

Lista 2 | Paulo Akira 2016.2

1) a) $f(x,y) = \operatorname{arctg} \left(\frac{y}{x} \right)$

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2} \cdot \frac{1}{1 + (\frac{y}{x})^2} = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x} \cdot \frac{1}{1 + (\frac{y}{x})^2} = \frac{x}{x^2 + y^2}$$

b) $f(x,y) = \ln(1 + \cos^2(xy^3))$

$$\frac{\partial f}{\partial x} = \frac{y^3(-\operatorname{sen}(xy^3)) \cdot 2\cos(xy^3)}{1 + \cos^2(xy^3)} = \frac{-y^3 \operatorname{sen}(2xy^3)}{1 + \cos^2(xy^3)}$$

$$\frac{\partial f}{\partial y} = \frac{3xy^2 \cdot (-\operatorname{sen}(xy^3)) \cdot 2\cos(xy^3)}{1 + \cos^2(xy^3)} = \frac{-3xy^2 \operatorname{sen}(2xy^3)}{1 + \cos^2(xy^3)}$$

2) $\frac{\partial f}{\partial x}(1,0) = \lim_{h \rightarrow 0} \frac{f(1+h,0) - f(1,0)}{h} =$

$$= \lim_{h \rightarrow 0} \frac{(1+h) \cdot (1+h)^{-3} - 1}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^{-2} - 1}{h}.$$

$$= \lim_{h \rightarrow 0} \frac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h \rightarrow 0} \frac{1 - 1 - 2h - h^2}{h + 2h^2 + h^3} \stackrel{H}{=} 0$$

$$= \lim_{h \rightarrow 0} \frac{-2 - 2h}{1 + 4h + 3h^2} = -2$$

3) $\frac{\partial u}{\partial x} = \frac{x \cdot (x^2 + y^2)^{-1/2}}{\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2}$

$$\frac{\partial u}{\partial y} = \frac{y \cdot (x^2 + y^2)^{-1/2}}{\sqrt{x^2 + y^2}} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{2(x^2 + y^2) - 2(x^2 + y^2)}{(x^2 + y^2)^2} = 0 \therefore u(x,y) = \ln \sqrt{x^2 + y^2} \text{ é}$$

Solução da equação de Laplace bidimensional

4) a) $f(x,y) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & se (x,y) \neq (0,0) \\ 0, & se (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x} = \frac{y^2 \cdot (x^2 + y^4) - xy^2 \cdot 2x}{(x^2 + y^4)^2} = \frac{y^2(x^2 + y^4)}{(x^2 + y^4)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2yx \cdot (x^2 + y^4) - xy^2 \cdot 4y^3}{(x^2 + y^4)^2} = \frac{2yx(x^2 + y^4) - 4xy^5}{(x^2 + y^4)^2}$$

ii) $(x,y) = (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ existem em todos os pontos de \mathbb{R}^2

b) $M_1(1,0) : f \circ M_1(+)=0$

$$M_2(1^2, 1) : f \circ M_2(+)=\frac{+^4}{2^4}=\frac{1}{2}$$

Como $\lim_{t \rightarrow 0} f \circ M_1(t) + \lim_{t \rightarrow 0} f \circ M_2(t)$ então a função

f não é contínua em $(0,0)$

c) Como a função não é contínua em $(0,0)$, logo
não é diferenciável em $(0,0)$.

5) a) Para que f seja contínua devemos ter $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$

$$= 0:$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot \frac{x^2}{x^2 + y^2}}{x^2 + y^2} = 0$$

$$b) \frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$c) \lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0) - h - ok}{\sqrt{h^2 + k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{\frac{h^3}{h^2 + k^2} - h}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{h^3 - h^3 - hk^2}{(h^2 + k^2)\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{-hk^2}{(h^2 + k^2)\sqrt{h^2 + k^2}}$$

Escolhendo $M(+)= (+,+)$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{-+^3}{2^{+2} \cdot (2^{+2})^{1/2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{-+^3}{2^{+2} + \sqrt{2}} =$$

$$= -\frac{1}{2\sqrt{2}} \therefore f \text{ não é diferenciável em } (0,0).$$

$$d) \frac{\partial f}{\partial x} = \frac{3x^2 \cdot (x^2 + y^2) - 2x \cdot x^3}{(x^2 + y^2)^2} = \frac{3x^4 + 3x^2y^2 - 2x^4}{(x^2 + y^2)^2}$$

$$M_1(+)= (0,+): \left(\frac{\partial f}{\partial x} \circ M_1\right)(+) = 0$$

$$M_2(+)= (+,+): \left(\frac{\partial f}{\partial x} \circ M_2\right)(+) = 1$$

Como $\lim_{t \rightarrow 0} \left(\frac{\partial f}{\partial x} \circ M_1 \right)(t) \neq \lim_{t \rightarrow 0} \left(\frac{\partial f}{\partial y} \circ M_2 \right)(t)$, então

$\frac{\partial f}{\partial x}$ não é contínua em $(0,0)$.

$$\frac{\partial f}{\partial y} = \frac{-2y^3}{(x^2+y^2)^2}$$

$$M_1(t) = (t,0) \therefore \left(\frac{\partial f}{\partial y} \circ M_1 \right)(t) = 0$$

$$M_2(t) = (t,t) \therefore \left(\frac{\partial f}{\partial y} \circ M_2 \right)(t) = -\frac{1}{2}$$

Como $\lim_{t \rightarrow 0} \left(\frac{\partial f}{\partial y} \circ M_1 \right)(t) \neq \lim_{t \rightarrow 0} \left(\frac{\partial f}{\partial y} \circ M_2 \right)(t)$, então

$\frac{\partial f}{\partial y}$ não é contínua em $(0,0)$

$$6) g(x,y) = (3x^4 + 2y^4)^{1/3}$$

$$\frac{\partial g}{\partial x} = 12x^3 (3x^4 + 2y^4)^{-2/3} = \frac{12x^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}}$$

$$\frac{\partial g}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{3h^{4/3}}{h} = 0$$

$$\frac{\partial g}{\partial y} = 8y^3 (3x^4 + 2y^4)^{-2/3} = \frac{8y^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}}$$

$$\frac{\partial g}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = \frac{2k^{4/3}}{k} = 0$$

$$\frac{\partial g}{\partial x} = \begin{cases} \frac{4x^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

Para que $\frac{\partial g}{\partial x}$ seja contínua devemos dar $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial g}{\partial x} = 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4x^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}} \cdot \lim_{(x,y) \rightarrow (0,0)} \frac{4}{3\sqrt[3]{(3x^4 + 2y^4)^2}} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} 4 \sqrt[3]{\underbrace{\frac{x^4}{(3x^4 + 2y^4)^2}}_{\text{limite}} \cdot x} = 0 \therefore \text{é contínua}$$

$$\frac{\partial g}{\partial y} = \begin{cases} \frac{8y^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

Para que $\frac{\partial g}{\partial y}$ seja contínua devemos dar $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial g}{\partial y} = 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{8y^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{8}{3\sqrt[3]{\frac{y^4}{(3x^4 + 2y^4)^2}}} y = 0$$

$$\textcircled{1} \quad x^4 \leq 3x^4 + 2y^4 \Rightarrow 0 \leq \frac{x^4}{3x^4 + 2y^4} \leq 1$$

Como $\frac{\partial g}{\partial x}$ e $\frac{\partial g}{\partial y}$ são contínuas em todo domínio \mathbb{R}^2 então g é de classe C^1 .

$$7) c) f(x,y) = e^{\sqrt{x^4+y^4}}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x^3 \cdot \frac{1}{2} \cdot (x^4+y^4)^{-1/2} \cdot e^{\sqrt{x^4+y^4}} = \\ &= \frac{2x^3 e^{\sqrt{x^4+y^4}}}{\sqrt{x^4+y^4}}, \text{ se } (x,y) \neq (0,0) \end{aligned}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{e^{h^2} - 1}{h}$$

$$\stackrel{4}{=} \lim_{h \rightarrow 0} 2he^{h^2} \cdot 0, \text{ se } (x,y) = (0,0)$$

$$\frac{\partial f}{\partial y} = \frac{2y^3 \cdot e^{\sqrt{x^4+y^4}}}{\sqrt{x^4+y^4}}, \text{ se } (x,y) \neq (0,0)$$

Analogamente a $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^3 e^{\sqrt{x^4+y^4}}}{\sqrt{x^4+y^4}} = \lim_{(x,y) \rightarrow (0,0)} \frac{2e^{\sqrt{x^4+y^4}}}{\sqrt{x^4+y^4}} \xrightarrow[\text{limite}]{\frac{x^4 \cdot x^2}{x^4+y^4} \rightarrow 0}$$

$$= 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2y^3 e^{\sqrt{x^4+y^4}}}{\sqrt{x^4+y^4}} = 0$$

A função f é de classe C^1 em \mathbb{R}^2 então é diferenciável em todo \mathbb{R}^2 .

$$d) f(x,y) = \cos(\sqrt{x^2+y^2})$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\cos \sqrt{h^2} - 1}{h}$$

$$\stackrel{4}{=} \lim_{h \rightarrow 0} \frac{2h}{2\sqrt{h^2}} \cdot (-\sin \sqrt{h^2}) = \lim_{h \rightarrow 0} \frac{2h}{2|h|} \cdot (-\sin \sqrt{h^2}) \stackrel{\textcircled{2}}{=} 0$$

$$\lim_{h \rightarrow 0^+} \frac{2h}{2h} \cdot (-\sin \sqrt{h^2}) = \lim_{h \rightarrow 0^+} -\frac{2h}{2h} \cdot (-\sin \sqrt{h^2}) = 0$$

Analogamente, $\frac{\partial f}{\partial y}(0,0) = 0$.

Como não sabemos nada sobre a continuidade de $\frac{\partial f}{\partial x}$

$\frac{\partial f}{\partial y}$ em $(0,0)$ então este ponto acaba se tornando

um "ponto problema" da função, pois, não podemos dizer nada sobre sua diferenciabilidade nesse ponto. Portanto devemos utilizar o "limite" neste ponto.

$$\frac{\partial f}{\partial x} = \frac{2x}{2} \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot (-\sin \sqrt{x^2+y^2}) =$$

$$= -x \cdot \frac{\sin \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad \text{Analogamente } \frac{\partial f}{\partial y} = -y \cdot \frac{\sin \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y}(x,y) = 0$$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ são contínuas. Assim f é de classe C^1
 $\Rightarrow f$ é diferenciável.

$$8) a) y \cdot f(x,y) \Rightarrow f(0,0) = 1$$

$$\frac{\partial f}{\partial x} = 2x e^{x^2+y^2} \therefore \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = 2y e^{x^2+y^2} \therefore \frac{\partial f}{\partial y}(0,0) = 0$$

$$\pi: z = 1 + 0(x-0) + 0(y-0) = 1$$

$$\vec{m} = (0, 0, -1) \therefore r: \infty = (0, 0, 1) + \lambda(0, 0, -1), \lambda \in \mathbb{R}$$

$$b) y = f(x,y) \therefore f(3,2) = 0$$

$$\frac{\partial f}{\partial x} = \ln\left(\frac{y}{2}\right) \cdot e^x \therefore \frac{\partial f}{\partial x}(3,2) = 0$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{2} \cdot \frac{1}{(y/2)} = \frac{e^x}{y} \therefore \frac{\partial f}{\partial y}(3,2) = \frac{e^3}{2}$$

$$\pi: z = 0 + \frac{\partial f}{\partial x}(3,2)(x-3) + \frac{\partial f}{\partial y}(3,2)(y-2)$$

$$\pi: z = \frac{e^3 y}{2} - e^3 = \frac{e^3 y - 2e^3}{2}$$

$$\pi: 2z - e^3 y + 2e^3 = 0$$

$$\vec{n} = (0, e^{3/2}, 0) \therefore r: \infty = (3, 2, 0) + \lambda(0, e^{3/2}, -1), \lambda \in \mathbb{R}$$

$$g) f(3,4) = \sqrt{9+16} = 5 \quad e \quad g(3,4) = \frac{1}{10}(9+16) + \frac{5}{2} = 5$$

Como $f(3,4) = g(3,4)$ vemos que as funções se intersectam no ponto $(3,4,5)$.

$$\frac{\partial f}{\partial x} = 2x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \therefore \frac{\partial f}{\partial x}(3,4) = \frac{3}{5}$$

$$\text{Analogamente, } \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \therefore \frac{\partial f}{\partial y}(3,4) = \frac{4}{5}$$

$$\frac{\partial g}{\partial x} = \frac{2x}{10} = \frac{x}{5} \therefore \frac{\partial g}{\partial x}(3,4) = \frac{3}{5}$$

$$\frac{\partial g}{\partial y} = \frac{2y}{10} = \frac{y}{5} \therefore \frac{\partial g}{\partial y}(3,4) = \frac{4}{5}$$

$$\text{Como } \frac{\partial f}{\partial x}(3,4) = \frac{\partial g}{\partial x}(3,4) \leftarrow \frac{\partial f}{\partial y}(3,4) = \frac{\partial g}{\partial y}(3,4)$$

então o plano tangente neste ponto é o mesmo.

$$11) \frac{\partial f}{\partial x} = 2x \cdot \frac{1}{x^2+ky^2} \therefore \frac{\partial f}{\partial x}(2,1) = \frac{4}{4+k}$$

$$\frac{\partial f}{\partial y} = 2ky \cdot \frac{1}{x^2+ky^2} \therefore \frac{\partial f}{\partial y}(2,1) = \frac{2k}{4+k}$$

$$n_1 = \left(\frac{4}{4+k}, \frac{4k}{4+k}, -1 \right)$$

$$n_2 = (-3, 0, -1)$$

$$\langle n_1, n_2 \rangle = 0 \Rightarrow -\left(\frac{12}{4+k}\right) + 1 = 0 \Rightarrow 4+k = 12 \Rightarrow$$

$$\Rightarrow k = 8$$

$$10) \frac{\partial g}{\partial x} = 3x^2y \therefore \frac{\partial g}{\partial x}(x_0, y_0) = 3x_0^2y_0$$

$$\frac{\partial g}{\partial y} = x^3 \therefore \frac{\partial g}{\partial y}(x_0, y_0) = x_0^3$$

$$\begin{cases} 5 = x_0^3 y_0 + 3x_0^2 y_0 (0-x_0) + x_0^3 (1-y_0) \\ 6 = x_0^3 y_0 + 3x_0^2 y_0 (0-x_0) + x_0^3 (0-y_0) \end{cases} \Rightarrow$$

$$\begin{cases} 5 = x_0^3 y_0 - 3x_0^3 y_0 + x_0^3 - x_0^3/y_0 \\ 6 = x_0^3 y_0 - 3x_0^3 y_0 - y_0^3 y_0 \end{cases} \Rightarrow$$

$$\begin{cases} 5 = x_0^3 - 3x_0^3 y_0 \\ 6 = -3x_0^3 y_0 \end{cases} \Rightarrow x_0 = -1 \therefore y_0 = 2$$

$$\pi: z = -2 + 6(x+1) - (y-2)$$

Existe apenas um plano.

$$F(t,u) = \omega(x(t,u), y(t,u))$$

$$\frac{\partial F}{\partial t} = \frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial t} \Rightarrow$$

$$\Rightarrow \frac{\partial \omega}{\partial t} = 2x \cdot 2t + 2y \cdot 2u \Rightarrow$$

$$\Rightarrow \frac{\partial \omega}{\partial t} = 4t(1^2 + u^2) + 8tu^2 = 4t^3 + 12tu^2$$

$$\frac{\partial F}{\partial u} = \frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial u} \Rightarrow$$

$$\Rightarrow \frac{\partial \omega}{\partial u} = 2x \cdot 2u + 2y \cdot 2t \Rightarrow \frac{\partial \omega}{\partial u} = 4(1^2 + u^2) + 8t^2 u$$

Verificando pela substituição:

$$\omega = (t^2 + u^2)^2 + (2tu)^2 = t^4 + 2t^2u^2 + u^4 + 4t^2u^2$$

$$\frac{\partial \omega}{\partial t} = 4t^3 + 4tu^2 + 8tu^2 = 4t^3 + 12tu^2$$

$$\frac{\partial \omega}{\partial u} = 4t^2u + 4u^3 + 8t^2u = 4u^3 + 12t^2u$$

$$b) G(t, u) = \omega(x(t, u), y(t, u))$$

$$\frac{\partial \omega}{\partial x} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} =$$

$$= \frac{t^2 \sin^2 u - t^2 \cos^2 u}{(t^2 \cos^2 u + t^2 \sin^2 u)^2} = \frac{t^2 (\sin^2 u - \cos^2 u)}{t^4} =$$

$$= \frac{\sin^2 u - \cos^2 u}{t^2}$$

$$\frac{\partial \omega}{\partial y} = \frac{-2y \cdot x}{(x^2 + y^2)^2} = \frac{-2t^2 \sin u \cos u}{t^4} =$$

$$= -\frac{2 \sin u \cos u}{t^2}$$

$$\frac{\partial G}{\partial t}(t, u) = \frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial t} =$$

$$= \left(\frac{\sin^2 u - \cos^2 u}{t^2} \right) \cdot (\cos u) + \left(-\frac{2 \sin u \cos u}{t^2} \right) \cdot (\sin u) =$$

$$= \frac{\sin^2 u \cos u - \cos^3 u}{t^2} - \frac{2 \sin^2 u \cos u}{t^2} =$$

$$= -\frac{\sin^2 u \cos u - \cos^3 u}{t^2} - \frac{\cos u (\sin^2 u + \cos^2 u)}{t^2} - \frac{\cos u}{t^2} =$$

$$\frac{\partial G}{\partial \theta}(t, u) = \frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial u} =$$

$$= \left(\frac{\sin^2 u - \cos^2 u}{t^2} \right) (-\sin u) + \left(-\frac{2 \sin u \cos u}{t^2} \right) (\cos u) =$$

$$= -\frac{\sin^3 u + \sin u \cos^2 u}{t^2} - \frac{2 \sin u \cos^2 u}{t^2} =$$

$$= -\frac{\sin^3 u - \sin u \cos^2 u}{t^2} = -\frac{\sin u}{t^2}$$

Verificando por substituição:

$$\omega = \frac{\cos u}{t^2 \cos^2 u + t^2 \sin^2 u} = \frac{\cos u}{t^2} = \frac{\cos u}{t^2}$$

$$\frac{\partial \omega}{\partial t} = -\frac{\sin u}{t^2} + \frac{\partial \omega}{\partial u} = -\frac{\sin u}{t^2}$$

$$17) \frac{\partial v}{\partial r}(r, \theta) = \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r} =$$

$$= \frac{\partial v}{\partial x} \cdot \cos \theta + \frac{\partial v}{\partial y} \cdot \sin \theta$$

$$\frac{\partial v}{\partial \theta}(r, \theta) = \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} =$$

$$= \frac{\partial v}{\partial x} \cdot (-r \sin \theta) + \frac{\partial v}{\partial y} \cdot (r \cos \theta)$$

$$i) \frac{\partial^2 v}{\partial r^2}(r, \theta) = \left(\frac{\partial}{\partial r} \left(\frac{\partial v}{\partial x} \right) \right) \frac{\partial x}{\partial r} + \frac{\partial v}{\partial x} \cdot \frac{\partial^2 x}{\partial r^2} +$$

$$+ \frac{\partial}{\partial r} \left(\frac{\partial v}{\partial y} \right) \frac{\partial y}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2} =$$

$$= \left(\frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y^2} \cdot \frac{\partial y}{\partial r} \right) \frac{\partial x}{\partial r} + \frac{\partial v}{\partial x} \cdot \frac{\partial^2 x}{\partial r^2} +$$

$$+ \left(\frac{\partial^2 v}{\partial x \partial y} \cdot \frac{\partial x}{\partial r} + \frac{\partial^2 v}{\partial y^2} \cdot \frac{\partial y}{\partial r} \right) \frac{\partial y}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2} =$$

$$= (u_{xx} \cdot x_r + u_{yy} \cdot y_r) x_r + u_x \cdot x_{rr} + (u_{xy} \cdot x_r + u_{yy} \cdot y_r) y_r$$

$$+ u_y \cdot y_{rr} = u_{xx} (x_r)^2 + 2 u_{xy} \cdot x_r \cdot y_r + u_x \cdot x_{rr} + u_y \cdot y_{rr} +$$

$$+ u_{yy} \cdot (y_r)^2 =$$

$$= \frac{\partial^2 v}{\partial x^2} \cdot \cos^2 \theta + 2 \cdot \frac{\partial v}{\partial x \partial y} \cdot \cos \theta \sin \theta + \frac{\partial v}{\partial y^2} \cdot \sin^2 \theta =$$

$$= \frac{\partial^2 v}{\partial x^2} \cdot \cos^2 \theta + \frac{\partial v}{\partial x \partial y} \cdot \sin 2\theta + \frac{\partial v}{\partial y^2} \cdot \sin^2 \theta \quad (a)$$

$$ii) \frac{\partial^2 v}{\partial \theta^2}(r, \theta) = \left(\frac{\partial}{\partial \theta} \left(\frac{\partial v}{\partial x} \right) \right) \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial x} \cdot \frac{\partial^2 x}{\partial \theta^2} +$$

$$+ \frac{\partial}{\partial \theta} \left(\frac{\partial v}{\partial y} \right) \frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial^2 y}{\partial \theta^2} =$$

$$= u_{xx} (x_\theta)^2 + 2 u_{xy} \cdot x_\theta \cdot y_\theta + u_x \cdot x_{\theta\theta} + u_y \cdot y_{\theta\theta} + u_{yy} (y_\theta)^2$$

$$= \frac{\partial^2 v}{\partial x^2} \cdot (r^2 \sin^2 \theta) + 2 \frac{\partial v}{\partial x \partial y} \cdot (-r \sin \theta) (r \cos \theta) + \frac{\partial v}{\partial x} \cdot (-r \cos \theta) +$$

$$+ \frac{\partial v}{\partial y} \cdot (-r \sin \theta) + \frac{\partial^2 v}{\partial y^2} \cdot (r^2 \cos^2 \theta) \quad (b)$$

$$iii) \Delta v = u_{xx} + u_{yy} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

$$iv) \frac{1}{r} \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial x} \cdot \cos \theta + \frac{1}{r} \frac{\partial v}{\partial y} \cdot \sin \theta \quad (c)$$

$$\text{Note que } (a) + \frac{1}{r^2} \cdot (b) + (c) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \Delta v$$

$$18) \frac{\partial v}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} (\cos s \cos t) + \frac{\partial f}{\partial y} (\cos s \sin t)$$

$$= e^s \left(\frac{\partial f}{\partial x} \cos t + \frac{\partial f}{\partial y} \sin t \right) \quad (d)$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial v} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} - e^s \sin t + \frac{\partial f}{\partial y} (e^s \cos t) =$$

$$= e^s \left(-\frac{\partial f}{\partial x} \sin t + \frac{\partial f}{\partial y} \cos t \right) \quad (2)$$

$$(1)^2 = e^{2s} \left[\left(\frac{\partial f}{\partial x} \right)^2 \cos^2 t + \left(\frac{\partial f}{\partial y} \right)^2 \sin^2 t + \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} \sin 2t \right] \quad (3)$$

$$(2)^2 = e^{2s} \left[\left(\frac{\partial f}{\partial x} \right)^2 \sin^2 t + \left(\frac{\partial f}{\partial y} \right)^2 \cos^2 t - \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} \sin 2t \right] \quad (4)$$

De (3) + (4):

$$\left(\frac{\partial u}{\partial s} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 = e^{2s} \left[\left(\frac{\partial f}{\partial x} \right)^2 (\cos^2 t + \sin^2 t) + \left(\frac{\partial f}{\partial y} \right)^2 (\sin^2 t + \cos^2 t) \right]$$

$$\left(\frac{\partial u}{\partial s} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 = e^{2s} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]$$

$$\frac{\partial^2 u}{\partial s^2} = \left(\frac{\partial}{\partial s} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial s} + \frac{\partial^2 x}{\partial s^2} \frac{\partial f}{\partial x} + \frac{\partial}{\partial s} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial s} + \right.$$

$$\left. + \frac{\partial^2 y}{\partial s^2} \frac{\partial f}{\partial y} \right) =$$

$$= \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial s} \right) \frac{\partial x}{\partial s} + \frac{\partial^2 x}{\partial s^2} \frac{\partial f}{\partial x} +$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial s} \right) \frac{\partial y}{\partial s} + \frac{\partial^2 y}{\partial s^2} \frac{\partial f}{\partial y} =$$

$$= (f_{xx} \cdot x_s + f_{xy} \cdot y_s) x_s + x_{ss} \cdot f_{xx} +$$

$$+ (f_{xy} \cdot x_s + f_{yy} \cdot y_s) y_s + y_{ss} \cdot f_{yy} =$$

$$= x_{ss} f_{xx} + f_{xx} (x_s)^2 + 2 f_{xy} \cdot x_s \cdot y_s + f_{yy} (y_s)^2 +$$

$$+ y_{ss} \cdot f_{yy} \quad (6)$$

Analogamente a (6),

$$\frac{\partial^2 u}{\partial t^2} = x_{tt} f_{xx} + f_{xx} (x_{tt})^2 + 2 f_{xy} \cdot x_{tt} \cdot y_{tt} + f_{yy} (y_{tt})^2 +$$

$$+ y_{tt} \cdot f_{yy} \quad (7)$$

$$x_s = \frac{\partial x}{\partial s} = e^s \cos t \quad \therefore x_{ss} = \frac{\partial^2 x}{\partial s^2} = e^s \cos t$$

$$y_s = \frac{\partial y}{\partial s} = e^s \sin t \quad \therefore y_{ss} = \frac{\partial^2 y}{\partial s^2} = e^s \sin t$$

$$x_{tt} = \frac{\partial x}{\partial t} = -e^s \sin t = -y_s \quad x_{tt} = -e^s \cos t = -x_{ss}$$

$$y_{tt} = \frac{\partial y}{\partial t} = e^s \cos t = x_s \quad \therefore y_{tt} = -e^s \sin t = -y_{ss}$$

De (6) + (7):

$$\frac{\partial^2 u}{\partial s^2} (s, t) + \frac{\partial^2 u}{\partial t^2} (s, t) = e^{2s} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

14) Como g é paralela a relativa no ponto $t=1$

$$\text{então } g'(1) = 2.$$

$$\frac{dg}{dt} (t) = \frac{\partial f}{\partial x} (2t^3 - 4t, t^4 - 3t) \frac{dx}{dt} + \frac{\partial f}{\partial y} (2t^3 - 4t, t^4 - 3t) \frac{dy}{dt}$$

$$g'(t) = (2t^3 - 4t, t^4 - 3t) \quad \Rightarrow \quad g'(t) = (6t^2 - 4, 4t^3 - 3) \rightarrow$$

$$\Rightarrow g'(1) = (2, 1) \quad (2)$$

$$\nabla f(-2, -2) = \left(\frac{\partial f}{\partial x} (-2, -2), \frac{\partial f}{\partial y} (-2, -2) \right) = (a, -4)$$

Tomando $t=1$ e substituindo (2) em (3) em (1):

$$g'(1) = \frac{\partial f}{\partial x} (-2, -2) \cdot 2 + \frac{\partial f}{\partial y} (-2, -2) \cdot 1 = 2 \Rightarrow$$

$$\Rightarrow 2a - 4 = 2 \Rightarrow 2a = 6 \Rightarrow a = 3$$

15) Sabendo que o plano tangente ao gráfico de f no ponto $(0, 2, f(0, 2))$ é $2x + y + z = 7$, portanto,

$$\frac{\partial f}{\partial x} (0, 2) = 2 \quad \text{e} \quad \frac{\partial f}{\partial y} (0, 2) = -1.$$

Devemos encontrar o valor da $f(0, 2)$:

$$z = \frac{\partial f}{\partial x} (0, 2) \cdot (x-0) + \frac{\partial f}{\partial y} (0, 2) (y-2) + f(0, 2) \Rightarrow$$

$$\Rightarrow z = 2x - y + 2 + f(0, 2) \Rightarrow z + 2x + y = \underbrace{2 + f(0, 2)}_{= 7}$$

$$\therefore f(0, 2) = 7 - 2 = 5$$

As derivadas parciais de f serão:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} =$$

$$= \frac{\partial f}{\partial x} \cdot (2u \cos(v^2 - v^3)) + \frac{\partial f}{\partial y} \cdot (4uv)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} =$$

$$= \frac{\partial f}{\partial x} (-3v^2 \cos(v^2 - v^3)) + \frac{\partial f}{\partial y} (2u^2)$$

As derivadas parciais de g serão:

$$\frac{\partial g}{\partial u} = f(\sin(u^2-v^3), 2u^2v) + u \cdot \frac{\partial f}{\partial u}$$

$$\frac{\partial g}{\partial v} = u \cdot \frac{\partial f}{\partial v}$$

No ponto $(1, 1, g(1,1))$ teremos:

$$\begin{aligned}\frac{\partial g}{\partial u}(1,1) &= 5 + 1 \cdot \frac{\partial f}{\partial x}(0,2) \cdot 2 + \frac{\partial f}{\partial y}(0,2) \cdot 4 = \\ &= 5 + (-4) + (-4) = -3\end{aligned}$$

$$\frac{\partial g}{\partial v} = 1 \cdot \frac{\partial f}{\partial y}(0,2) \cdot (-3) + \frac{\partial f}{\partial x}(0,2) \cdot 2 = 4$$

O vetor normal será: $\vec{m} = (-3, 4, -1)$.

Para que o plano tangente ao gráfico g no ponto $(1, 1, g(1,1))$ seja paralelo ao vetor $v = (4, 2, \alpha)$ devemos ter:

$$\langle \vec{n}, \vec{v} \rangle = 0 \Rightarrow -12 + 8 - \alpha = 0 \Rightarrow \alpha = -4$$

$$20) F(r, s) = G(r e^s, r^3 \cos(s))$$

$$\frac{\partial G}{\partial r} = \frac{\partial G}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial G}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial}{\partial r} \left(\frac{\partial G}{\partial r} \right) = \frac{\partial^2 G}{\partial r^2} =$$

$$= \frac{\partial}{\partial r} \left(\frac{\partial G}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial G}{\partial x} \cdot \frac{\partial^2 x}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{\partial G}{\partial y} \right) \frac{\partial y}{\partial r} +$$

$$+ \frac{\partial G}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2} =$$

$$= \left(\frac{\partial^2 G}{\partial x^2} \cdot \frac{\partial x}{\partial r} + \frac{\partial^2 G}{\partial y \partial x} \cdot \frac{\partial y}{\partial r} \right) \frac{\partial x}{\partial r} + \frac{\partial G}{\partial r} \cdot \frac{\partial^2 x}{\partial r^2} +$$

$$+ \left(\frac{\partial^2 G}{\partial x \partial y} \cdot \frac{\partial x}{\partial r} + \frac{\partial^2 G}{\partial y^2} \cdot \frac{\partial y}{\partial r} \right) \frac{\partial y}{\partial r} + \frac{\partial G}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2}$$

Notamos que

$$\frac{\partial x}{\partial r} = s \cdot e^{rs} \therefore \frac{\partial x}{\partial r}(1,0) = 0$$

$$\frac{\partial^2 x}{\partial r^2} = s^2 \cdot e^{rs} \therefore \frac{\partial^2 x}{\partial r^2}(1,0) = 0$$

$$\frac{\partial y}{\partial r} = 3r^2 \cos(s) \therefore \frac{\partial y}{\partial r}(1,0) = 3$$

$$\frac{\partial^2 y}{\partial r^2} = 6r \cos(s) \therefore \frac{\partial^2 y}{\partial r^2}(1,0) = 6$$

$$(r, s) = (1, 0) \Rightarrow G(1, 1) \Rightarrow t^2 - 1 = 1 \Rightarrow t = 0$$

$$\frac{\partial G}{\partial y}(1,1) = 3 \in \frac{\partial^2 G}{\partial y^2} = 2t - 2 \therefore \frac{\partial^2 G}{\partial y^2}(1,1) = -2$$

$$\begin{aligned}\frac{\partial^2 F}{\partial r^2}(1,0) &= \left(\frac{\partial^2 G}{\partial y^2} \cdot \frac{\partial y}{\partial r} \right) \frac{\partial y}{\partial r} + \frac{\partial G}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2} = \\ &= (-2) \cdot 3 + 3 \cdot 6 = 0\end{aligned}$$

$$\begin{aligned}21) \nabla f(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 8y) \\ \nabla f(2,1) &= (4, 8)\end{aligned}$$

Seja t o vetor tangente à curva de nível 8 no ponto $(2,1)$:

$$\nabla f(2,1) \cdot t = 0 \Leftrightarrow t = \lambda v, v \perp \nabla f(2,1)$$

Para obter $v = (a, b)$ temos que:

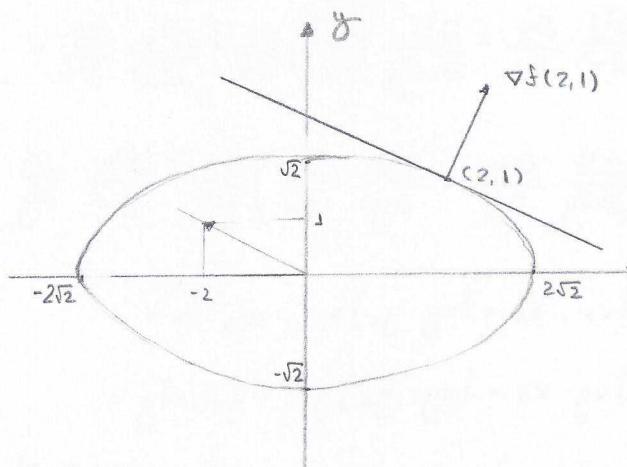
$$\nabla f(2,1) \cdot v = 0 \Rightarrow 4a + 8b = 0 \Rightarrow a = -2b$$

$$v = (a, b) = (a, b) = (-2b, b) = b(-2, 1)$$

Portanto, a reta tangente no ponto $(2,1)$ será dada por:
 $r: x = (2,1) + \lambda(-2,1)$

Para a curva de nível 8 temos:

$$x^2 + 4y^2 = 8 \Rightarrow \frac{x^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \left(\frac{x}{2\sqrt{2}} \right)^2 + \left(\frac{y}{\sqrt{2}} \right)^2 = 1$$



22) Calculando a reta r :

$$f(x, y) = x^3 + 3xy + y^3 + 3x$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y + 3 \quad \therefore \nabla f(1,2) = (12, 15)$$

$$\frac{\partial f}{\partial y} = 3x + 3y^2$$

$$\int f(r(t)) dt = c \therefore \frac{d f(r(t))}{dt} = 0 \Rightarrow \langle \nabla f(x, y), f'(t) \rangle = 0$$

No ponto $(1,2)$:

$$(12, 15) \cdot (x-1, y-2) = 0 \Rightarrow 12x - 12 + 15y - 30 = 0 \Rightarrow$$

$$\rightarrow 12x + 15y - 42 = 0 \Rightarrow 3x + 5y - 14 = 0$$

Para a outra curva vamos considerar o ponto (x_0, y_0)

$$g(x, y) = x^2 + xy - y^2$$

$$g(\mathbf{f}(t)) = 7 \Rightarrow \frac{dg(\mathbf{f}(t))}{dt} = 0 \Rightarrow \langle \nabla g(x_0, y_0), \mathbf{f}'(t_0) \rangle = 0$$

$$\Rightarrow \langle (2x_0 + y_0, x_0 + 2y_0), (x - x_0, y - y_0) \rangle = 0 \quad (1)$$

Como queremos as retas paralelas a r :

$$\nabla g(x_0, y_0) = \lambda \nabla f(1, 2) \Rightarrow (2x_0 + y_0, x_0 + 2y_0) =$$

$$= (12\lambda, 15\lambda)$$

$$\begin{cases} 2x_0 + y_0 = 12\lambda \\ x_0 + 2y_0 = 15\lambda \end{cases} \Rightarrow \begin{cases} x_0 = 15\lambda - 2y_0 \\ 30\lambda - 4y_0 + y_0 = 12\lambda \end{cases} \Rightarrow$$

$$\Rightarrow 3y_0 = 18\lambda \Rightarrow y_0 = 6\lambda \quad \therefore x_0 = 3\lambda$$

$$g(x_0, y_0) = 7 \Rightarrow 9\lambda^2 + 18\lambda^2 + 36\lambda^2 = 7 \Rightarrow \lambda^2 = 1/9 \Rightarrow$$

$$\Rightarrow \lambda = 1/3 \text{ ou } \lambda = -1/3$$

Substituindo em (1):

$$\langle (12\lambda, 15\lambda), (x - 3\lambda, y - 6\lambda) \rangle = 0$$

$$\lambda = \frac{1}{3} : \langle (4, 5), (x - 1, y - 2) \rangle = 0 \Rightarrow$$

$$\Rightarrow 4(x - 1) + 5(y - 2) = 0$$

$$\lambda = -\frac{1}{3} : \langle (-4, -5), (x + 1, y + 2) \rangle = 0 \Rightarrow$$

$$\Rightarrow -4(x + 1) - 5(y + 2) = 0$$

b) a) Da equação do plano $z = -2x + 2y + 3$ temos que

$$\frac{\partial f}{\partial x}(x_0, y_0) = -2 \quad \& \quad \frac{\partial f}{\partial y}(x_0, y_0) = 2.$$

$$\mathbf{f}'(t) = \left(-\frac{1}{t}, 1\right) \quad \& \quad \mathbf{f}'(t) = \left(\frac{1}{t^2}, 1\right)$$

Como f é uma função diferenciável e sendo $t \in \mathbb{R}$ uma constante que representa a curva de nível:

$$f(\mathbf{f}(t)) = c \quad \& \quad f'(\mathbf{f}(t)) = 0 \Rightarrow \langle \nabla f(x_0, y_0), \mathbf{f}'(t) \rangle = 0 \Rightarrow$$

$$\Rightarrow -\frac{2}{t^2} + 2 = 0 \Rightarrow t^2 = 1 \Rightarrow t = 1 \text{ ou } t = -1$$

Portanto, a curva $\mathbf{f}(t) = \left(-\frac{1}{t}, t\right)$ pode ser uma curva de nível de f que contém P.

$$b) \mathbf{f}'(t) = \left(\frac{1}{t^2}, -\frac{2}{t^3} + 3\right) \quad \& \quad \mathbf{f}'(t) = \left(t^4, -2t^2 + 3\right)$$

$$\langle \nabla f(x_0, y_0), \mathbf{f}'(t) \rangle = 0 \Rightarrow -2t^4 - 4t^2 + 6 = 0 \Rightarrow$$

$$\Rightarrow t^4 + 2t^2 - 3 = 0$$

$$t^2 = \frac{-2 \pm \sqrt{4 + 12}}{2} \quad \left\{ \begin{array}{l} t^2 = 1 \Rightarrow t = 1 \text{ ou } t = -1 \\ t^2 = -3 \text{ (não } t \in \mathbb{R}) \end{array} \right.$$

Portanto, a curva $\mathbf{f}(t) = \left(\frac{t^5}{5}, -\frac{2t^3}{3} + 3t\right)$ satisfaz a condição pedida.

$$c) \mathbf{f}'(t) = (t^2, t^3 + t) \quad \& \quad \mathbf{f}'(t) = (2t, 3t + 1)$$

$$\langle \nabla f(x_0, y_0), \mathbf{f}'(t) \rangle = 0 \Rightarrow -4t + 6t + 2 = 0 \Rightarrow$$

$$\Rightarrow t = -1 \quad (\text{não } t \in \mathbb{R})$$

Não pode ser a curva de nível f que contém o ponto P.

$$24) a) \nabla f(1, 2) = (a, b)$$

$$\begin{cases} \operatorname{colg} t = 1 \Rightarrow \frac{\sin t}{\tan t} = 1 \Rightarrow \cos t = \sin t \\ \sec^2 t = 2 \Rightarrow \cos^2 t = \frac{1}{2} \end{cases} \quad \therefore t = \pi/4$$

$$\begin{cases} \sqrt[3]{u} = 1 \Rightarrow u = 1 \\ u^2 + 1 = 2 \Rightarrow u = 1 \end{cases}$$

$$\mathbf{f}'(t) = (\operatorname{colg} t, \sec^2 t) \quad \& \quad \mathbf{f}'(t) = \left(-\frac{1}{\sin^2 t}, 2\sec^2 t \operatorname{lg} t\right)$$

$$\sigma(u) = \left(\sqrt[3]{u}, u^2 + 1, \frac{u^3}{2} - \frac{\sqrt[3]{u}}{2} + 1\right) \quad \&$$

$$\sigma'(u) = \left(\frac{1}{3\sqrt[3]{u^2}}, 2u, \frac{3u^2}{2} - \frac{1}{6\sqrt[3]{u^2}}\right)$$

$$\langle \nabla f(1, 2), \sigma'(\pi/4) \rangle = 0 \Rightarrow \langle (a, b), (-2, 4) \rangle = 0 \Rightarrow$$

$$\Rightarrow -2a + 4b = 0 \Rightarrow a = 2b$$

$$\langle \nabla f(1, 2), \sigma'(1) \rangle = \frac{4}{3} \Rightarrow \langle (a, b), (1/3, 2) \rangle = \frac{4}{3} \Rightarrow$$

$$\Rightarrow \frac{a}{3} + 2b = \frac{4}{3}$$

$$\begin{cases} a = 2b \\ \frac{a}{3} + 2b = \frac{4}{3} \end{cases} \Rightarrow \frac{2b}{3} + 2b = \frac{4}{3} \Rightarrow \frac{8b}{3} = \frac{4}{3} \Rightarrow b = 1/2 \quad \& \quad a =$$

$$\therefore \nabla f(1, 2) = (1, 1/2)$$

b) Como f é de classe C^1 então f é diferenciável, portanto, vale que:

$$\frac{\partial f}{\partial v}(1, 2) = \langle \nabla f(1, 2), v \rangle = \frac{1}{2} + \frac{\sqrt{3}}{4} = \frac{2+\sqrt{3}}{4}$$

$$c) f(\sigma(1)) = f(1, 2) = \frac{1}{2} - \frac{1}{2} + 1 = 1$$

O plano tangente será dado por:

$$\pi: z = f(1, 2) + \frac{\partial f}{\partial x}(1, 2)(x - 1) + \frac{\partial f}{\partial y}(1, 2)(y - 2)$$

$$\pi: z = 1 + x - \frac{1}{2}(y-2) \Rightarrow \pi: z = x + \frac{1}{2}y - 1$$

$$5) \nabla f(x,y) = (2x, 4y^3) \Leftrightarrow f'(t) = (2t, 1)$$

Como $\nabla f(t) = (t_0^2, t_0) = (x(t_0), y(t_0))$ então $x(t_0) = t_0^2$

$$\text{e } y(t_0) = t_0.$$

O gradiente é paralelo à imagem da curva γ^1 (então f não é diferenciável), portanto, $\nabla f(x(t_0), y(t_0)) = \lambda \gamma^1(t_0)$

$$\begin{cases} 2t_0 = \lambda \cdot 2t_0 \Rightarrow \lambda = 1 \\ 4t_0^3 = \lambda t_0 \Rightarrow 4t_0^2 = 1 \Rightarrow t_0 = \frac{1}{2} \quad (t_0 > 0) \end{cases} \quad \begin{array}{l} x(t_0) = 1 \\ y(t_0) = \frac{1}{2} \end{array}$$

$$r: x = (t_0^2, t_0) + t \nabla f(x(t_0), y(t_0))$$

$$r: x = (1/4, 1/2) + t(1, 1/2)$$

28) Para mostrar que o função f é contínua basta mostrar que o limite ao seguir lente ao valor $f(0,0) = 0$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \sqrt[3]{xy} = 0 \therefore \text{o função é contínua.}$$

Para provar que f tem todas as derivadas direcionais em $(0,0)$, vamos tomar $v = (a,b)$ tq. $a^2+b^2=1$:

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t \rightarrow 0} \frac{f(at, bt) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{\sqrt[3]{at^2bt} - 0}{t} =$$

$$= \lim_{t \rightarrow 0} \frac{\sqrt[3]{a^2b} - 0}{t} = \lim_{t \rightarrow 0} \sqrt[3]{a^2b} = \sqrt[3]{a^2b}, \forall a, b \in \mathbb{R}$$

Para mostrar que a função é diferenciável em $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0) - \frac{\partial f}{\partial x}(0,0)h - \frac{\partial f}{\partial y}(0,0)k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{\sqrt[3]{h^2k}}{\sqrt{h^2+k^2}}$$

Colhendo as curvas $\gamma(t) = (t, t)$:

$$\lim_{t \rightarrow 0} \frac{\sqrt[3]{t^3}}{\sqrt{t^2+t^2}} = \lim_{t \rightarrow 0} \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}} \therefore \text{a função não é}$$

diferenciável em $(0,0)$.

$$29) \nabla f(x,y) = (e^{-y}, -xe^{-y}+3) \therefore \nabla f(1,0) = (1, 2)$$

$$\text{O versor } v \text{ é dado por: } v = \frac{\nabla f(1,0)}{\|\nabla f(1,0)\|} = \frac{1}{\sqrt{5}} (1,2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

A derivada direcional máxima de f será (a fórmula vale porque f é diferenciável):

$$\frac{\partial f}{\partial v} = \langle \nabla f(1,0), v \rangle = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$b) \nabla f(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right) \therefore \nabla f(1,2) = \left(\frac{2}{5}, \frac{4}{5}\right)$$

$$\text{O versor } v \text{ é dado por: } v = \frac{\nabla f(1,2)}{\|\nabla f(1,2)\|} = \frac{\sqrt{5}}{2} \left(\frac{2}{5}, \frac{4}{5}\right) = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

A derivada direcional máxima de f será:

$$\frac{\partial f}{\partial v} = \langle \nabla f(1,2), v \rangle = \frac{2\sqrt{5}}{25} + \frac{8\sqrt{5}}{25} = \frac{2\sqrt{5}}{5}$$

30) Assumindo que a função f seja diferenciável:

$$\nabla f(x,y) = (2x-2, 2y-4)$$

$$\nabla f(x_0, y_0) = (1,1) \Rightarrow (2x_0-2, 2y_0-4) = (1,1) \Rightarrow$$

$$\Rightarrow \begin{cases} 2x_0-2 = 1 \Rightarrow x_0 = \frac{3}{2} \\ 2y_0-4 = 1 \Rightarrow y_0 = \frac{5}{2} \end{cases} \quad D = (3/2, 5/2)$$

$$r: x = D + t(1,1) = (3/2, 5/2) + t(1,1)$$

$$r: x-y+1=0$$

$$31) \gamma(t_0) = (t_0+1, -t_0^2) = (-1, -4), t_0 \in \mathbb{R}$$

$$\begin{cases} t_0+1 = -1 \Rightarrow t_0 = -2 \\ -t_0^2 = -4 \Rightarrow t_0 = -2 \end{cases}$$

$$\gamma'(t) = (1, -2t) \therefore \gamma'(t_0) = (1, 4)$$

Como a função f é diferenciável:

$f(\gamma(t_0)) = c, c \in \mathbb{R}$ e representa a curva de nível c

$$f'(\gamma(t_0)) = 0 \Rightarrow \langle \nabla f(\gamma(t_0)), \gamma'(t_0) \rangle = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial f}{\partial x}(-1, -4) \cdot 1 + \frac{\partial f}{\partial y}(-1, -4) \cdot 4 = 0 \Rightarrow \frac{\partial f}{\partial y}(-1, -4) = -\frac{1}{2}$$

$$\therefore \nabla f(-1, -4) = (2, -1/2)$$

A derivada direcional será dada por:

$$\frac{\partial f}{\partial v}(-1, -4) = \langle \nabla f(-1, -4), v \rangle = 2 \cdot \frac{3}{5} - \frac{1}{2} \cdot \frac{4}{5} = \frac{4}{5}$$

$$26) \nabla f(2,8) = (a, b) \Leftrightarrow f'(t) = (t, 2t^2, t^2)$$

Seja $\sigma(t) = (t, 2t^2)$ uma curva de nível cuja imagem está contida no gráfico de f .

$$f(\sigma(t)) = t^2 \therefore f(\sigma(1)) = 2t = 2$$

$$\Rightarrow \nabla f(x,y) \cdot \sigma'(t) = 2t \therefore \nabla f(2,8) \cdot \sigma'(2) = 4$$

$\frac{\partial f}{\partial y}$ é contínua em $(0,0)$.

2) a) $\frac{\partial f}{\partial x} = \frac{y^3(x^2+y^2) - 2x \cdot xy^3}{(x^2+y^2)^2} = \frac{y^3(x^2+y^2)-2x^2y^3}{(x^2+y^2)^2}$

$$\frac{\partial f}{\partial y} = \frac{3y^2x(x^2+y^2) - 2y \cdot xy^3}{(x^2+y^2)^2} = \frac{3y^2x(x^2+y^2)-2x^2y^4}{(x^2+y^2)^2}$$

$$\lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0 \quad \Rightarrow \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

Para $(x,0)$ com $x \neq 0$:

$$\frac{\partial f}{\partial x}(0,y) = \frac{y^5}{y^4} = y$$

$$\left| \begin{array}{l} \text{Para } (y,0) \text{ com } y \neq 0: \\ \frac{\partial f}{\partial y}(x,0) = 0 \end{array} \right.$$

Para $(0,y)$ com $y \neq 0$:

$$\frac{\partial f}{\partial y}(0,0) = 0 = y$$

$$\left| \begin{array}{l} \text{Para } (x,0) \text{ com } x \neq 0: \\ \frac{\partial f}{\partial y}(0,y) = 0 \end{array} \right.$$

b) $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 0$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k \rightarrow 0} \frac{\frac{\partial f}{\partial x}(0,k) - \frac{\partial f}{\partial x}(0,0)}{k} = \frac{k^5}{k^5} = 1$$

3) a) $\frac{\partial f}{\partial x}(0,0) = \frac{f(h,0) - f(0,0)}{h} = \frac{h}{h} = 1$

$$\frac{\partial f}{\partial y}(0,0) = \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\therefore \nabla f(0,0) = (1,0)$$

b) $f(\mathbf{M}(t)) = f(-t, -t) = \frac{-t^3}{t^2+t^2} = \frac{1}{2} \quad \therefore f'(\mathbf{M}(t)) = \frac{1}{2}$

$$\therefore f'(\mathbf{M}(0)) = 0$$

$$\mathbf{M}'(t) = (-1, -1) \quad \therefore \mathbf{M}'(0) = (-1, -1)$$

$$\langle \nabla f(0,0), \mathbf{M}'(0) \rangle = -1 \cdot 1 + 0 \cdot (-1) = -1$$

Então $f'(M(0)) \neq \langle \nabla f(0,0), M'(0) \rangle$

c) $\frac{\partial f}{\partial u}(0,0) = \lim_{t \rightarrow 0} \frac{f(mt, nt) - f(0,0)}{t} =$

$$= \lim_{t \rightarrow 0} \frac{m^3 t^3}{m^2 t^2 + n^2 t^2} \cdot \frac{1}{t} = \lim_{t \rightarrow 0} \frac{m^3 t^4}{t^2 (m^2 + n^2)} \cdot \frac{1}{t} = \frac{m^3}{m^2 + n^2} = m^3$$

d) f não é diferenciável em $(0,0)$ porque não vale

$$\langle \nabla f(0,0), M'(0) \rangle = f'(\mathbf{M}(0)).$$

4) $\frac{\partial f}{\partial v}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\therefore \nabla f(0,0) = (0,0)$$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0) + f_x(0,0) \cdot h + f_y(0,0) \cdot k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{h^3 k}{h^4+k^2} \cdot \frac{1}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} h \cdot \underbrace{\frac{h^2}{h^4+k^2}}_{\not=} \cdot \frac{k}{\sqrt{h^2+k^2}} \Rightarrow \text{"}\not\text{ limição"}$$

④ $\lim_{(h,k) \rightarrow (0,0)} \frac{h^2}{h^4+k^2}$, escolhendo $M_1(t) = (t,0) \in M_2(t) = (0,t)$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{t^2}{t^4+t^2} + \lim_{(h,k) \rightarrow (0,0)} 0$$

Para $u = (a,b)$ com $\|u\| = 1$ temos:

$$\frac{\partial f}{\partial u}(0,0) = \lim_{t \rightarrow 0} \frac{f(a+bt, bt) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{a^3 b \cdot t^4}{a^4 t^4 + b^2 t^2} \cdot \frac{1}{t} =$$

$$= \lim_{t \rightarrow 0} \frac{a^3 b t^4}{t^4 (a^4 t^2 + b^2)} = \lim_{t \rightarrow 0} \frac{a^3 b}{a^4 t^2 + b^2} = \frac{a^3 b}{b^2} = \frac{a^3}{b}$$

Como f não é diferenciável não vale $\langle \nabla f(0,0), u \rangle$.

$$a + 8b = 4, \text{ eis, } \sigma'(t) = (1, 4t)$$

Como sabemos que $\sigma(t) = (x(t), y(t)) \Rightarrow$

$$\sigma'(t) = (x'(t), y'(t)) = (x - x_0, y - y_0). \text{ Para } t=2$$

temos que $\sigma'(2) = (x-2, y-8).$ A reta tangente é dada por

$$\nabla f(2,8) \cdot (x-2, y-8) = 0 \Rightarrow (a,b)(1-2, 4-8) = 0 \Rightarrow$$

$$\Rightarrow -a - 12b = 0 \Rightarrow a = -12b$$

$$\begin{cases} a + 8b = 4 \\ a = -12b \end{cases} \Rightarrow b = -\frac{1}{4} \text{ e } a = 3 \therefore \nabla f(2,8) = (12, -3)$$

A equação da plana tangente será dada por:

$$\pi: y = 12(x-2) + (-1)(y-8) + 4$$

$$16) a) \frac{\partial u}{\partial t} = c^2 f'(x+ct) + c g'(x+ct)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x+ct) + c^2 g''(x+ct)$$

$$\frac{\partial u}{\partial x} = f'(x+ct) + g'(x+ct)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x+ct)$$

A igualdade $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ é válida, como verificando

pelas equações acima.

$$b) \frac{\partial u}{\partial x} = f(x+y) + x f'(x+y) + y g'(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} = \underbrace{f'(x+y)}_{2f'(x+y)} + \underbrace{f'(x+y)}_{2f'(x+y)} + x f''(x+y) + y g''(x+y)$$

$$\frac{\partial u}{\partial y} = g(x+y) + y g'(x+y) + x f(x+y)$$

$$\frac{\partial^2 u}{\partial y^2} = \underbrace{g'(x+y)}_{2g'(x+y)} + g'(x+y) + y g''(x+y) + x f''(x+y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = g'(x+y) + y g''(x+y) + f'(x+y) + x f''(x+y)$$

Conclui-se que $u(x,y) = x f(x+y) + y g(x+y)$ é solução

$$\text{da equação } \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Mais alguns exemplos:

$$3) a) \frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0) - 0 \cdot h - 0 \cdot k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{h^2+k^2}{h^2+k^2} \cdot \frac{1}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \underbrace{\frac{h^2}{h^2+k^2}}_{\text{limítado}} \cdot \underbrace{\frac{k}{h^2+k^2}}_{\text{limítado}} \cdot \underbrace{\frac{h^2}{h^2+k^2}}_{\text{limítado}} \cdot \underbrace{\frac{k^2}{h^2+k^2}}_{\text{limítado}} = 0$$

$\therefore f$ é diferenciável em $(0,0)$

b) Para $(x,y) \neq (0,0):$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy^2(x^2+y^4) - 2x \cdot x^2y^2}{(x^2+y^4)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2yx^2(x^2+y^4) - 4y^3 \cdot x^2y^2}{(x^2+y^4)^2}$$

$$\text{Então: } \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{2xy^2(x^2+y^4) - 2x^3y^2}{(x^2+y^4)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{2yx^2(x^2+y^4) - 4y^5x^2}{(x^2+y^4)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2(x^2+y^4) - 2x^3y^2}{(x^2+y^4)^2} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2+y^4} - \frac{2x^3y^2}{(x^2+y^4)^2} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \underbrace{\frac{2xy^2}{x^2+y^4}}_{\text{-1/2}} - \lim_{(x,y) \rightarrow (0,0)} \frac{2x^3y^2}{(x^2+y^4)^2} \therefore \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x}(x,y)$$

$\Rightarrow \frac{\partial f}{\partial x}$ não é contínua em $(0,0)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2yx^2}{x^2+y^4} - \frac{4y^5x^2}{(x^2+y^4)^2} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \underbrace{\frac{2y}{x^2+y^4}}_{\text{limítado}} \cdot \underbrace{\frac{x^2}{x^2+y^4}}_{\text{limítado}} - \lim_{(x,y) \rightarrow (0,0)} \underbrace{\frac{4y}{x^2+y^4}}_{\text{limítado}} \cdot \underbrace{\frac{x^2}{x^2+y^4}}_{\text{limítado}} \cdot \underbrace{\frac{y^4}{x^2+y^4}}_{\text{limítado}} = 0$$

$$1) f(x,y) = \cos(\sqrt[3]{x^2+y^2})$$

$$\frac{\partial f}{\partial x}(x,y) = 2x \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(x^2+y^2)}} \cdot (-\sin(\sqrt[3]{x^2+y^2})) =$$

$$= -\frac{2x \cdot \sin(\sqrt[3]{x^2+y^2})}{3\sqrt[3]{(x^2+y^2)^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = 2y \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(x^2+y^2)}} \cdot (-\sin(\sqrt[3]{x^2+y^2})) =$$

$$= -\frac{2y \cdot \sin(\sqrt[3]{x^2+y^2})}{3\sqrt[3]{(x^2+y^2)^2}}$$

Para $(x,y) \neq (0,0)$ $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas então f é de classe C^1 em $\mathbb{R}^2 \setminus \{(0,0)\}$, e, portanto, é diferenciável em $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\cos(h^{2/3}) - 1}{h} \stackrel{H}{=} \lim_{h \rightarrow 0} \frac{\frac{d}{dh} \cos(h^{2/3})}{1} \cdot \frac{1}{h} =$$

$$\stackrel{H}{=} \lim_{h \rightarrow 0} \frac{2/3 \cdot h^{-1/3} (-\sin(h^{2/3}))}{1} = \lim_{h \rightarrow 0} -\frac{2}{3} \frac{1}{\sqrt[3]{h}} \cdot \sin(\sqrt[3]{h^2}) =$$

$$= \lim_{h \rightarrow 0} -\frac{2}{3} \cdot \frac{1}{\sqrt[3]{h}} \cdot \frac{1}{\sqrt[3]{h}} \cdot \sin(\sqrt[3]{h^2}) \cdot \sqrt[3]{h} =$$

$$= \lim_{h \rightarrow 0} -\frac{2}{3} \cdot \underbrace{\frac{\sin(\sqrt[3]{h^2})}{\sqrt[3]{h^2}}}_{\text{limite fundamental}} \cdot \frac{\sqrt[3]{h}}{\sqrt[3]{h^2}} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{\cos(k^{2/3}) - 1}{k} = 0$$

Vamos verificar a diferenciabilidade em $(0,0)$. Se

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0) \cdot h - f_y(0,0) \cdot k}{\sqrt{h^2+k^2}} = 0$$

então a função é diferenciável em $(0,0)$.

$$\lim_{(h,k) \rightarrow (0,0)} \frac{\cos(\sqrt[3]{h^2+k^2}) - 1}{\sqrt{h^2+k^2}} \cdot \frac{\cos(\sqrt[3]{h^2+k^2}) + 1}{\cos(\sqrt[3]{h^2+k^2} + 1)} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{\sin^2(\sqrt[3]{h^2+k^2})}{(\sqrt[3]{h^2+k^2})^2} \cdot \frac{(\sqrt[3]{h^2+k^2})^2}{\sqrt{h^2+k^2}} \cdot \frac{1}{\cos(\sqrt[3]{h^2+k^2} + 1)} =$$

$$= \lim_{(h,k) \rightarrow (0,0)} \underbrace{\frac{\sin^2(\sqrt[3]{h^2+k^2})}{(\sqrt[3]{h^2+k^2})^2}}_{\rightarrow 1} \cdot \underbrace{\frac{(h^2+k^2)^{1/2}}{\sqrt{h^2+k^2}}}_{\rightarrow 0} \cdot \underbrace{\frac{1}{\cos(\sqrt[3]{h^2+k^2} + 1)}}_{\frac{1}{\cos 1}} = 0$$

$\therefore f$ é diferenciável em $(0,0)$

$$2) a) (I) \lim_{(x,y) \rightarrow (0,0)} \frac{y \cdot \frac{y^2}{x^2+y^2}}{\frac{x^2+y^2}{\text{diminuta}}} = 0 \quad \because \text{é contínua}$$

$$(II) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

Vamos escolher $M_1(t) = (t,0) \in \mathbb{M}_2(t) = (t,t^2)$

$$\lim_{t \rightarrow 0} (f \circ M_1)(t) = 0$$

$$\lim_{t \rightarrow 0} (f \circ M_2)(t) = \lim_{t \rightarrow 0} \frac{t^4}{t^4+t^4} = \frac{1}{2}$$

\therefore como $\lim_{t \rightarrow 0} (f \circ M_1)(t) \neq \lim_{t \rightarrow 0} (f \circ M_2)(t)$ então f não é contínua em $(0,0)$.

$$(III) f(0,0) = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \sqrt{2x^2+5y^2} = 0 = f(0,0) \quad \therefore \text{é contínua.}$$

$$b) (I) \frac{\partial f}{\partial u}(0,0) = \lim_{t \rightarrow 0} \frac{f(at,bt) - f(0,0)}{t} =$$

$$= \lim_{t \rightarrow 0} \frac{\frac{d}{dt} \cos(\sqrt[3]{a^2t^2+b^2t^2})}{1} \cdot \frac{1}{t} = \lim_{t \rightarrow 0} \frac{a^2t \sqrt[3]{a^2+b^2}}{\sqrt[3]{(a^2+b^2)^2}} = \frac{a^3}{a^2+b^2} = a^3$$

$$(II) \frac{\partial f}{\partial v}(0,0) = \lim_{t \rightarrow 0} \frac{\frac{d}{dt} \cos(\sqrt[3]{a^2t^2+b^2t^2})}{1} \cdot \frac{1}{t} = \lim_{t \rightarrow 0} \frac{a^2b t \sqrt[3]{a^2+b^2}}{\sqrt[3]{(a^2+b^2)^2}} =$$

$$= \lim_{t \rightarrow 0} \frac{a^2b}{(a^2+b^2)} = \frac{a^2b}{b^2} = \frac{a^2}{b} \quad \left\{ \begin{array}{l} \frac{\partial f}{\partial v}(0,0) = \frac{a^2}{b}, \text{ se } b \neq 0 \\ 0, \text{ se } b = 0 \end{array} \right.$$

$$(III) \frac{\partial f}{\partial u}(0,0) = \lim_{t \rightarrow 0} \sqrt{2a^2t^2+5b^2t^2} = 0$$

$$3) a) \text{Seja } \nabla f(3,4) = \left(\frac{\partial f}{\partial x}(3,4), \frac{\partial f}{\partial y}(3,4) \right) = (a,b).$$

$$\{ (M(t)) = 2t^5 + t^4 - 2t^3 + t^2 \quad \therefore f'(M(t)) = 10t^4 + 4t^3 + 6t^2 - 6t$$

Para o ponto $(3,4)$ temos $t = 2$.

Como a função f é diferenciável vale que:

$$f'(M(1)) = \langle \nabla f(x,y), M'(1) \rangle \quad \therefore$$

$$\langle \nabla f(3,4), M'(2) \rangle = 164 \Rightarrow \langle (a,b), (1,4) \rangle = 164 \Rightarrow$$

$$\Rightarrow a + 4b = 164 \quad (1)$$

$$\frac{\partial f}{\partial u}(3,4) = \frac{31\sqrt{2}}{2} \quad \therefore \langle \nabla f(3,4), u \rangle = \frac{31\sqrt{2}}{2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{2}}{2}(-a+b) = \frac{31\sqrt{2}}{2} \Rightarrow b = 31 + a \quad (2)$$

Substituindo (2) em (1):

$$a + 124 + 4a = 164 \Rightarrow 5a = 40 \Rightarrow a = 8 \quad \therefore b = 39$$

$$\nabla f(3,4) = (8,39)$$

$$f(3,4) = f(M(2)) = 64 + 16 - 16 - 4 = 60$$

$$\therefore 3 = 60 + 8(x-3) + 39(y-4)$$

b) Como a função f é diferenciável ontem sabemos que:

$\langle \nabla f(3,4), t \rangle = 0$, sendo t é um vetor tangente à curva tal que $t = (c,d)$

$$\langle (8,39), (c,d) \rangle = 0 \Rightarrow 8c + 39d = 0 \Rightarrow c = -\frac{39d}{8}$$

$$\therefore r: \mathbb{R} \rightarrow (3,4) + t(-39/8, 1)$$

4) Como f é de classe C^2 ontem $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$:

$$\frac{\partial F}{\partial t} = 2t \cdot f(t^2 u, 5t + 3u) + t^2 \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \right) =$$

$$= 2t f(t^2 u, 5t + 3u) + t^2 \left(\frac{\partial f}{\partial x} \cdot 2tu + \frac{\partial f}{\partial y} \cdot 5 \right) =$$

$$= 2t f(t^2 u, 5t + 3u) + \frac{\partial f}{\partial x} \cdot 2t^3 u + \frac{\partial f}{\partial y} \cdot 5t$$

$$\frac{\partial^2 F}{\partial u \partial t} = 2t \cdot \left(\frac{\partial f}{\partial x} + t^2 \cdot \frac{\partial f}{\partial y} \cdot 3 \right) + \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial x} \right) 2t^3 u +$$

$$+ \frac{\partial f}{\partial x} \cdot 2t^3 + \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial y} \right) \cdot 5t =$$

$$= 2t^3 \cdot \frac{\partial f}{\partial x} + 6t \cdot \frac{\partial f}{\partial y} + \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial u} \right) \cdot 2t^3 u +$$

$$+ \frac{\partial f}{\partial x} \cdot 2t^3 + \left(\frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial u} \right) \cdot 5t =$$

$$= 2t^3 \cdot \frac{\partial f}{\partial x} + 6t \cdot \frac{\partial f}{\partial y} + 2t^4 \cdot \frac{\partial^2 f}{\partial x^2} + 6t^3 u \cdot \frac{\partial^2 f}{\partial y \partial x} +$$

$$+ 2t^3 \cdot \frac{\partial f}{\partial x} + 5t^3 \frac{\partial^2 f}{\partial y \partial x} + 15t \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 F}{\partial u \partial t}(1,2) = 2 \frac{\partial f}{\partial x} + 6 \frac{\partial f}{\partial y} + 4 \frac{\partial^2 f}{\partial x^2} + 12 \frac{\partial^2 f}{\partial y \partial x} +$$

$$+ 2 \frac{\partial f}{\partial x} + 5 \frac{\partial^2 f}{\partial y \partial x} + 15 \frac{\partial^2 f}{\partial y^2} =$$

$$= 4 \frac{\partial f}{\partial x} + 17 \frac{\partial^2 f}{\partial y \partial x} + 6 \frac{\partial f}{\partial y} + 4 \frac{\partial^2 f}{\partial x^2} + 15 \frac{\partial^2 f}{\partial y^2}$$

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$$1) f(x,y) = x e^{\sqrt[3]{x^2+y^4}}$$

$$\frac{\partial f}{\partial x}(x,y) = e^{\sqrt[3]{x^2+y^4}} + \frac{2x^2 \cdot e^{\sqrt[3]{x^2+y^4}}}{\sqrt[3]{(x^2+y^4)^2}}, (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \frac{x \cdot e^{\sqrt[3]{h^2}}}{h} = 1$$

Portanto,

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} e^{\sqrt[3]{x^2+y^4}} + \frac{2x^2 \cdot e^{\sqrt[3]{x^2+y^4}}}{\sqrt[3]{(x^2+y^4)^2}}, (x,y) \neq (0,0) \\ 1, (x,y) = (0,0) \end{cases}$$

$$\text{Dom } \frac{\partial f}{\partial x} = \mathbb{R}^2$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{(x,y) \rightarrow (0,0)} e^{\sqrt[3]{x^2+y^4}} \left(1 + \frac{2x^2}{\sqrt[3]{(x^2+y^4)^2}} \right) =$$

$$= \lim_{(x,y) \rightarrow (0,0)} e^{\sqrt[3]{x^2+y^4}} \left(1 + \frac{1}{\sqrt[3]{\frac{8x^6}{(x^2+y^4)^2}}} \right) =$$

$$= \lim_{(x,y) \rightarrow (0,0)} e^{\sqrt[3]{x^2+y^4}} \left(1 + \frac{2}{\sqrt[3]{\frac{x^2 \cdot \frac{x^2}{x^2+y^4} \cdot \frac{x^2}{x^2+y^4}}}} \right) =$$

$$\textcircled{*} x^2 \leq x^2+y^4 \Rightarrow \frac{x^2}{x^2+y^4} \leq 1$$

Conclui-se que $\frac{\partial f}{\partial x}$ é contínua em $(0,0)$

$$b) \frac{\partial f}{\partial y}(x,y) = \frac{4xy^3 \cdot e^{\sqrt[3]{x^2+y^4}}}{\sqrt[3]{(x^2+y^4)^2}}, (x,y) \neq (0,0)$$

Como $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas para $(x,y) \neq (0,0)$

então f é diferenciável em $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

Vamos verificar se $\frac{\partial f}{\partial y}$ é contínua em $(0,0)$.

$$\frac{\partial f}{\partial y}(0,0) = \lim_{(x,y) \rightarrow (0,0)} \frac{4xy^3 e^{\sqrt[3]{x^2+y^4}}}{\sqrt[3]{(x^2+y^4)^2}}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{4x}{3} \cdot \sqrt[3]{y \left(\frac{y^4}{x^2+y^4} \right)^2} \cdot e^{\sqrt[3]{x^2+y^4}} = 0$$

$$\textcircled{*} y^4 \leq x^2+y^4 \Rightarrow \frac{y^4}{x^2+y^4} \leq 1$$

Como f é de classe C^1 em todo \mathbb{R}^2 então f é diferenciável em todo \mathbb{R}^2 .

$$2) g(u,v) = f(u^2 - v^3, 2u^2v)$$

$$a) \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} \cdot 2u + \frac{\partial f}{\partial y} \cdot 4uv$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial f}{\partial x} \cdot (-3v^2) + \frac{\partial f}{\partial y} \cdot 2u^2$$

$$\begin{aligned} \frac{\partial^2 g}{\partial u \partial v} &= \frac{\partial}{\partial v} \left(\frac{\partial f}{\partial x} \right) \cdot (-3v) + 0 \cdot \frac{\partial f}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial f}{\partial y} \right) 2u^3 \\ + \frac{\partial f}{\partial y} \cdot 4u &= \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial v} \right) \cdot (-3v) + \\ + 0 \frac{\partial f}{\partial x} + \left(\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial v} \right) \cdot (2u^2) \\ + \frac{\partial f}{\partial y} \cdot 4u &= -6uv \cdot \frac{\partial^2 f}{\partial x^2} - 12uv^2 \frac{\partial^2 f}{\partial y \partial x} + \\ + 4u^3 \cdot \frac{\partial^2 f}{\partial y \partial x} + 8u^3v \frac{\partial^2 f}{\partial y^2} + 4u \frac{\partial f}{\partial y} = \\ = -6uv^2 \frac{\partial^2 f}{\partial x^2} + (4u^3 - 12uv^3) \frac{\partial^2 f}{\partial x \partial y} + 8u^3v \frac{\partial^2 f}{\partial y^2} + 4u \frac{\partial f}{\partial y} \end{aligned}$$

b) $2x + 4y + 3z = 4 \Rightarrow z = \frac{4}{3} - \frac{2}{3}x - \frac{4}{3}y$
 $\exists = f_{(0,2)} - \frac{2}{3}(x-0) - \frac{4}{3}(y-2)$
 $\therefore f_{(0,2)} + \frac{8}{3} = \frac{4}{3} \Rightarrow f_{(0,2)} = -\frac{4}{3}$
 $\nabla f_{(0,2)} = \left(-\frac{2}{3}, -\frac{4}{3}\right)$
 $g_{(1,1)} = f_{(0,2)} \therefore$

$$\frac{\partial g}{\partial u}(1,1) = \frac{\partial f}{\partial x}(0,2) \cdot 2 + \frac{\partial f}{\partial y}(0,2) \cdot 4 = -\frac{4}{3} - \frac{16}{3} = -\frac{20}{3}$$

$$\frac{\partial g}{\partial v}(1,1) = \frac{\partial f}{\partial x}(0,2) \cdot (-3) + \frac{\partial f}{\partial y}(0,2) \cdot 2 = \frac{6}{3} - \frac{8}{3} = -\frac{2}{3}$$

$$\nabla g_{(1,1)} = \left(-\frac{20}{3}, -\frac{2}{3}\right)$$

$$c) \frac{\partial^2 g}{\partial u \partial v}(1,1) = -6 \cdot 2 + 8 \cdot 4 - 8 \cdot 1 - 4 \cdot 4 =$$

$$= -12 + 32 - 8 - \frac{16}{3} = 12 - \frac{16}{3} = \frac{20}{3}$$

$$3) \gamma'(1) = \left(1 - \frac{1}{t_0^2}, 1\right) \therefore \gamma'(t_0) = \left(-\frac{1}{t_0^2}, 1\right)$$

$$f(\gamma(t_0)) = 1 \Rightarrow f'(\gamma(t_0)) \circ 0 \Rightarrow$$

$$\Rightarrow \langle \nabla f(x_0, y_0), \gamma'(t_0) \rangle = 0$$

Dado $v = (a, b) \perp \gamma'(t_0)$:

$$-\frac{a}{t_0^2} + b = 0 \Rightarrow b = \frac{a}{t_0^2} \therefore v = \left(1, \frac{1}{t_0^2}\right)$$

$$\nabla f(x_0, y_0) = \lambda \left(1, \frac{1}{t_0^2}\right)$$

$$\text{Como } \gamma(t_0) = (x_0, y_0) \Rightarrow \left(1 + \frac{1}{t_0}, t_0\right) = (x_0, y_0) :$$

$$\exists = 1 + \lambda \left(x - \frac{1}{t_0}\right) + \frac{\lambda}{t_0^2} (y - t_0)$$

$$\begin{cases} \frac{1}{2} = 1 + \lambda \left(1 - \frac{1}{t_0}\right) + \frac{\lambda}{t_0^2} (1 - t_0) \\ 2 = 1 + \lambda \left(4 - \frac{1}{t_0}\right) + \frac{\lambda}{t_0^2} (1 - t_0) \end{cases} \quad (1)$$

$$\begin{cases} \frac{1}{2} = 1 - \frac{\lambda}{t_0} + \frac{\lambda}{t_0^2} (1 - t_0) \\ 2 = 1 + 3\lambda - \frac{\lambda}{t_0} + \frac{\lambda}{t_0^2} (1 - t_0) \end{cases} \quad (2)$$

De (2) - (1):

$$3\lambda = \frac{3}{2} \Rightarrow \lambda = \frac{1}{2}$$

De (3):

$$\begin{aligned} \frac{1}{2} - \frac{1}{2t_0} + \frac{1}{2t_0^2} - \frac{1}{2t_0} &= 0 \Rightarrow \\ \Rightarrow \frac{1}{2} - \frac{1}{t_0} + \frac{1}{2t_0^2} &= 0 \Rightarrow \frac{1}{2} - k + \frac{k^2}{2} = 0 \Rightarrow \end{aligned}$$

$$\Rightarrow k^2 - 2k + 1 = 0 \Rightarrow (k-1)^2 = 0$$

$$k = 1 \therefore 1 = \frac{1}{t_0} \Rightarrow t_0 = 1$$

A equação do plano será:

$$\pi: \exists = 1 + \frac{1}{2}(x-2) + \frac{1}{2}(y-1)$$

$$4) \nabla f(1,0) = (a, b)$$

$$\begin{cases} 1+1=1 \\ 1^2=0 \end{cases} \Rightarrow 1 \cdot 0 \in \begin{cases} \sin u=1 \\ \cos u=0 \end{cases} \therefore u = \frac{\pi}{2}$$

$$f(\delta(1)) = 1^4 + 2 \cdot 3 + 1^2 - 1 - 1$$

$$f'(\delta(1)) = \langle \nabla f(x,y), \delta'(1) \rangle = 4 \cdot 1^3 + 6 \cdot 2^2 + 2 \cdot 1 - 1$$

$$\langle (a,b), (1,0) \rangle = -1 \Rightarrow a = -1$$

$$f(\psi(u)) = \cos u - \cos^3 u - \sin u$$

$$f(\psi(u)) = -\sin u + 3 \sin u \cdot \cos^2 u - \cos u$$

$$\langle (a,b), (0, -1) \rangle = -1 \Rightarrow b = 1 \therefore$$

$$\nabla f(1,0) = (-1, 1)$$

$$f(\delta(0)) = f(1,0) = -1$$

$$\pi: \exists = -1 - 1(x-1) + y$$

$$\pi: \exists = -x + y$$

D2 de 2014

$$1) a) f(x,y) = x \sqrt[3]{x^2 + y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \sqrt[3]{x^2+y^2} + \frac{2x^2}{3\sqrt[3]{(x^2+y^2)^2}}, (x,y) \neq (0,0)$$

$$\lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \frac{\sqrt[3]{h^2}}{h} = 0, (x, y) = (0, 0)$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{2yx}{3\sqrt[3]{(x^2+y^2)^2}}, (x, y) \neq (0, 0)$$

$$\lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = 0, (x, y) = (0, 0)$$

$$\text{Dom } \frac{\partial f}{\partial x} = \mathbb{R}^2 \quad \text{e} \quad \text{Dom } \frac{\partial f}{\partial y} = \mathbb{R}^2$$

$$b) \frac{\partial f}{\partial x} = \begin{cases} \sqrt[3]{x^2+y^2} + \frac{2x^2}{3\sqrt[3]{(x^2+y^2)^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{2xy}{3\sqrt[3]{(x^2+y^2)^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Notamos que para $(x, y) \neq (0, 0)$ $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas. Devemos verificar para $(x, y) = (0, 0)$

$$\lim_{(x,y) \rightarrow (0,0)} \sqrt[3]{x^2+y^2} + \frac{2 \cdot x^2}{3\sqrt[3]{(x^2+y^2)^2}} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \sqrt[3]{x^2+y^2} + \frac{2}{3} \cdot \sqrt[3]{\underbrace{x^2}_{\rightarrow 0} \cdot \underbrace{\left(\frac{x^2}{x^2+y^2}\right)^2}_{\text{limítada}}} = 0$$

$\frac{\partial f}{\partial x}$ é contínua em $(0, 0)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2}{3} \cdot \frac{xy}{\sqrt[3]{(x^2+y^2)^2}} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2}{3} \cdot \sqrt[3]{\underbrace{xy}_{\rightarrow 0} \cdot \underbrace{\frac{x^2}{x^2+y^2} \cdot \frac{y^2}{x^2+y^2}}_{\text{limítada}}} = 0$$

$\frac{\partial f}{\partial y}$ é contínua em $(0, 0)$

c) Como f é de classe $C^1(\mathbb{R}^2)$ então f é diferenciável em \mathbb{R}^2

$$2) a) \nabla f(2, 8) = (a, b)$$

$$\psi(+) = (+, 2+)^T \therefore \psi'(+) = (1, 4+)^T \cdot \psi'(2) = (1, 8)$$

$$f(\psi(1)) = 1^2 \Rightarrow f'(\psi(+)) \cdot 2+ \Rightarrow f'(\psi(2)) = 4 \Rightarrow$$

$$\Rightarrow \langle \nabla f(2, 8), \psi'(2) \rangle = 4 \Rightarrow a + 8b = 4 \quad (1)$$

A reta tangente será $r: \langle \nabla f(2, 8), (x-2, y-8) \rangle = 0$. Como

r passa por $(1, -4)$:

$$\langle (a, b), (-1, -12) \rangle = 0 \Rightarrow -a - 12b = 0 \Rightarrow$$

$$\Rightarrow a = -12b \quad (2)$$

De (2) em (1):

$$-12b + 8b = 4 \Rightarrow -4b = 4 \Rightarrow b = -1 \therefore a = 12$$

$$\nabla f(2, 8) = (12, -1)$$

b) Como $\nabla f(2, 8) = (12, -1)$ sabemos que

$$\frac{\partial f}{\partial x}(2, 8) = 12 \quad \text{e} \quad \frac{\partial f}{\partial y}(2, 8) = -1$$

Como $f(\psi(2)) = f(2, 8) = 4$, a equação do plano será:

$$\pi: y = 4 + 12(x-2) - (y-8)$$

$$3) a) F(s, t) = sG(st, -s)$$

$$\frac{\partial F}{\partial s} = G(st-s) + s \cdot \left(\frac{\partial G}{\partial x} \cdot 1 - \frac{\partial G}{\partial y} \right) =$$

$$= G(st-s) + st \cdot \frac{\partial G}{\partial x} - s \frac{\partial G}{\partial y}$$

$$\frac{\partial F}{\partial t} = s \left(\frac{\partial G}{\partial y} \cdot s \right) = s^2 \cdot \frac{\partial G}{\partial y}$$

$$\frac{\partial^2 F}{\partial s \partial t} = 2s \cdot \frac{\partial G}{\partial x} + s^2 \cdot \frac{\partial}{\partial s} \left(\frac{\partial G}{\partial x} \right) =$$

$$= 2s \cdot \frac{\partial G}{\partial x} + s^2 \left(\frac{\partial^2 G}{\partial x^2} \cdot 1 - \frac{\partial G}{\partial y \partial x} \right) =$$

$$= 2s \frac{\partial G}{\partial x} + s^2 + \frac{\partial^2 G}{\partial x^2} - s^2 \frac{\partial G}{\partial y \partial x}$$

$$b) x-y+2z+1=0 \Rightarrow 2z = -x+y-1 \Rightarrow$$

$$\Rightarrow z = -\frac{x}{2} + \frac{y}{2} - \frac{1}{2} \therefore \nabla G(-2, -2) = (-\frac{1}{2}, \frac{1}{2})$$

$$z = -\frac{1}{2}(x+2) + \frac{1}{2}(y+2) + G(-2, -2)$$

$$\therefore -1+1+G(-2, -2) = -\frac{1}{2} \Rightarrow G(-2, -2) = -\frac{1}{2}$$

$$c) \nabla F(2, -1) = (-\frac{1}{2}, -2) \quad \frac{1}{4} \cdot 4 = \frac{1}{4}$$

Como F é de C^2 então é diferenciável

$$\frac{\partial F}{\partial y} = \langle (-\frac{1}{2}, -2), (a, b) \rangle = \|\nabla F(2, -1)\| \cdot \|u\| \cdot \cos \theta$$

$\frac{\partial F}{\partial y}$ será máximo quando $\cos \theta = 1$.

$$u = \frac{\nabla F(2, -1)}{\|\nabla F(2, -1)\|} = \frac{2}{\sqrt{17}} \left(-\frac{1}{2}, -2 \right)$$

P2 de 2015 :

$$1) a) f(x,y) = \sqrt[3]{3x^4 + 2y^4}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{12x^3}{3\sqrt[3]{(3x^4 + 2y^4)^2}}, \text{ se } (x,y) \neq (0,0)$$

$$\lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{3h^4}}{h} = \lim_{h \rightarrow 0} \frac{(3h^4)^{1/3}}{h} =$$

$$= \lim_{h \rightarrow 0} \sqrt[3]{3} \cdot \sqrt[3]{h} = 0, \text{ se } (x,y) = (0,0)$$

$$b) \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4x^3}{\sqrt[3]{(3x^4 + 2y^4)^2}}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4x^3}{\sqrt[3]{(3x^4 + 2y^4)^2}} = \lim_{(x,y) \rightarrow (0,0)} 4 \sqrt[3]{x \cdot \left(\frac{x^4}{3x^4 + 2y^4}\right)^2} =$$

$$= 0$$

$$\textcircled{4} \quad x^4 \leq 3x^4 + 2y^4 \Rightarrow \frac{x^4}{3x^4 + 2y^4} \leq 1$$

Portanto, $\frac{\partial f}{\partial x}$ é contínua em $(0,0)$

$$2) \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot 2u + \frac{\partial f}{\partial y} \cdot 2t$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \cdot (-2t) + \frac{\partial f}{\partial y} \cdot 2u$$

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial x} \right) \cdot 2u + 2 \cdot \frac{\partial f}{\partial x} + \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial y} \right) \cdot 2t =$$

$$= \left(\frac{\partial^2 f}{\partial x^2} \cdot 2u + \frac{\partial f}{\partial x} \cdot 2t \right) 2u + 2 \frac{\partial f}{\partial x} +$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} \cdot 2u + \frac{\partial^2 f}{\partial y^2} \cdot 2t \right) \cdot 2t =$$

$$= 4u^2 \cdot \frac{\partial^2 f}{\partial x^2} + 4tu \cdot \frac{\partial^2 f}{\partial x \partial y} + 2t \frac{\partial^2 f}{\partial y^2} +$$

$$+ 4tu \cdot \frac{\partial^2 f}{\partial x \partial y} + 4t^2 \cdot \frac{\partial^2 f}{\partial y^2} =$$

$$= 4u^2 \cdot \frac{\partial^2 f}{\partial x^2} + 8tu \cdot \frac{\partial^2 f}{\partial x \partial y} + 2 \cdot \frac{\partial^2 f}{\partial x^2} + 4t^2 \cdot \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 g}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) \cdot (-2t) - 2 \cdot \frac{\partial f}{\partial x} + 2u \cdot \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) =$$

$$= (-2t) \left(\frac{\partial^2 f}{\partial x^2} \cdot (-2t) + \frac{\partial^2 f}{\partial y \partial x} \cdot (2u) \right) - 2 \frac{\partial f}{\partial x} +$$

$$+ 2u \left(\frac{\partial^2 f}{\partial x \partial y} \cdot (-2t) + \frac{\partial^2 f}{\partial y^2} \cdot 2u \right) =$$

$$= 4t^2 \cdot \frac{\partial^2 f}{\partial x^2} - 4tu \cdot \frac{\partial^2 f}{\partial x \partial y} - 2 \frac{\partial f}{\partial x} + 4u^2 \frac{\partial^2 f}{\partial y^2} + 4u^2 \frac{\partial^2 f}{\partial y^2} =$$

$$= 4t^2 \frac{\partial^2 f}{\partial x^2} - 8tu \cdot \frac{\partial^2 f}{\partial x \partial y} - 2 \frac{\partial f}{\partial x} + 4u^2 \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial t^2} = (4u^2 + 4t^2) \cdot \frac{\partial^2 f}{\partial x^2} + (4u^2 + 4t^2) \frac{\partial^2 f}{\partial y^2} =$$

$$= (4u^2 + 4t^2) \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right]$$

$$4u^2 + 4t^2 = 48 \Rightarrow u^2 + t^2 = 12 \therefore r = \sqrt{12}$$

$$3) a) \nabla f(1,2) = (a,b)$$

$$\begin{cases} \cos(t) = 1 \Rightarrow \cos t = \text{const} \\ \sin^2 t = 2 \Rightarrow \cos t = \frac{\sqrt{2}}{2} \end{cases} \therefore t = \frac{\pi}{4}$$

$$\begin{cases} \sqrt[3]{u} = 1 \Rightarrow u = 1 \\ u^2 + 1 = 2 \end{cases} \therefore u = 1$$

$$\mathbf{v}'(t) = \left(-\frac{1}{\cos^2 t}, 2\sin t \cdot \frac{1}{\cos^2 t} \right) : \mathbf{v}'(\pi/4) = (-2, 4)$$

$$\mathbf{v}(t) = (\sqrt[3]{u}, u^2 + 1) \therefore \mathbf{v}'(t) = \left(\frac{1}{3\sqrt[3]{u^2}}, 2u \right) \Rightarrow \mathbf{v}(t) = \left(\frac{1}{3}, 2 \right)$$

Como f é de classe C^1 então é diferenciável:

$$f(\mathbf{v}(t)) = c \Rightarrow f(\mathbf{v}(2)) = c \Rightarrow$$

$$\Rightarrow f'(\mathbf{v}(2)) = 0 \Rightarrow \langle (a,b), (-2,4) \rangle = 0 \Rightarrow$$

$$\Rightarrow -2a + 4b \Rightarrow a = 2b \text{ (1)}$$

$$f(\mathbf{v}(t)) = \frac{u^3}{2} - \frac{\sqrt[3]{u}}{2} + 1 \Rightarrow f'(\mathbf{v}(1)) = \frac{3u^2}{2} - \frac{1}{6\sqrt[3]{u^2}} \Rightarrow$$

$$\Rightarrow f'(\mathbf{v}(1)) = \frac{3}{2} - \frac{1}{6} = \frac{8}{6} \Rightarrow$$

$$\Rightarrow \langle (a,b), (1/3, 2) \rangle = \frac{8}{6} \Rightarrow$$

$$\Rightarrow \frac{a}{3} + 2b = \frac{8}{6} \text{ (2)}$$

De (1) em (2):

$$\frac{2b}{3} + 2b = \frac{8}{6} \Rightarrow \frac{8b}{3} = \frac{8}{6} \Rightarrow b = 1/2 \therefore a = 1$$

$$\nabla f(1,2) = (1, 1/2)$$

b) Como f é diferenciável:

$$\frac{\partial f}{\partial u}(1,2) = \langle (1, 1/2), (1/2, \sqrt{3}/2) \rangle = \frac{1}{2} + \frac{\sqrt{3}}{4} =$$

$$= \frac{2 + \sqrt{3}}{4}$$

$$o) f(\psi(1)) = f(1, 2) = \frac{1}{2} - \frac{1}{2} + 1 = 1$$

$$\pi: g = 1 + (x-1) + \frac{1}{2}(y-1/2)$$