MAT2455 - Cálculo III - POLI
Prova Substitutiva - 05.07.2005

Turma B

Nome : _____
NºUSP : _____
Professor(a): _____
Turma _____

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1	
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Total	

1. Calcule a massa do sólido contido no interior da esfera $x^2 + y^2 + z^2 = a^2$, com $x \geq 0$, $z \geq 0$, $y \geq 0$ e $y \geq x\sqrt{3}$, e com densidade $z = x^2 + y^2 + 1$.

$$\text{Massa} = \iiint_R (x^2+y^2+1)\,dx\,dy\,dz.$$

Coordenadas esféricas:

$$\begin{cases} x = \rho\cos\theta\,\text{sen}\varphi \\ y = \rho\,\text{sen}\theta\,\text{sen}\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$\left.\begin{array}{l} x \geq 0 \rightarrow \cos\theta \geq 0 \\ y \geq 0 \rightarrow \text{sen}\theta \geq 0 \\ y \geq x\sqrt{3} \rightarrow \text{tg}\theta \geq \sqrt{3} \end{array}\right\} \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$$

$$z \geq 0 \rightarrow \cos\varphi \geq 0 \rightarrow \varphi \in \left[0, \frac{\pi}{2}\right]$$

Jacobiano: $|J| = \rho^2\,\text{sen}\varphi$

$$0 \leq \rho \leq a.$$

$$\text{Massa} = \int_0^{\pi/2}\int_0^{a}\int_{\pi/3}^{\pi/2} (\rho^2\text{sen}^2\varphi + 1)\rho^2\text{sen}\varphi\,d\theta\,d\rho\,d\varphi =$$

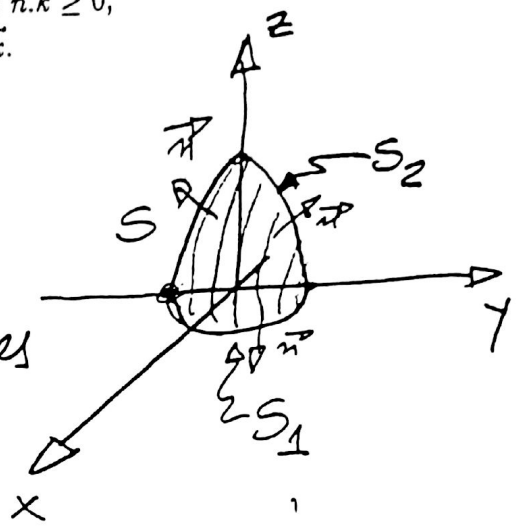
$$= \int_0^{\pi/2}\int_0^{a} \frac{\pi}{6}\left[\rho^4\text{sen}^3\varphi + \rho^2\text{sen}\varphi\right]d\rho\,d\varphi =$$

$$= \int_0^{\pi/2} \frac{\pi}{6}\left[\frac{a^5}{5}\text{sen}^3\varphi + \frac{a^3}{3}\text{sen}\varphi\right]d\varphi =$$

$$= \int_0^{\pi/2} \frac{\pi}{6}\left[\frac{a^5}{5}\text{sen}\varphi - \frac{a^5}{5}\text{sen}\varphi\cos^2\varphi + \frac{a^3}{3}\text{sen}\varphi\right]d\varphi =$$

$$= \frac{\pi}{6}\left[\frac{a^5}{5} - \frac{a^5}{15} + \frac{a^3}{3}\right] = \frac{\pi}{6}\left[\frac{2a^5}{15} + \frac{a^3}{3}\right].$$

2. Calcule $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$, onde S é a superfície do parabolóide $z = 9 - x^2 - y^2$ com $z \geq 0$ e $x \geq 0$, orientada por $\vec{n}$, tal que $\vec{n}.\vec{k} \geq 0$, e $\vec{F} = (y^2 + z)\vec{i} + e^{z^3} \cos(z + x)\vec{j} + (z^2 + 1)\vec{k}$.



Vamos usar o Teorema de Gauss. Para isto, vamos fechar uma Região R com mais duas superfícies S_1 (com $z=0$, $x \geq 0$ e $x^2+y^2 \leq 9$, $\vec{n} = -\vec{k}$) e S_2 (com $x=0$, $0 \leq z \leq 9-x^2-y^2$, $\vec{n} = -\vec{i}$), fechando R $(x \geq 0,\ x^2+y^2 \leq 9,\ 0 \leq z \leq 9-x^2-y^2)$

Temos $\operatorname{div} \vec{F} = 2z$. Pelo T. Gauss:

$$\iiint_R 2z\, dx\,dy\,dz = \iint_S \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma$$

$$\iiint_R 2z\, dx\,dy\,dz = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} 2z\, r\, dz\, dr\, d\theta =$$

Coord. cilíndricas

$$= \int_0^{2\pi}\int_0^3 (9-r^2)^2\, r\, dr\, d\theta =$$

$$= \int_0^{2\pi} \frac{243}{2}\, d\theta = 243\pi$$

S_1: $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = 0 \end{cases}$ $\quad g_r \times g_\theta = (0,0,r) \rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma = \int_{-\pi/2}^{\pi/2}\int_0^3 -r\, dr\, d\theta = -\frac{9\pi}{2}$

$r \in [0,3]$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

S_2: $\begin{cases} x = 0 \\ y = u \\ z = v \end{cases}$ $\quad -3 \leq u \leq 3$, $0 \leq v \leq 9-u^2$, $g_u \times g_v = (-1,0,0)$

$\iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma = \int^3 \int^{9-u^2} -(u^2+v)\, dv\, du = -\frac{972}{5}$

$\therefore \iint_S \vec{F}\cdot\vec{n}\, d\sigma = 243\pi + \frac{9\pi}{2} + \frac{972}{5}$

3. Calcule $\int_\gamma \vec{F}.d\vec{r}$, sendo γ a curva $x^4+y^4=10$ e $z=x^2+y^2$, cuja projeção no plano xy é percorrida no sentido anti-horário, e

$$\vec{F}=\left(\frac{x+y}{x^2+y^2},\frac{y-x}{x^2+y^2},\frac{z^8}{1+z^6}\right).$$

Vamos usar o Teorema de Stokes:

$$\vec{F}=(P,Q,R)$$

$$\text{rot}\,\vec{F}=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ P & Q & R\end{pmatrix}$$

$$\left.\begin{array}{l}\dfrac{\partial R}{\partial y}=0=\dfrac{\partial Q}{\partial z},\quad \dfrac{\partial P}{\partial z}=0=\dfrac{\partial R}{\partial x}\quad e\\[2ex] \dfrac{\partial Q}{\partial x}=\dfrac{-1.(x^2+y^2)-(y-x)\cdot(2x)}{(x^2+y^2)^2}=\dfrac{-y^2-2xy+x^2}{(x^2+y^2)^2}\\[2ex] \dfrac{\partial P}{\partial y}=\dfrac{1.(x^2+y^2)-(x+y)\cdot(2y)}{(x^2+y^2)^2}=\dfrac{-y^2-2xy+x^2}{(x^2+y^2)^2}\end{array}\right\}\text{rot}\,\vec{F}=(0,0,0)$$

Vamos tomar a superfície $S: z=x^2+y^2$, com $x^4+y^4\leq 10$ e $x^2+y^2\geq 1$, que tem como bordo as curvas γ e γ_1 (que é $x^2+y^2=1$, $z=x^2+y^2$, orientada no "sentido horário"):

$$0=\iint_S \underbrace{(\text{rot}\,\vec{F})\cdot\vec{n}}_{=0}\,d\sigma=\int_\gamma \vec{F}.d\vec{r}+\int_{\gamma_1}\vec{F}.d\vec{r}$$

$$\rightsquigarrow \boxed{\int_\gamma \vec{F}.d\vec{r}}=-\int_{\gamma_1}\vec{F}.d\vec{r}=-\int_0^{2\pi}1.dt=\boxed{-2\pi}$$

$\gamma_1(t)=(\text{sen}\,t,\cos t,1)$

$\gamma_1'(t)=(\cos t,-\text{sen}\,t,0)$