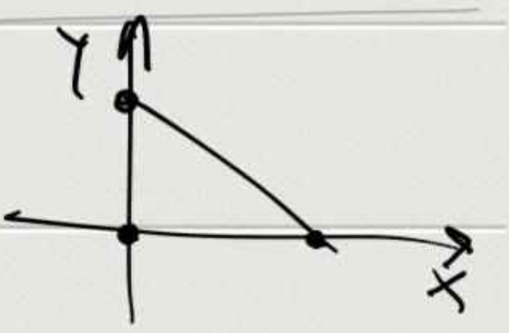
Bom, eu acho que não errei, se errei, por favor, me avisem.

(37) $f \in \mathcal{C}[0,1]$

$$R = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le 1-x\}$$

ou

$$R = \{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 1,\ 0\le x\le 1-y\}$$




Seja $F: \mathbb{R}^2 \to \mathbb{R} \mid F(x,y) = f(x+y)$ e $f: \mathbb{R}\to\mathbb{R}$

Sendo $f \in \mathcal{C}[0,1]$, então para $(x,y)\in[0,1]\times[0,1]$

$$\iint_R f(x+y)\cdot dx\,dy = \iint_R F(x,y)\cdot dx\,dy = \int_0^1\int_0^{1-x} F(x,y)\cdot dy\,dx = \int_0^1\int_0^{1-y} F(x,y)\cdot dx\,dy \Leftrightarrow$$

$$\int_\alpha^1 (1-x)\cdot dx = 1 - \frac{1}{2} = \frac{1}{2}$$

$$u = \frac{1+x^2}{2}$$
$$du = \frac{2x}{2}dx$$
$$d - x\,dx$$

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39 a) $\int_\gamma x\,ds$ , $\gamma(t) = (t^3, t)$ , $0 \le t \le 1$

$\gamma'(t) = (3t^2, 1)$

$\int_\gamma x\,ds = \int_0^1 t^3 \,\|(3t^2, 1)\|\,dt = \int_0^1 t^3 \sqrt{9t^4+1}\,dt$ ;

$tg\theta = 3t^2 \Rightarrow \sec^2\theta\,d\theta = 6\,t\,dt$

$\int_\gamma x\,ds = \int_0^{\arctg 3} \frac{t^3}{6t} \cdot \sec\theta \cdot \sec^2\theta\,d\theta = \int_0^{\arctg 3} \frac{1}{6} \cdot \frac{tg\theta}{3} \cdot \sec^3\theta \cdot d\theta =$

$\frac{1}{18} \cdot \int_0^{\arctg 3} \frac{\operatorname{sen}\theta}{\cos^4\theta} \cdot d\theta = -\frac{1}{18} \cdot \int_1^{\frac{1}{\sqrt{10}}} \frac{1}{v^4} \cdot dv$

$-\frac{1}{18} \cdot -\frac{1}{3} \cdot \left[\frac{1}{v^3}\right]_1^{\frac{1}{\sqrt{10}}} = \frac{1}{54} \cdot \left(10\sqrt{10} - 1\right)$

$V = \cos\theta \Leftrightarrow dv = -\operatorname{sen}\theta \cdot d\theta$

$tg\,0 \to \cos\theta = 1$

$tg\,\theta = 3 \to \cos\theta = \frac{1}{\sqrt{10}}$

outro jeito de resolver a integral :

$I = \int tg\theta \cdot \sec\theta \cdot \sec^2\theta \cdot d\theta = \sec\theta \cdot \sec^2\theta - \int \sec\theta \cdot 2 \cdot \sec\theta \cdot \sec\theta \cdot tg\theta \cdot d\theta =$

$\sec^3\theta - 2I \quad \Delta \Rightarrow \quad I = \sec^3\theta - 2I \quad \Delta \Rightarrow \quad \boxed{I = \frac{\sec^3\theta}{3}}$

b) $\int_\gamma xy^4 \cdot ds$ , $\gamma$ é a curva $x^2 + y^2 = 16$ (circun. de $r = 4$)

parametrizando-se $\gamma$ : $\gamma(t) = (4\cos t, 4\operatorname{sen} t) \mid t \in [0, 2\pi]$

$x = 4\cos t \ge 0 \quad \Delta \Rightarrow \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\gamma'(t) = 4(-\operatorname{sen} t, \cos t)$ $\therefore$ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cdot \cos t \cdot (4 \cdot \operatorname{sen} t)^4 \,\|4(-\operatorname{sen} t, \cos t)\|\,dt =$

$4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \operatorname{sen}^4 t \cdot dt = 4^6 \int_{-1}^{1} v^4 \cdot dv = \frac{4^6}{5}\left(1^5 - (-1)^5\right) =$

$V = \operatorname{sen} t$

$dv = \cos t \cdot dt$

1638,4

(48)

c) $\int_\gamma xyz \cdot ds$, $\gamma(t) = (2t, 3\operatorname{sen} t, 3\cos t) \mid 0 \le t \le \frac{\pi}{2}$

↳ curva na superfície de um cilindro de raio 3 e eixo axial no x

$\gamma'(t) = (2, 3\cos t, -3\operatorname{sen} t)$

$$\int_\gamma xyz\,ds = \int_0^{\frac{\pi}{2}} 2t \cdot 3\operatorname{sen} t \cdot 3\cos t \, \|(2, 3\cos t, -3\operatorname{sen} t)\| \cdot dt =$$

$$\int_0^{\frac{\pi}{2}} 9t \cdot \operatorname{sen} 2t \cdot \sqrt{4+9} \cdot dt = 9\sqrt{13}\left( \left[ t \cdot \frac{(-\cos 2t)}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos 2t}{2} \cdot dt \right) =$$

$$9\sqrt{13}\left( \frac{\pi}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \left( \operatorname{sen} 2 \cdot \frac{\pi}{2} - \operatorname{sen} 0 \right) \right) = \frac{9\pi\sqrt{13}}{4}$$

d) $\int_\gamma xy^2z \cdot ds$, $\gamma$ segmento de reta $(1,0,1)$ a $(0,3,6)$

$\gamma(t) = (1,0,1) + t[(0,3,6) - (1,0,1)] = (1,0,1) + t(-1,3,5)$

$\gamma(t) = (1-t, 3t, 1+5t) \mid t \in [0,1]$

$\gamma'(t) = (-1, 3, 5)$

$$\int_\gamma xy^2z \cdot ds = \int_0^1 (1-t) \cdot (3t)^2 \cdot (1+5t) \cdot \|(-1,3,5)\| \cdot dt =$$

$$\int_0^1 9t^2(1+4t-5t^2)\sqrt{35} \cdot dt = 9\sqrt{35} \int_0^1 (t^2+4t^3-5t^4) \cdot dt = 9\sqrt{35}\left( \frac{1}{3} + \frac{4}{4} - \frac{5}{5} \right)$$

$= 3\sqrt{35}$

(40) a) $\gamma(t) = (a(t-\operatorname{sen} t), a(1-\cos t)) \mid 0 \le t \le 2\pi$ e $a > 0$

$\gamma'(t) = a(1-\cos t, \operatorname{sen} t)$

$$\Delta S = \int_0^{2\pi} \|a(1-\cos t, \operatorname{sen} t)\| \cdot dt = \int_0^{2\pi} |a| \sqrt{1-2\cos t+1} \cdot dt = \int_0^{2\pi} a\sqrt{2}\sqrt{1-\cos t} \cdot dt$$

$$= a\sqrt{2} \int_0^{2\pi} \sqrt{1-\cos t} \cdot dt = a\sqrt{2} \int_0^{2\pi} \sqrt{\frac{1-\cos^2 t}{1+\cos t}} \cdot dt = a\sqrt{2} \int_0^{2\pi} \frac{|\operatorname{sen} t|}{\sqrt{1+\cos t}} \cdot dt =$$

$$a\sqrt{2}\left( \int_0^{\pi} \frac{\operatorname{sen} t}{\sqrt{1+\cos t}} \cdot dt + \int_\pi^{2\pi} \frac{-\operatorname{sen} t}{\sqrt{1+\cos t}} \cdot dt \right); \quad u = 1+\cos t \Rightarrow du = -\operatorname{sen} t \cdot dt$$

$$a\sqrt{2}\left( \int_2^0 \frac{-1}{u^{1/2}} \cdot du + \int_0^2 \frac{1}{u^{1/2}} \cdot du \right) = a\sqrt{2} \cdot 2 \int_0^2 u^{-1/2} \cdot du = a\sqrt{2} \cdot 2 \frac{1}{1/2} \cdot \sqrt{2}$$

$= 8a$ u.c.

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b) $\gamma(t) = (t.\cos t, t\,\text{sen}\,t, t)$, $0 \le t \le \sqrt{2}$

$\gamma'(t) = (\cos t - t.\text{sen}\,t, \text{sen}\,t + t.\cos t, 1)$

$$\Delta S = \int_\gamma dS = \int_0^t \|(\cos t - t.\text{sen}\,t, \text{sen}\,t + t\cos t, 1)\|.dt =$$

$$\int_0^{\sqrt{2}} \sqrt{\cos^2 t - 2t.\cos t.\text{sen}\,t + t^2\text{sen}^2 t + \text{sen}^2 t + 2t.\text{sen}\,t.\cos t. + t^2.\cos^2 t + 1}\ .dt =$$

$$\int_0^{\sqrt{2}} \sqrt{2 + t^2}\ .dt = \int_0^{\sqrt{2}} \sqrt{2}\sqrt{1 + \frac{t^2}{2}}.dt \quad \Big| \ tg\,\theta = \frac{t}{\sqrt{2}} \Leftrightarrow \sec^2\theta.d\theta = \frac{dt}{\sqrt{2}}$$

$$= \int_0^{\text{arctg}\,L} 2.\sec^3\theta.d\theta = 2\left(\left[\frac{tg\,\theta.\sec\theta}{2}\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec\theta.d\theta\right)$$

$$= 2\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\left[\ln(tg\,\theta + \sec\theta)\right]_0^{\text{arctg}\,\frac{1}{\sqrt{2}}}\right)$$

$$= \sqrt{2} + 1.\ln(1 + \sqrt{2}) - \ln(1) = \sqrt{2} + \ln(\sqrt{2}+1)\ \text{u.c.}$$

Só lembrando:

$$\int \sec x\,dx = \int \frac{1}{\sqrt{1+u^2}}.du =$$

$$\int \frac{1}{a}.da = \ln a + K =$$

$$\ln(u + \sqrt{1+u^2}) + K =$$

$$\ln(tg\,x + \sec x) + K$$

1º

$\sec x = \sqrt{1+u^2} \Leftrightarrow \sec x\,tg\,x\,dx = \frac{u}{\sqrt{1+u^2}}.du$

$\Leftrightarrow dx = \frac{u}{\sqrt{1+u^2}} \cdot \frac{du}{\sec x.tg\,x}$ ; $tg\,x = u$

$$dx = \frac{1}{\sqrt{1+u^2}} \cdot \frac{du}{\sec x}$$

2º

$a = u + \sqrt{1+u^2}$

$$da = \left(1 + \frac{u}{\sqrt{1+u^2}}\right)du = \frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}}.du$$

$$du = \frac{\sqrt{1+u^2}\ .da}{\sqrt{1+u^2}+u} = \frac{\sqrt{1+u^2}}{a}.da$$

Essas integrais tem que saber de cor: $\int \sec\theta.d\theta$ e $\int \frac{1}{\sqrt{x^2+K}}.dx$

c) $\gamma(t) = \left(t, \frac{3t^2}{2}, \frac{3t^2}{2}\right)$, onde $0 \le t \le 2$

$\gamma'(t) = (1, 3t, 3t)$

$\sec\theta = \sqrt{73}$

$$\Delta S = \int_\gamma dS = \int_0^2 \|(1, 3t, 3t)\|.dt = \int_0^2 \sqrt{1+18t^2}.dt \ ; \ tg\,\theta = 3\sqrt{2}t$$

$$\Leftrightarrow \sec^2\theta.d\theta = 3\sqrt{2}.dt \ ; \ \int_0^{\text{arctg}\,6\sqrt{2}} \sec\theta \cdot \frac{\sec^2\theta}{3\sqrt{2}}.d\theta = \frac{1}{3\sqrt{2}}\int_0^{\text{arctg}\,6\sqrt{2}} \sec^3\theta.d\theta =$$

$$\frac{1}{3\sqrt{2}}\left(\frac{3\sqrt{2}.2.\sqrt{73}}{2} + \frac{1}{2}\ln\left(6\sqrt{2} + \sqrt{73}\right)\right) \neq 14$$

errei alguma coisa? Se errei, me avise

(50)

(41) a) $\gamma(t) = (2t, t^2, t^2)$, $0 \le t \le 1$, $\delta(x,y,z) = x$

$\gamma'(t) = (2, 2t, 2t) = 2(1, t, t)$

$$m = \int_\gamma dm = \int_\gamma \delta(x,y,z) \cdot ds = \int_\gamma x \cdot ds = \int_0^1 2t \, \|(2, 2t, 2t)\| \cdot dt$$

$$m = 4\int_0^1 t\sqrt{1+2t^2} \cdot dt = 4\int_1^3 t \; u^{1/2} \cdot \frac{du}{2t}$$

$$\begin{cases} u = 1+2t^2 \\ du = 4t \cdot dt \end{cases}$$

$$m = \frac{4}{4}\int_1^3 u^{1/2} \cdot du = \frac{1}{3/2} \cdot \left[3^{3/2} - 1^{3/2}\right] = \frac{2}{3}\left[3\sqrt{3} - 1\right] \text{ u.m.}$$

b) $x^2 + y^2 = 4$, $x \ge 0$ $\quad \delta(x,y) = x^2$

$\gamma(t) = (2\cos t, 2\operatorname{sen} t) \mid t \in [-\pi/2, \pi/2]$ (x ≥ 0)

$\gamma'(t) = 2(-\operatorname{sen} t, \cos t)$

$$m = \int_\gamma dm = \int_\gamma \delta(x,y) \cdot ds = \int_\gamma x^2 \cdot ds = \int_{-\pi/2}^{\pi/2} (2\cos t)^2 \cdot \|2(-\operatorname{sen} t, \cos t)\| \, dt =$$

$$\int_{-\pi/2}^{\pi/2} 8 \cdot \cos^2 t \cdot \sqrt{1} \cdot dt = 8 \cdot \left( \left[ \frac{\operatorname{sen} t \cdot \cos t}{2} \right]_{-\pi/2}^{\pi/2} + \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 \cdot dt \right) = 8 \cdot \frac{1}{2} \cdot \pi = 4\pi \text{ u.m}$$

$$\vec{R} = \frac{\sum_{i=1}^n dm_i \cdot \vec{r}_i}{\sum_{i=1}^n dm_i} = \frac{\sum_{i=1}^n dm_i (x_i, y_i)}{\sum_{i=1}^n dm_i} = \frac{\sum_{i=1}^n (dm_i \cdot x_i, dm_i \, y_i)}{M}$$

$$\vec{R} = \frac{1}{M} \cdot \left( \sum_{i=1}^n dm_i \cdot x_i, \sum_{i=1}^n dm_i \cdot y_i \right) \Rightarrow \vec{R} = \frac{1}{M}\left( \int_\gamma dm \, x, \int_\gamma dm \, y \right)$$

$$\int_\gamma dm \cdot x = \int_\gamma \delta(x,y) \cdot ds \cdot x \Leftrightarrow \int_\gamma x \cdot \delta(x,y) \cdot ds \text{ e } \int_\gamma y \cdot \delta(x,y) \cdot ds$$

Extendendo-se para cordas tridimensionais:

$$\vec{R} = \frac{1}{M}\left( \int_\gamma x \cdot \delta(x,y,z) \cdot ds, \int_\gamma y \, \delta(x,y,z) \cdot ds, \int_\gamma z \cdot \delta(x,y,z) \cdot ds \right)$$

Voltando ao exercício:

$$\int_\gamma x \cdot x^2 \cdot ds = \int_{-\pi/2}^{\pi/2} (2\cos t)^3 \cdot \|2(-\operatorname{sen} t, \cos t)\| \cdot dt = 16\int_{-\pi/2}^{\pi/2} \cos^3 t \cdot dt =$$

$$16\left( \left[ \frac{\operatorname{sen} t \cdot \cos^2 t}{3} \right]_{-\pi/2}^{\pi/2} + \frac{2}{3}\int_{-\pi/2}^{\pi/2} \cos t \cdot dt \right) = 16\left[ 0 + \frac{2}{3}\left( \operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\left(-\frac{\pi}{2}\right) \right) \right] = \frac{64}{3}$$

$$\int_\gamma y \cdot x^2 \cdot ds = \int_{-\pi/2}^{\pi/2} 2\operatorname{sen} t \cdot (2\cos t)^2 \cdot \|2(-\operatorname{sen} t, \cos t)\| \cdot dt = 16\int_{-\pi/2}^{\pi/2} \operatorname{sen} t \cdot \cos^2 t \cdot dt$$

$v = \cos t \Leftrightarrow dv = -\operatorname{sen} t \, dt$

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$$-16\int_0^0 v^2\,dv = \underline{0}$$

$$\therefore \vec{R} = \frac{1}{4\pi}\left(\frac{64}{3}, 0\right) = \left(\frac{16}{3\pi}, 0\right) \mu.c$$

c) Pensei que era o formato de um helicoide hahaha

$\gamma(t) = (2\sin t, 2\cos t, 3t)$ $\quad t \in [0, 2\pi]$

$\gamma'(t) = (2\cos t, -2\sin t, 3)$ $\quad \delta(x,y,z) = k$ (uma densidade linear)

$$m = \int_\gamma dm = \int_\gamma \delta(x,y,z).dS = \int_\gamma k\,dS = \int_0^{2\pi} k\,\|(2\cos, -2\sin t, 3)\|.dt =$$

$$k\int_0^{2\pi}\sqrt{4+9}\,.dt \Rightarrow \boxed{m = 2\pi\sqrt{13}\,k\ \mu.m.}$$

$$\vec{R} = \frac{1}{m}\left(\int_\gamma x\,dm, \int_\gamma y\,dm, \int_\gamma z\,dm\right)$$

$$\int_\gamma x\,dm = \int_\gamma x.K.dS = \int_0^{2\pi} 2\sin t.K.\|(2\cos t, -2\sin t, 3)\|.dt =$$

$$2\sqrt{13}.K\int_0^{2\pi}\sin t.dt = \underline{0}$$

$$\int_\gamma y.dm = \int_\gamma y\,k.dS = \int_0^{2\pi} 2\cos t.k\,\|(2\cos t, -2\sin t, 3)\|.dt = \underline{0}$$

$$\int_\gamma z\,dm = \int_\gamma z.k.dS = \int_0^{2\pi} 3t.k\,\|(2\cos t, -2\sin t, 3)\|.dt = 3\sqrt{13}k.\int_0^{2\pi} t.dt$$

$$3\sqrt{13}k\ \frac{1}{2}\ 4\pi^2 = 2\pi.\sqrt{13}.k.3\pi$$

$$\boxed{\vec{R} = (0, 0, 3\pi)\ \mu.c.}$$

(42) Bata seu gabarito com o meu. E perceba que:

$dI = r^2.dm = r^2.\delta(x,y,z).(dA, dV \text{ ou } dS)$

densidade areolar volumétrica linear — infinitesimais de alguma dimensão

$r = r(x,y,z) = \|\vec{r}\|$

→ A ideia é igualzinha a do exercício 18, só que agora se descreve uma corda.

$\gamma(t) = (2\sin t, 2\cos t, 3t)$ $\quad \gamma'(t) = (2\cos t, -2\sin t, 3)$

$$I_x = \int_\gamma (y^2+z^2)\,\delta(x,y,z).dS = \int_\gamma y^2.k.dS + \int_\gamma z^2.k.dS =$$

$$\int_0^{2\pi}(2\cos t)^2.k\,\|(2\cos t, -2\sin t, 3)\|.dt + \int_0^{2\pi}(3t)^2.K\,\|(2\cos t, -2\sin t, 3)\|.dt =$$

$$4\sqrt{13}.k\int_0^{2\pi}\cos^2 t.dt + 9\sqrt{13}\,k\int_0^{2\pi} t^2.dt = 4\sqrt{13}\,k\left(\left[\frac{\sin t.\cos t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1.dt\right) + 9.\sqrt{13}k\left[\frac{t^3}{3}\right]_0^{2\pi}$$

$$= 4\pi\sqrt{13}.K + 3\sqrt{13}.k\ \ 8\pi^3 = \underline{4\pi\sqrt{13}\,k\left(1 + 6\pi^2\right)}$$

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já' fiz

$$I_y = \int_\gamma (x^2+z^2).K.dS = \int_\gamma x^2.K.dS + \int_\gamma z^2.K.dS =$$

$$\int_0^{2\pi} K.\sqrt{13}.(2\operatorname{sen} t)^2.dt + 24\pi^3 K\sqrt{13} = 4K\sqrt{13}\left(\left[-\frac{\cos t.\operatorname{sen} t}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} .dt\right) +$$

$$24\pi^3 K\sqrt{13} = 4K\sqrt{13}.\pi + 4.6K\sqrt{13}\pi^3$$

$$\therefore \boxed{I_y = 4\pi K\sqrt{13}\left(1+6\pi^2\right)}$$

$$I_z = \int_\gamma (x^2+y^2)\,K.dS = \int_\gamma x^2 K.dS + \int_\gamma y^2 K.dS$$

$$I_z = 4\pi K\sqrt{13} + 4\pi K\sqrt{13} \Leftrightarrow \boxed{I_z = 8\pi K\sqrt{13}}$$