MAT 2455 - Cálculo Diferencial e Integral III para Engenharia
1ª Prova - 1o semestre de 2007

Questão 1.

a) (valor: 1,5) Calcule $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \operatorname{sen} y^3 dydx$.

b) (valor: 1,5) Calcule $\displaystyle\int\int_D xdxdy$ sendo $D \subset \mathbb{R}^2$ a região limitada pelas curvas $x = y^2$ e $x = 1 + \frac{3y^2}{4}$.

Solução:

a) A região de integração, esboçada na Figura 1, é

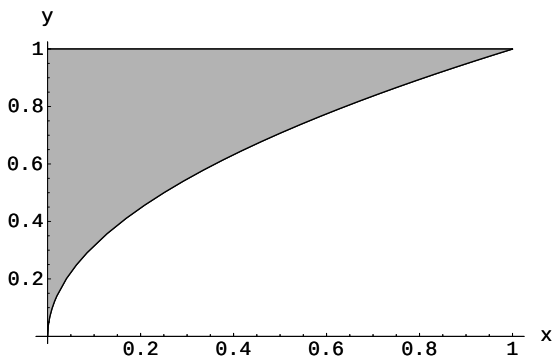

Figura 1. Região de integração.

Pelo Teorema de Fubini, temos:

$$\begin{aligned}\int_0^1 \int_{\sqrt{x}}^1 \operatorname{sen} y^3 dydx &= \int_0^1 \int_0^{y^2} \operatorname{sen} y^3 dxdy = \int_0^1 \operatorname{sen} y^3\, x\Big|_0^{y^2} dy = \int_0^1 y^2 \operatorname{sen} y^3 dy \\ &= \frac{1}{3}\int_0^1 \operatorname{sen} y^3 dy^3 = -\frac{1}{3}\cos y^3\Big|_0^1 = \frac{1}{3}(1-\cos 1).\end{aligned}$$

b) A região de integração, esboçada na Figura 2, é dada por

$$D = \left\{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2 \quad \text{e} \quad y^2 \le x \le 1 + \frac{3y^2}{4}\right\}$$

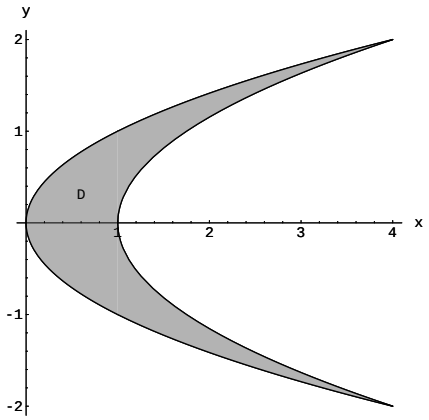

Figura 2. Região de integração D.

$$\begin{aligned}\int\int_D xdxdy &= 2\int_0^2\int_{y^2}^{1+\frac{3y^2}{4}} xdxdy = 2\int_0^2 \frac{x^2}{2}\Big|_{y^2}^{1+\frac{3y^2}{4}} dy = \int_0^2 \left(1+\frac{3y^2}{2}-\frac{7y^4}{16}\right) dy \\ &= \left(y+\frac{y^3}{2}-\frac{7y^5}{80}\right)\Big|_0^2 = 2+4-\frac{14}{5} = \frac{16}{5}.\end{aligned}$$

Questão 2. (valor 3,5) Calcule a massa do sólido dado por $x^2+y^2 \leq 16$, $x^2+y^2 \geq 2y$ e $0 \leq z \leq 3$ com densidade $\delta(x,y,z) = \sqrt{x^2+y^2}$.

Solução: Defina-se os seguintes sólidos:

$$\begin{aligned} S_1 &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 16, \quad 0 \leq z \leq 3\} \\ S_2 &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \geq 2y, \quad 0 \leq z \leq 3\} \end{aligned}$$

Logo o sólido em questão é dado por: $S = S_1 - S_2$ e esboçado na Fig. 3.

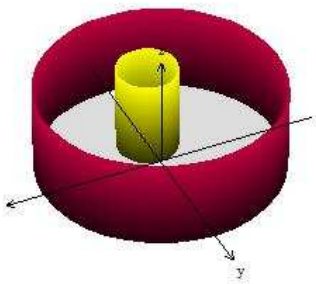

Figura 3. Sólido de integração S.

Seja M_1 a massa do sólido S_1 e M_2 a massa do sólido S_2. Por definição

$$M_1 = \int\int\int_{S_1} \sqrt{x^2+y^2}dxdydz, \quad M_2 = \int\int\int_{S_2} \sqrt{x^2+y^2}dxdydz$$

Os sólidos de integração em coordenadas cilíndricas: $x = r\cos\theta,\ y = r\text{sen}\,\theta,\ z = z$ são:

$$S_1^* = \{(r,\theta,z) \in \mathbb{R}^3 : 0 \le \theta \le 2\pi, \quad 0 \le r \le 4, \quad 0 \le z \le 3\}$$
$$S_2^* = \{(r,\theta,z) \in \mathbb{R}^3 : 0 \le \theta \le \pi, \quad 0 \le r \le 2\,\text{sen}\,\theta, \quad 0 \le z \le 3\}$$

e o jacobinao é r. Logo a massa M_1 em coordenadas cilíndricas é:

$$\begin{aligned} M_1 &= \int\int\int_{S_1^*} r^2 dzdrd\theta = \int_0^{2\pi}\int_0^4\int_0^3 r^2 dzdrd\theta = \int_0^{2\pi}\int_0^4 r^2\, z\Big|_0^3 drd\theta \\ &= z\Big|_0^3 \int_0^{2\pi} \frac{r^3}{3}\Big|_0^4 d\theta = z\Big|_0^3 \frac{r^3}{3}\Big|_0^4 \theta\Big|_0^{2\pi} = 3\left(\frac{4^3}{3}\right)2\pi = 128\pi. \end{aligned}$$

e a massa M_2 por:

$$\begin{aligned} M_2 &= \int\int\int_{S_2^*} r^2 dzdrd\theta = \int_0^{\pi}\int_0^{2\,\text{sen}\,\theta}\int_0^3 r^2 dzdrd\theta = \int_0^{\pi}\int_0^{2\,\text{sen}\,\theta} r^2\, z\Big|_0^3 drd\theta \\ &= z\Big|_0^3 \int_0^{\pi} \frac{r^3}{3}\Big|_0^{2\,\text{sen}\,\theta} d\theta = z\Big|_0^3 \frac{1}{3}\int_0^{\pi} (2\,\text{sen}\,\theta)^3\, d\theta = 8\int_0^{\pi} \text{sen}^3\theta d\theta \\ &= 8\int_0^{\pi} (1-\cos^2\theta)\text{sen}\,\theta d\theta \end{aligned}$$

Seja $u = \cos\theta$ então $du = -\text{sen}\,\theta d\theta$. Logo:

$$M_2 = -8\int_1^{-1}(1-u^2)du = 8\int_{-1}^{1}(1-u^2)du = 8\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = \frac{32}{3}.$$

Portanto a massa M do sólido S é dado por:

$$M = M_1 - M_2 = 128\pi - \frac{32}{3}.$$

Questão 3. (valor 3,5) Use coordenadas esféricas para calcular a integral

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{\sqrt{x+y^2}}^{\sqrt{2a^2-x^2-y^2}} (x^2+y^2+z^2)dzdydx \quad (a > 0, \text{ fixo})$$

Solução: O domínio de integração E é:

$$E = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le a,\ 0 \le y \le \sqrt{a^2-x^2},\ \sqrt{x^2+y^2} \le z \le \sqrt{2a^2-x^2-y^2}\right\}$$

e para $a = 1$ esboçamos seu gráfico na Fig. 4.

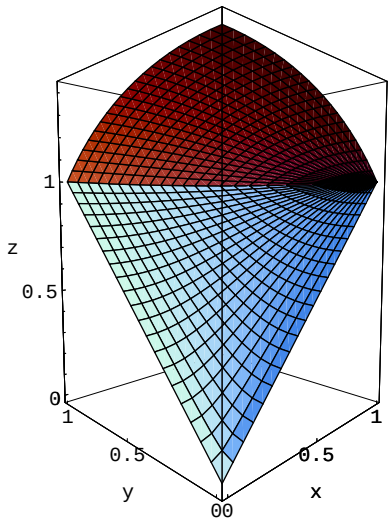

Figura 4. Domínio de integração E para $a = 1$.

Usando coordenadas esféricas: $x = \rho \operatorname{sen}\varphi \cos\theta$, $y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta$, $z = \cos\varphi$ temos que o novo domínio é:

$$E^* = \left\{(\rho, \theta, \varphi) \in \mathbb{R}^3 : 0 \leq \rho \leq \sqrt{2}a,\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq \varphi \leq \frac{\pi}{4}\right\}$$

O jacobiano é $\rho^2 \operatorname{sen}\varphi$. Logo

$$\begin{aligned}
I &= \int\int\int_E (x^2 + y^2 + z^2)dxdydz = \int\int\int_{E^*} \rho^2(\rho^2 \operatorname{sen}\phi)d\rho d\theta d\phi \\
&= \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}a} \rho^4 \operatorname{sen}\varphi d\rho d\varphi d\theta = \frac{\rho^5}{5}\Big|_0^{\sqrt{2}a} \int_0^{\pi/2}\int_0^{\pi/4} \operatorname{sen}\varphi\ d\varphi d\theta \\
&= \frac{\rho^5}{5}\Big|_0^{\sqrt{2}a}\left(-\cos\varphi\Big|_0^{\pi/4}\right)\int_0^{\pi/2} d\theta = \frac{\rho^5}{5}\Big|_0^{\sqrt{2}a}\left(-\cos\varphi\Big|_0^{\pi/4}\right)\theta\Big|_0^{\pi/2} \\
&= \frac{(\sqrt{2}a)^5}{5}\left(1 - \frac{\sqrt{2}}{2}\right)\frac{\pi}{2} = \frac{2a^5\pi}{5}(\sqrt{2} - 1).
\end{aligned}$$