

## CÁLCULO II - Lista 2

1a)  $f(x,y) = \arctg\left(\frac{y}{x}\right)$

$$\bullet \quad \sin^2 x + \cos^2 x = 1 \Rightarrow \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \tan^2 x + 1 = \sec^2 x \quad (1)$$

$$\bullet \quad \tan' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1 + \tan^2 x}{\cos^2 x} \stackrel{(1)}{=} \sec^2 x \quad (2)$$

$$\bullet \quad \begin{cases} f(x) = \arctg x \\ g(y) = \tan y \end{cases} \Rightarrow (g \circ f)(x) = \tan(\arctg(x)) = x \quad (3)$$

Derivando todos os lados da igualdade

$$\begin{aligned} (g \circ f)'(x) &= \tan(\arctg(x))' = 1 \stackrel{(2)}{\Rightarrow} \sec^2(\arctg(x)) = 1 \\ \stackrel{(1)}{\Rightarrow} \arctg'(x) \cdot (\tan^2(\arctg(x)) + 1) &= 1 \stackrel{(3)}{\Rightarrow} \arctg'(x)(x^2 + 1) = 1 \\ \Rightarrow \arctg'(x) &= \frac{1}{1+x^2} \end{aligned}$$

Voltando ao problema original:

$$\frac{\partial f(x,y)}{\partial x} = -\frac{y}{x^2} \cdot \frac{1}{1+(y/x)^2} = -\frac{y}{x^2} \cdot \frac{x^2}{x^2+y^2} \Rightarrow \frac{\partial f(x,y)}{\partial x} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{x} \cdot \frac{1}{1+(y/x)^2} = \frac{1}{x} \cdot \frac{x^2}{x^2+y^2} \Rightarrow \frac{\partial f(x,y)}{\partial y} = \frac{x}{x^2+y^2}$$

b)  $f(x,y) = \ln(1 + \cos^2(xy^3))$

$$\frac{\partial f(x,y)}{\partial x} = \underbrace{2\cos(xy^3) \cdot y^3 \cdot (-\sin(xy^3))}_{\text{regra da cadeia}} \cdot \frac{1}{1 + \cos^2(xy^3)} \quad \ln' x = 1/x$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = -\frac{y^3 \cdot \sin(2xy^3)}{1 + \cos^2(xy^3)} \quad 2\sin x \cos x = \sin(2x)$$

$$\frac{\partial f(x,y)}{\partial y} = 2\cos(xy^3) \cdot 3xy^2 \cdot (-\sin(xy^3)) \cdot \frac{1}{1 + \cos^2(xy^3)}$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial y} = -\frac{3xy^2 \cdot \sin(2xy^3)}{1 + \cos^2(xy^3)}$$

/ / ②

$$2. f(x,y) = x(x^2+y^2)^{-3/2} e^{\sin(x^2y)}$$

$$\begin{aligned}\frac{\partial f(1,0)}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(1+h,0) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)[(1+h)^2]^{-3/2} \cdot e^0 - 1[1^2]^{-3/2} \cdot e^0}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^{-2} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(1+h)^2} - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h \rightarrow 0} \frac{-h(2+h)}{h(1+h)^2} = -2\end{aligned}$$

$$\therefore \frac{\partial f(1,0)}{\partial x} = -2$$

$$3. u(x,y) = \ln \sqrt{x^2+y^2} \quad (\sqrt{x} = (x^{1/2})^1 = \frac{1}{2} \cdot \frac{d}{dx} x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot 2x \cdot \frac{1}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\text{Analogamente: } \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0 \Rightarrow \text{solução da equação de Laplace bidimensional}$$

$$4a) u(x,t) = f(x+ct) + g(x-ct)$$

$$\frac{\partial u}{\partial t} = c f'(x+ct) - c g'(x-ct)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 f''(x+ct) + c^2 g''(x-ct) = c^2 (f''(x+ct) + g''(x-ct))$$

$$\frac{\partial u}{\partial x} = f'(x+ct) + g'(x-ct)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x-ct)$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \blacksquare$$

$$\text{b) } u(x, y) = x \cdot f(x+y) + y \cdot g(x+y) \Rightarrow (0,0) \text{ é uma das fixas?}$$

$$\frac{\partial u}{\partial x} = f(x+y) + x \cdot f'(x+y) + y \cdot g'(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(x+y) + f'(x+y) + x f''(x+y) + y g''(x+y) \quad ①$$

$$\frac{\partial u}{\partial y} = x f'(x+y) + g(x+y) + y g'(x+y)$$

$$\frac{\partial^2 u}{\partial y \partial x} = f'(x+y) + x f''(x+y) + g'(x+y) + y g''(x+y) \quad ② = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = x f''(x+y) + g'(x+y) + g'(x+y) + y g''(x+y) \quad ③$$

$$\text{Logo: } ① - 2② + ③ = 2f'(x+y) + x f''(x+y) + y g''(x+y) \\ - 2[f'(x+y) + x f''(x+y) + g'(x+y) + y g''(x+y)] \\ + 2g'(x+y) + x f''(x+y) + y g''(x+y) = 0$$

5.

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{se } (x, y) \neq (0,0) \\ 0, & \text{se } (x, y) = (0,0) \end{cases}$$

a) 1º caso:  $(x, y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{y^2(x^2+y^4) - 2x(xy^2)}{(x^2+y^4)^2} = \frac{y^2(y^4-x^2)}{(x^2+y^4)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2xy(x^2+y^4) - 2y(xy^2)}{(x^2+y^4)^2} = \frac{2xy(x^2+y^4-y^2)}{(x^2+y^4)^2}$$

2º caso:  $(x, y) = (0,0)$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^2} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^4} - 0}{h} = 0$$

∴ as derivadas parciais de primeira ordem existem em todo o  $\mathbb{R}^2$

b)  $f(x,y)$  é contínua em  $(0,0) \Leftrightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0)$

Mas não existe  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ , pois dadas  $\gamma_1(t) = (t^2, t)$  e  $\gamma_2(t) = (0, t)$

$$\lim_{t \rightarrow 0} f(\gamma_1(t)) = \lim_{t \rightarrow 0} \frac{t^2 \cdot t^2}{(t^2)^2 + t^4} = \lim_{t \rightarrow 0} \frac{t^4}{2t^4} = \frac{1}{2} = L_1$$

$$\lim_{t \rightarrow 0} f(\gamma_2(t)) = \lim_{t \rightarrow 0} \frac{0 \cdot t^2}{0^2 + t^4} = \lim_{t \rightarrow 0} \frac{0}{t^4} = 0 = L_2$$

c)  $f(x,y)$  não é contínua em  $(0,0) \Rightarrow f(x,y)$  não é diferenciável em  $(0,0)$

6.

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

a)  $f(x,y)$  é contínua em  $(0,0) \Leftrightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0)$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} x \cdot \frac{x^2}{x^2+y^2} \xrightarrow{\text{limite da}} 0 = f(0,0)$$

Se  $(x,y) \neq (0,0)$   
 $x^2+y^2 > x^2 > 0$   
 $\Rightarrow 0 < \frac{x^2}{x^2+y^2} < 1$

$\therefore f(x,y)$  é contínua em  $(0,0)$

b)  $\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h^3} = \lim_{h \rightarrow 0} 1 = 1$

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h^2} = 0$$

c) Para que  $f$  seja diferenciável em  $(0,0)$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x}(0,0) \text{ e } \frac{\partial f}{\partial y}(0,0) \text{ devem existir (checkado no item b)} \\ \lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|} = 0 \end{array} \right.$$

em que  $E(h,k) = f(x_0+h, y_0+k) - f(x_0, y_0) - \frac{\partial f(x_0, y_0)}{\partial x}h - \frac{\partial f(x_0, y_0)}{\partial y}k$

Como  $(x_0, y_0) = (0,0)$  e  $\frac{\partial f}{\partial x}(0,0) = 1$  e  $\frac{\partial f}{\partial y}(0,0) = 0$ , vem que:

$$\lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|} = \lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - h \frac{\partial f(0,0)}{\partial x}}{\sqrt{h^2 + k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{h^3/h^2 + k^2 - h}{(h^2 + k^2)^{3/2}}$$

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{-hk^2}{(h^2 + k^2)^{3/2}}, \text{ que não existe, pois se } g(h,k) = \frac{-hk^2}{(h^2 + k^2)^{3/2}}$$

podemos escolher  $\gamma_1(t) = (0,t)$  e  $\gamma_2(t) = (t,t)$ , de tal forma que:

$$\left. \begin{aligned} \lim_{t \rightarrow 0} g(\gamma_1(t)) &= \lim_{t \rightarrow 0} \frac{0}{t^3} = 0 = L_1 \\ \lim_{t \rightarrow 0} g(\gamma_2(t)) &= \lim_{t \rightarrow 0} \frac{-t^3}{(2t^2)^{3/2}} = \frac{\sqrt{2}}{4} = L_2 \end{aligned} \right\} L_1 \neq L_2$$

Como  $L_1 \neq L_2$ , o limite não existe e  $f$  não é diferenciável em  $(0,0)$ .

a) se  $\frac{\partial f}{\partial x}$  e  $\frac{\partial f}{\partial y}$  forem contínuas em  $(0,0) \Rightarrow f$  é de classe  $C^1$

$f$  é de classe  $C^1 \Rightarrow f$  é diferenciável em  $(0,0)$

Mas  $f$  não é diferenciável em  $(0,0)$  (item a)  $\Rightarrow f$  não é de classe  $C^1$

$\Rightarrow \frac{\partial f}{\partial x}$  e  $\frac{\partial f}{\partial y}$  não são (ambas) contínuas em  $(0,0)$ .

$$7. f(x,y) = (x^2 + y^2)^{2/3}$$

$f$  é de classe  $C^1 \Leftrightarrow \frac{\partial f}{\partial x}$  e  $\frac{\partial f}{\partial y}$  em todo o  $\mathbb{R}^2$  são contínuas

• Para,  $(x,y) \neq (0,0)$ :

$$\frac{\partial f}{\partial x} = 2x \cdot \frac{2}{3} \cdot \frac{1}{(x^2 + y^2)^{1/3}} = \frac{4}{3} \cdot \frac{x}{(x^2 + y^2)^{1/3}}$$

$$\frac{\partial f}{\partial y} = 2y \cdot \frac{2}{3} \cdot \frac{1}{(x^2 + y^2)^{1/3}} = \frac{4}{3} \cdot \frac{y}{(x^2 + y^2)^{1/3}}$$

• Para,  $(x,y) = (0,0)$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{h^{4/3}}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{h^{4/3}}{h} = 0$$

Assim:

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{4}{3} \frac{x}{(x^2+y^2)^{1/3}}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{4}{3} \frac{y}{(x^2+y^2)^{1/3}}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

$\frac{\partial f}{\partial x}$  é contínua, em  $(0,0)$   $\Leftrightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(0,0)$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x} &= \lim_{(x,y) \rightarrow (0,0)} \frac{4}{3} \frac{x}{(x^2+y^2)^{1/3}} = \lim_{(x,y) \rightarrow (0,0)} \frac{4}{3} \frac{(x^2+y^2)^{3/2}}{(x^2+y^2)^{3/2}} \cdot \frac{x}{(x^2+y^2)^{1/3}} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{4}{3} (x^2+y^2)^{3/2} \cdot \frac{x}{\sqrt{x^2+y^2}} \stackrel{\text{limitada}}{\rightarrow} 0 = \frac{\partial f}{\partial x}(0,0) \Rightarrow \frac{\partial f}{\partial x} \text{ contínua no } \mathbb{R}^2 \end{aligned}$$

Analogamente,  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial y}(0,0) \Rightarrow \frac{\partial f}{\partial y} \text{ contínua no } \mathbb{R}^2$

Como as derivadas parciais de primeira ordem de  $f$  são contínuas em todo  $\mathbb{R}^2$ ,  $f$  é de classe  $C^1$ .

8a)  $f(x,y) = \sqrt[3]{x^3+y^3}$

Calculando as derivadas parciais pelas regras usuais de derivação:

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{3} \cdot 3x^2 \cdot \frac{1}{\sqrt[3]{x^3+y^3}^{2/3}} = \frac{1}{3} \cdot \frac{x^2}{\sqrt[3]{x^3+y^3}^{2/3}} \quad \left. \begin{array}{l} \text{contínua em} \\ \text{todos os pontos} \end{array} \right\}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{3} \cdot 3y^2 \cdot \frac{1}{\sqrt[3]{x^3+y^3}^{2/3}} = \frac{1}{3} \cdot \frac{y^2}{\sqrt[3]{x^3+y^3}^{2/3}} \quad \left. \begin{array}{l} \text{em que está} \\ \text{definida} \Rightarrow f \end{array} \right\}$$

Se  $(x^3+y^3)^{2/3}=0 \Rightarrow x^3+y^3=0 \Rightarrow x=-y$ , deve-nos calcular  $\frac{\partial f}{\partial x}$  e  $\frac{\partial f}{\partial y}$  pela definição:

$$x=-y \rightarrow \gamma(t) = (t, -t)$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(\gamma(t)) = \lim_{h \rightarrow 0} \frac{f(t+h, -t) - f(t, -t)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(t+h)^3 + (-t)^3}{h}^{1/3} = \lim_{h \rightarrow 0} \frac{\cancel{h}\sqrt[3]{1+3t/h+3t^2/h^2}}{\cancel{h}} \Rightarrow \text{não existe limite}$$

Se  $x_0 = 0 \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1$ , análogamente  $\Rightarrow \frac{\partial f}{\partial y}(0,0) = 1$

$$\lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|} = \lim_{(h,k) \rightarrow (0,0)} \frac{f(0+h, 0+k) - f(0,0) - h-k}{\sqrt{h^2+k^2}} = \lim_{(h,k) \rightarrow (0,0)} \frac{\sqrt{h^2+k^2} - h - k}{\sqrt{h^2+k^2}}$$

que não existe (tome  $\gamma_1 = (t,0)$  e  $\gamma_2 = (t,t)$ )  $\Rightarrow f$  dif. em  $x \neq -y$

b)  $f(x,y) = x \cdot |y|$ . Pelas regras usuais de derivação ( $|y| = \sqrt{y^2}$ )

$\frac{\partial f}{\partial x} = |y|$  } definida para  $D_f = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ , nesses pontos as derivadas parciais são contínuas

$\frac{\partial f}{\partial y} = \frac{x \cdot y}{\sqrt{y^2}}$  }  $\Rightarrow f$  é diferenciável nesses pontos

$$f(y) = \sqrt{y^2} \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{y^2}} \text{ (regras da cálculo)}$$

Em  $(x,0)$ :  $\frac{\partial f}{\partial y}(x,0) = \lim_{h \rightarrow 0} \frac{f(x,h) - f(x,0)}{h} = \lim_{h \rightarrow 0} \frac{x|h|}{h} \Rightarrow$  não existe limite,

$$\text{pois } \lim_{h \rightarrow 0^+} \frac{x|h|}{h} = x \text{ e } \lim_{h \rightarrow 0^-} \frac{x|h|}{h} = -x \quad (*)$$

$$(*) \text{ Mas } \lim_{h \rightarrow 0^+} \frac{x|h|}{h} = \lim_{h \rightarrow 0^-} \frac{x|h|}{h} = 0, \text{ se } x=0$$

Nesse caso, temos  $\frac{\partial f}{\partial y}(x,y) = \begin{cases} xy, & \text{se } y \neq 0 \\ \sqrt{y}, & \text{se } x=y=0 \\ 0, & \text{se } y=0 \text{ e } x \neq 0 \Rightarrow \text{não dif.} \end{cases}$

nesses pontos

Em  $(0,0)$ ,  $\frac{\partial f}{\partial y}$  é contínua se, e somente se:  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(0,0)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot y}{|y|} \Rightarrow \begin{cases} \lim_{(x,y) \rightarrow (0,0^+)} \frac{x \cdot y}{|y|} = \lim_{(x,y) \rightarrow (0,0^+)} x \cdot 1 = 0 \\ \lim_{(x,y) \rightarrow (0,0^-)} \frac{x \cdot y}{|y|} = \lim_{(x,y) \rightarrow (0,0^-)} x \cdot (-1) = 0 \end{cases}$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial y}(0,0) \Rightarrow \frac{\partial f}{\partial y} \text{ contínua (e } f \text{ diferenciável)}$$

em todos os pontos que não são da forma  $(0,w)$ ,  $w \neq 0$

/ / 8

c)  $f(x,y) = e^{\sqrt{x^4+y^4}}$ . Pelas regras usuais de derivação:

$$\left. \begin{array}{l} \frac{\partial f}{\partial x}(x,y) = \frac{2x^3}{\sqrt{x^4+y^4}} \\ \frac{\partial f}{\partial y}(x,y) = \frac{2y^3}{\sqrt{x^4+y^4}} \end{array} \right\} \begin{array}{l} \text{Derivadas parciais definidas e contínuas em } (x,y) \neq (0,0) \Rightarrow f \text{ diferenciável} \\ \text{nesses pontos.} \end{array}$$

Em  $(0,0)$ , devemos calcular  $\frac{\partial f}{\partial x}$  e  $\frac{\partial f}{\partial y}$  pela definição:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{e^{\sqrt{h^4}} - 1}{h} \stackrel{L'H}{=} \lim_{h \rightarrow 0} \frac{2h \cdot e^{h^2}}{h} = 0$$

Analogamente  $\frac{\partial f}{\partial y}(0,0) = 0$  e, em síntese:

$$\frac{\partial f}{\partial x} = \begin{cases} 2x^3/\sqrt{x^4+y^4}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}; \quad \frac{\partial f}{\partial y} = \begin{cases} 2y^3/\sqrt{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Se  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(0,0)$  e  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(0,0) \Rightarrow f \text{ é de}$

classe  $C^1$  em todo o  $\mathbb{R}^2$  (as derivadas parciais são contínuas) e  $f$  é diferenciável (se não fosse  $C^1 \Rightarrow$  calcular  $\lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|}$  para verificar diferenciabilidade)

Note que  $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^3}{\sqrt{x^4+y^4}} = \lim_{(x,y) \rightarrow (0,0)} \frac{2x^3 \cdot x^4}{x^4+y^4} = 0 = \frac{\partial f}{\partial x}(0,0)$

Analogamente  $\lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(0,0) = 0$

Como as derivadas parciais de primeira ordem existem e são contínuas em todo o  $\mathbb{R}^2$  (classe  $C^1$ )  $\Rightarrow f$  é diferenciável em todo o  $\mathbb{R}^2$ .

d)  $f(x,y) = \cos(\sqrt{x^2+y^2}) \Rightarrow$  o raciocínio é idêntico ao item anterior.

Pelas regras usuais de derivação:

$$\left. \begin{array}{l} \frac{\partial f}{\partial x}(x,y) = \frac{-x \cdot \operatorname{sen}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \\ \frac{\partial f}{\partial y}(x,y) = \frac{-y \cdot \operatorname{sen}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \end{array} \right\} \begin{array}{l} \text{Válido se } (x,y) \neq (0,0) \\ \text{Nesses pontos } \frac{\partial f}{\partial x} \text{ e } \frac{\partial f}{\partial y} \text{ são} \\ \text{contínuas e } f \text{ é diferenciável (I)} \end{array}$$

Em  $(0,0)$ , pela definição:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\cos(\sqrt{h^2}) - 1}{h} = 0$$

Analogamente  $\frac{\partial f}{\partial y}(0,0) = 0$

$(*)$

Desta vez vamos usar  $\lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|}$  para verificar a diferenciabilidade.  $E(h,k) = f(x_0+h, x_0+k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k$

Como  $(x_0, y_0) = (0,0)$  e  $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ , temos que:

$$\lim_{(h,k) \rightarrow (0,0)} \frac{E(h,k)}{\|(h,k)\|} = \lim_{(h,k) \rightarrow (0,0)} \frac{\cos(\sqrt{h^2+k^2}) - 1}{\sqrt{h^2+k^2}}$$

Fazendo  $\sqrt{h^2+k^2} = \omega$ , podemos reescrever o limite como:

$$\lim_{\omega \rightarrow 0^+} \frac{\cos \omega - 1}{\omega} \stackrel{L'H}{=} \lim_{\omega \rightarrow 0^+} -\sin \omega = 0$$

Como  $\frac{\partial f(0,0)}{\partial x}$  e  $\frac{\partial f(0,0)}{\partial y}$  existem e  $(*) = 0$ ,  $f$  é diferenciável em  $\mathbb{R}^2 - \{(0,0)\}$ . JÁ havia sido verificado que  $f$  é diferenciável em  $\mathbb{R}^2 - \{(0,0)\}$ . De (I) e (II)  $\Rightarrow f$  é diferenciável em todo  $\mathbb{R}^2$ .

9(a)  $g = e^{x^2+y^2}$ , ponto  $P = (0,0,1)$ ;  $f(x,y) = e^{x^2+y^2}$

O plano tangente  $\Pi$  é dado por:

$$\mathcal{G} = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0)$$

Nesse caso,  $(x_0, y_0) = (0,0)$

$$\left\{ \begin{array}{l} f(0,0) = 1 \\ \frac{\partial f}{\partial x}(x,y) = 2x \cdot e^{x^2+y^2} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 \end{array} \right.$$

$$\frac{\partial f}{\partial y}(x,y) = 2y \cdot e^{x^2+y^2} \Rightarrow \frac{\partial f}{\partial y}(0,0) = 0$$

$$\Rightarrow \Pi: g = 1$$

A reta normal é dada por:

$$x: p + \lambda \vec{n}, \lambda \in \mathbb{R}, \vec{n} = \left( \frac{\partial f(x_0, y_0)}{\partial x}, \frac{\partial f(x_0, y_0)}{\partial y}, -1 \right)$$

$$\left\{ \begin{array}{l} p = (0, 0, 1) \\ \vec{n} = (0, 0, -1), \text{ pois } \frac{\partial f(x_0, y_0)}{\partial x} = \frac{\partial f(x_0, y_0)}{\partial y} = 0 \end{array} \right.$$

$$\Rightarrow x: (0, 0, 1) + \lambda(0, 0, -1), \lambda \in \mathbb{R}$$

Note que como  $\lambda \in \mathbb{R}$ , podemos escrever  $\lambda(a, b, c)$ , por exemplo, por  $\lambda(a, b, c) = \lambda_1(\lambda a, \lambda b, \lambda c)$ ,  $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\text{Em outras "palavras"} \left\{ \begin{array}{l} \lambda_1(0, 0, 1) = \lambda_2(0, 0, -1) \\ \lambda_1(1, 2, 3) = \lambda_2(3, 6, 9), \lambda_1, \lambda_2 \in \mathbb{R} \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda_1(\sqrt{2}/2, \sqrt{2}, -\sqrt{2}/2) = \lambda_2(1, 2, -1) \end{array} \right.$$

b)  $g = \ln(2x+y)$ ,  $P = (-1, 3; 0) \Rightarrow f(x, y) = \ln(2x+y)$ ,  $(x_0, y_0) = (-1, 3)$

Plano tangente em  $(-1, 3; 0)$

$$f(-1, 3) = 0$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{2}{2x+y} \Rightarrow \frac{\partial f}{\partial x}(-1, 3) = 2$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{2x+y} \Rightarrow \frac{\partial f}{\partial y}(-1, 3) = 1$$

Se  $x = -y/2 \Rightarrow$  aplicar

a definição

$$\pi: g = f(-1, 3) + \frac{\partial f}{\partial x}(-1, 3) \cdot (x - (-1)) + \frac{\partial f}{\partial y}(-1, 3) \cdot (y - 3)$$

$$\pi: g = 0 + 2x + 2 + y - 3 \Rightarrow \pi: 2x + y - 3 - 1 = 0$$

Reta normal em  $(-1, 3; 0)$ :

$$\left\{ \begin{array}{l} \vec{n} = \left( \frac{\partial f}{\partial x}(-1, 3), \frac{\partial f}{\partial y}(-1, 3), -1 \right) = (2, 1, -1) \\ P = (-1, 3, 0) \end{array} \right.$$

$$x = P + \lambda \vec{n}, \lambda \in \mathbb{R} \Rightarrow$$

$$\Rightarrow x = (-1, 3, 0) + \lambda(2, 1, -1), \lambda \in \mathbb{R}$$

$$\text{c) } z = x^2 - y^2, P = (-3, -2, 5) \Rightarrow f(x, y) = x^2 - y^2; (x_0, y_0) = (-3, -2)$$

Plano tangente em  $(-3, -2, 5)$

$$\left\{ \begin{array}{l} f(-3, -2) = 5 \\ \frac{\partial f}{\partial x}(x, y) = 2x \Rightarrow \frac{\partial f}{\partial x}(-3, -2) = -6 \\ \frac{\partial f}{\partial y}(x, y) = -2y \Rightarrow \frac{\partial f}{\partial y}(-3, -2) = 4 \end{array} \right.$$

$$\Pi: z = f(-3, -2) + \frac{\partial f}{\partial x}(-3, -2)(x - (-3)) + \frac{\partial f}{\partial y}(-3, -2)(y - (-2))$$

$$\Pi: z = 5 - 6x - 18 + 4y + 8 \Rightarrow \Pi: -6x + 4y - z - 5 = 0$$

Reta normal em  $(-3, -2, 5)$

$$\left\{ \begin{array}{l} \vec{n} = \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right) = (-6, 4, -1) \\ P = (-3, -2, 5) \end{array} \right.$$

$$x = P + \lambda \vec{n}, \lambda \in \mathbb{R} \Rightarrow x = (-3, -2, 5) + \lambda(-6, 4, -1), \lambda \in \mathbb{R}$$

$$\text{d) } z = e^x \cdot \ln y, P = (3, 1, 0) \quad f(x, y) = e^x \cdot \ln y; (x_0, y_0) = (3, 1)$$

Plano tangente em  $(3, 1, 0)$

$$\left\{ \begin{array}{l} f(3, 1) = 0 \\ \frac{\partial f}{\partial x}(x, y) = e^x \cdot \ln y \Rightarrow \frac{\partial f}{\partial x}(3, 1) = 0 \\ \frac{\partial f}{\partial y}(x, y) = \frac{e^x}{y} \Rightarrow \frac{\partial f}{\partial y}(3, 1) = e^3 \end{array} \right.$$

$$\Pi: z = f(3, 1) + \frac{\partial f}{\partial x}(3, 1)(x - 3) + \frac{\partial f}{\partial y}(3, 1)(y - 1)$$

$$\Pi: z = 0 + 0x + e^3y - e^3 \Rightarrow \Pi: 0x + e^3y - z - e^3 = 0$$

Reta normal em  $(3, 1, 0)$

$$\left\{ \begin{array}{l} \vec{n} = \left( \frac{\partial f}{\partial x}(3, 1), \frac{\partial f}{\partial y}(3, 1), -1 \right) = (0, e^3, -1) \\ P = (3, 1, 0) \end{array} \right.$$

$$x = P + \lambda \vec{n}, \lambda \in \mathbb{R} \Rightarrow x = (3, 1, 0) + \lambda(0, e^3, -1), \lambda \in \mathbb{R}$$

10.  $f(x,y) = \sqrt{x^2+y^2}$ ;  $g(x,y) = \frac{1}{10}(x^2+y^2) + 5/2$ ;  $P = (3;4;5)$

$$\begin{cases} f(3,4) = \sqrt{3^2+4^2} = 5 \\ g(3,4) = \frac{1}{10}(3^2+4^2) + 5/2 = 5/2 + 5/2 = 5 \end{cases}$$

Como  $f(3,4) = g(3,4) = 5 \Rightarrow f \circ g$  se interceptam em  $P = (3;4;5)$

O plano tangente ao gráfico de uma função  $f(x,y)$  no ponto  $P = (x_0, y_0, z_0)$  é dado por:

$$\Pi: z = z_0 + \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0)$$

Para que o plano tangente ao gráfico de  $f(x,y)$  no ponto  $(3;4;5)$  seja o mesmo plano tangente ao gráfico de  $f(x,y)$  no mesmo ponto. Devemos ter:

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x}(3,4) = \frac{\partial g}{\partial x}(3,4) \\ \frac{\partial f}{\partial y}(3,4) = \frac{\partial g}{\partial y}(3,4) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x}(3,4) = \frac{3}{5} \\ \frac{\partial f}{\partial y}(3,4) = \frac{4}{5} \end{array} \right.$$

Calculando as derivadas parciais no ponto:

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial f}{\partial x}(3,4) = \frac{3}{5}$$

$$\frac{\partial g}{\partial x} = \frac{x}{5} \Rightarrow \frac{\partial g}{\partial x}(3,4) = \frac{3}{5}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial f}{\partial y}(3,4) = \frac{4}{5}$$

$$\frac{\partial g}{\partial y} = \frac{y}{5} \Rightarrow \frac{\partial g}{\partial y}(3,4) = \frac{4}{5}$$

em concordância à

condição imposta

Assim, os gráficos de  $f(x,y)$  e  $g(x,y)$  têm o mesmo plano tangente no ponto  $(3;4;5)$

11. Um plano tangente ao gráfico de  $f(x,y)$  é da forma:

$$\Pi: z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0)$$

Como  $f(x,y) = xy \Rightarrow \nabla f(x_0, y_0) = (y_0, x_0)$  e, portanto,

$$\mathcal{Z} = f(x_0, y_0) + y_0(x - x_0) + x_0(y - y_0)$$

Como o plano passa por  $P_1 = (1, 1, 2)$  e por  $P_2 = (-1, 1, 1)$ , vem que

$$\left\{ \begin{array}{l} 2 = f(x_0, y_0) + y_0(1-x_0) + x_0(1-y_0) \quad (\text{I}) \\ 1 = f(x_0, y_0) + y_0(-1-x_0) + x_0(1-y_0) \quad (\text{II}) \end{array} \right.$$

$$\text{De (I) - (II): } 1 = 2y_0 \Rightarrow y_0 = \frac{1}{2}$$

$$\text{De (I) + (II): } 3 = 2f(x_0, y_0) + 2x_0 - 4x_0y_0$$

$$(f(x_0, y_0) = x_0y_0) \Rightarrow 3 = 2x_0 - 2x_0y_0$$

$$y_0 = \frac{1}{2} \Rightarrow 3 = 2x_0 - x_0 \Rightarrow x_0 = 3 \Rightarrow f(x_0, y_0) = \frac{3}{2}$$

Voltando à equação do plano:

$$\begin{aligned} \mathcal{Z} &= \frac{3}{2} + \frac{1}{2}(x-3) + \frac{3}{2}(y-\frac{1}{2}) \Rightarrow 2\mathcal{Z} = 3 + x - 3 + 6y - 3 \\ &\Rightarrow x + 6y - 2\mathcal{Z} - 3 = 0 \quad (\text{sim, o plano é único} \Leftrightarrow (x_0, y_0) \text{ é único}) \end{aligned}$$

12. Um plano tangente ao gráfico de  $g(x, y)$  é dado por:

$$\text{II: } \mathcal{Z} = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y-y_0)$$

Como  $g(x, y) = x^3y \Rightarrow \nabla g(x_0, y_0) = (3x_0y_0, x_0^3)$  e:

$$\mathcal{Z} = g(x_0, y_0) + 3x_0y_0(x-x_0) + x_0^3(y-y_0)$$

Como o plano passa por  $P_1 = (0, 1, 5)$  e  $P_2 = (0, 0, 6)$ :

$$\left\{ \begin{array}{l} 5 = f(x_0, y_0) + 3x_0y_0(0-x_0) + x_0^3(1-y_0) \quad (\text{I}) \\ 6 = f(x_0, y_0) + 3x_0y_0(0-x_0) + x_0^3(0-y_0) \quad (\text{II}) \end{array} \right.$$

$$\text{De (I) - (II): } -1 = x_0^3 \Rightarrow x_0 = -1$$

$$\text{De (II) } \circ f(x_0, y_0) = x_0^3y : 6 = -1y_0 - 3y_0 + y_0 \Rightarrow y_0 = -2$$

Voltando à equação do plano:

$$\mathcal{Z} = (-1)^3(-2) + 3 \cdot (-1) \cdot (-2)(x - (-1)) + (-1)^3(y - (-2))$$

$$\mathcal{Z} = -2 + 6x + 6 - y - 2 \Rightarrow 6x - y - \mathcal{Z} + 6 = 0$$

13. Um plano tangente ao gráfico de  $f(x, y)$  é dado por:

$$\text{III: } \mathcal{Z} = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0), (x_0, y_0) \in \Gamma_1$$

Um vetor normal ao plano é dado por:  $\vec{n}_1 = \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right)$

(14)

Como  $f(x, y) = \ln(x^2 + ky^2)$  e  $(x_0, y_0) = (2, 1)$ , tem que:

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x}{x^2 + ky^2} \Rightarrow \frac{\partial f}{\partial x}(2, 1) = \frac{4}{4+k} \Rightarrow \vec{n}_1 = \left( \frac{4}{4+k}, \frac{2k}{4+k}, -1 \right)$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{2ky}{x^2 + ky^2} \Rightarrow \frac{\partial f}{\partial y}(2, 1) = \frac{2k}{4+k}$$

Pela equação do outro plano:  $\pi_2: 3x + 0y + 3 = 0 \Rightarrow \vec{n}_2 = (3, 0, 1)$   
em que  $\vec{n}_2$  é um vetor normal ao plano

Se os planos  $\pi_1$  e  $\pi_2$  são perpendiculares:

$$\vec{n}_1 \perp \vec{n}_2 \Rightarrow \langle \vec{n}_1, \vec{n}_2 \rangle = 0 \Rightarrow \left( \frac{4}{4+k}, \frac{2k}{4+k}, -1 \right) \cdot (3, 0, 1) = 0$$

$$\Rightarrow \frac{12}{4+k} + 0 - 1 = 0 \Rightarrow 12 = 4+k \Rightarrow k = 8$$

14. Um plano tangente ao gráfico de  $g(x, y) = g = xf(x/y)$  é:

$$g = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0)$$

As derivadas parciais de  $g(x, y)$  são dadas por:

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial x}{\partial x} \cdot f(x/y) + \frac{\partial f}{\partial x}(x/y) \cdot x = f(x/y) + \frac{1}{y} f'(x/y) \cdot x$$

$$\frac{\partial g}{\partial y}(x, y) = \frac{\partial x}{\partial y} f(x/y) + \frac{\partial f}{\partial y}(x/y) \cdot x = 0 + \frac{-x}{y^2} f'(x/y) \cdot x$$

Voltando na equação do plano:

$$g = \cancel{x_0 f(x_0/y_0)} + \left[ f(x_0/y_0) + \cancel{\frac{x_0}{y_0} f'(x_0/y_0)} \right] (x - x_0) - \cancel{\frac{x_0^2}{y_0^2} f'(x_0/y_0)} (y - y_0)$$

$$\Rightarrow g = \left[ f(x_0/y_0) + \frac{x_0}{y_0} f'(x_0/y_0) \right] x - \frac{x_0^2}{y_0^2} f'(x_0/y_0) y$$

• Para qualquer ponto  $(x_0, y_0)$  e  $f(x/y)$  derivável a origem  $(0, 0, 0)$  pertence ao plano (substituir  $(x, y, g)$  por  $(0, 0, 0)$  torna a expressão da equação do plano verdadeira)

$$15. \nabla f(x,y) = \left( \frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y} \right) \Rightarrow \int f'(x) dx = f(x) + C$$

Note que  $\int \frac{\partial f(x,y)}{\partial x} dx = f(x,y) - h(y)$   
onde em  $x \Rightarrow \frac{\partial h(y)}{\partial x} = 0$

e também  $\int \frac{\partial f(x,y)}{\partial y} dy = f(x,y) - k(x)$

Se  $\nabla f(x,y) = (x^2y, y^2)$  é o gradiente de uma função:

$$\int x^2y dx = \frac{x^3y}{3} + h(y) = f(x,y) \quad (I)$$

$$\int y^2 dy = \frac{y^3}{3} + k(x) = f(x,y) \quad (II)$$

De (I), temos que  $f(x,y) = \frac{x^3y}{3} + h(y)$ , derivando parcialmente em  $y$ :

$$\frac{\partial f(x,y)}{\partial y} = \frac{x^3}{3} + h'(y) = y^2 \rightarrow \text{pelo enunciado}$$

$$\Rightarrow \begin{cases} h'(y) = y^2 \\ x^3/3 = 0 \end{cases} \rightarrow \text{o que é um absurdo} \Rightarrow \nabla f(x,y) = (x^2y, y^2)$$

não pode ser gradiente.

Pode parecer confuso, mas tomemos uma função  $f(x,y)$  genérica:  $f(x,y) = x^2 + xy^2 + y^4 \rightarrow \nabla f(x,y) = (2x+y^2, 2xy+4y^3)$

$$\int \frac{\partial f(x,y)}{\partial x} dx = \int (2x+y^2) dx = x^2 + xy^2 = f(x,y) - h(y)$$

$$\int \frac{\partial f(x,y)}{\partial y} dy = \int (2xy+4y^3) dy = xy^2 + y^4 = f(x,y) - k(x)$$

Das duas integrais podemos concluir que  $h(y) = y^4$  e  $k(x) = x^2$ , de tal forma que:

$$f(x,y) = x^2 + xy^2 + h(y) = k(x) + xy^2 + y^4 = x^2 + xy^2 + y^4$$

(10)

$$16a) \omega = x^2 + y^2, \quad x = t^2 + u^2, \quad y = 2tu$$

Regras da cadeia

$$\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial t} = 2(t^2 + u^2) \cdot 2t + 2 \cdot (2tu) \cdot 2u = 4t^3 + 12tu^2$$

$$\frac{\partial \omega}{\partial x} = 2x = 2(t^2 + u^2), \quad \frac{\partial \omega}{\partial y} = 2y = 4tu$$

$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y} \frac{\partial y}{\partial u} = 2(t^2 + u^2) \cdot 2u + 2(2tu) \cdot 2t = 4u^3 + 12t^2u$$

Substituição e aplicação das derivadas parciais

$$\omega = x^2 + y^2 = (t^2 + u^2)^2 + (2tu)^2 = t^4 + 2t^2u^2 + u^4 + 4t^2u^2 \\ \Rightarrow \omega = t^4 + 6t^2u^2 + u^4$$

$$\frac{\partial \omega}{\partial t} = 4t^3 + 12tu^2, \quad \frac{\partial \omega}{\partial u} = 4u^3 + 12t^2u$$

$$b) \omega = \frac{x}{x^2 + y^2}, \quad x = t \cdot \cos u, \quad y = t \cdot \sin u$$

$$\frac{\partial \omega}{\partial x} = \frac{1(x^2 + y^2) - 2x(x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\sin^2 u - \cos^2 u}{t^2}$$

$$\frac{\partial \omega}{\partial y} = \frac{-2y(x)}{(x^2 + y^2)^2} = \frac{-2 \sin u \cdot \cos u}{t^2}$$

$$\Rightarrow \frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial \omega}{\partial y} \frac{\partial y}{\partial t} = \frac{(\sin^2 u - \cos^2 u) \cos u}{t^2} - \frac{2 \sin u \cos u}{t^2}$$

$$= - \frac{(\sin^2 u + \cos^2 u) \cos u}{t^2} = - \frac{\cos u}{t^2}$$

$$\Rightarrow \frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y} \frac{\partial y}{\partial u} = \frac{(\sin^2 u - \cos^2 u) \cdot (-t \sin u)}{t^2} - \frac{2t \sin u \cos^2 u}{t^2}$$

$$= - \frac{(\sin^2 u + \cos^2 u) \cdot \sin u}{t} = - \frac{\sin u}{t}$$

Substituição e aplicação das derivadas parciais

$$\omega = \frac{x}{x^2 + y^2} = \frac{t \cos u}{t^2 (\sin^2 u + \cos^2 u)} = \frac{\cos u}{t}$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\cos u}{t} \right) = -\frac{\cos u}{t^2}$$

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial u} \left( \frac{\cos u}{t} \right) = -\frac{\sin u}{t}$$

17.  $\nabla f(-2, -2) = (\omega, -4)$ ;  $g(t) = t(2t^3 - 4t, t^4 - 3t)$ ;  $\gamma(t) = (2t^3 - 4t, t^4 - 3t)$

• reta tangente ao gráfico de em  $x=1$  paralela a  $y=2x+3$

$$\Rightarrow g'(1) = 2$$

$$\bullet \gamma'(t) = (6t^2 - 4, 4t^3 - 3)$$

$$\bullet \text{para } t=1 \Rightarrow g(1) = t(3 \cdot 1 - 4 \cdot 1, 1^4 - 3 \cdot 1) = f(-2, -2)$$

$$\gamma'(1) = (6 \cdot 1 - 4, 4 \cdot 1 - 3) = (2, 1)$$

• pelas regras da cadeia:

$$g'(1) = \nabla f(-2, -2) \cdot \gamma'(1) \Rightarrow 2 = (\omega, -4)(2, 1)$$

$$\Rightarrow 2 = 2\omega - 4 \Rightarrow \omega = 3$$

18.  $u(x, t) = v(x + ct, x - ct)$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial t}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial t} \right) =$$

$$= \underbrace{\frac{\partial r}{\partial t} \cdot \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial r} \right) + \frac{\partial v}{\partial r} \cdot \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial t} \right)}_{\text{regras do produto}} + \underbrace{\frac{\partial s}{\partial t} \cdot \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial s} \right) + \frac{\partial v}{\partial s} \cdot \frac{\partial}{\partial t} \left( \frac{\partial s}{\partial t} \right)}_{\text{regras do produto}}$$

$$= \frac{\partial r}{\partial t} \left( \frac{\partial^2 v}{\partial r^2} \cdot \frac{\partial r}{\partial t} + \frac{\partial^2 v}{\partial s \partial r} \cdot \frac{\partial s}{\partial t} \right) + \frac{\partial v}{\partial r} \cdot \frac{\partial^2 r}{\partial t^2} + \frac{\partial s}{\partial t} \left( \frac{\partial^2 v}{\partial s^2} \cdot \frac{\partial s}{\partial t} + \frac{\partial^2 v}{\partial r \partial s} \cdot \frac{\partial r}{\partial t} \right) + \frac{\partial v}{\partial s} \cdot \frac{\partial^2 s}{\partial t^2}$$

$$\frac{\partial r}{\partial t} = c \quad \frac{\partial^2 r}{\partial t^2} = 0 \quad \frac{\partial s}{\partial t} = -c \quad \frac{\partial^2 s}{\partial t^2} = 0$$

$$= c \left( \vartheta_{rr} \cdot c + \vartheta_{rs} \cdot (-c) \right) - c \left( \vartheta_{sr} \cdot c + \vartheta_{ss} \cdot (-c) \right) = c^2 (\vartheta_{rr} + \vartheta_{ss} - 2\vartheta_{rs})$$

$$\Rightarrow U_{tt} = c^2 (\vartheta_{rr} + \vartheta_{ss} - 2\vartheta_{rs})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial r}{\partial x} \left( \frac{\partial^2 u}{\partial r^2} \frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s^2} \frac{\partial s}{\partial x} \right) + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial s}{\partial x} \left( \frac{\partial^2 u}{\partial r^2} \frac{\partial r}{\partial s} + \frac{\partial^2 u}{\partial s^2} \frac{\partial s}{\partial x} \right) + \frac{\partial^2 u}{\partial s^2}$$

$$\left\{ \begin{array}{l} \frac{\partial r}{\partial x} = 1 \\ \frac{\partial^2 r}{\partial x^2} = 0 \\ \frac{\partial s}{\partial x} = 1 \\ \frac{\partial^2 s}{\partial x^2} = 0 \end{array} \right.$$

$$= \vartheta_{rr} + \vartheta_{rs} + \vartheta_{sr} + \vartheta_{ss} = \vartheta_{rr} + 2\vartheta_{rs} + \vartheta_{ss}$$

$$\Rightarrow u_{xx} = \vartheta_{rr} + 2\vartheta_{rs} + \vartheta_{ss} \Rightarrow c^2 u_{xx} = c^2 (\vartheta_{rr} + 2\vartheta_{rs} + \vartheta_{ss})$$

$$u_{tt} - c^2 u_{xx} = c^2 (\vartheta_{rr} - 2\vartheta_{rs} + \vartheta_{ss}) - c^2 (\vartheta_{rr} + 2\vartheta_{rs} + \vartheta_{ss}) \\ = -4c^2 \vartheta_{rs} = \omega(r, s) = \omega(x+ct, x-ct)$$

b)  $u_{tt} = c^2 u_{xx} \Rightarrow -4c^2 \vartheta_{rs} = 0 \Rightarrow \vartheta_{rs} = 0 \Rightarrow \frac{\partial(\partial \vartheta)}{\partial s} = 0 \text{ e } \frac{\partial(\partial \vartheta)}{\partial r} = 0$

Seja  $F(\underbrace{x+ct}_r) \text{ e } G(\underbrace{x-ct}_s) \text{ e } \vartheta(r, s) = F(r) + G(s)$

$$\cdot \frac{\partial \vartheta(r, s)}{\partial r} = F'(r) \Rightarrow \frac{\partial}{\partial s} \left( \frac{\partial \vartheta(r, s)}{\partial r} \right) = \frac{\partial F'(r)}{\partial s} = 0$$

$$\cdot \frac{\partial \vartheta(r, s)}{\partial s} = G'(s) \Rightarrow \frac{\partial}{\partial r} \left( \frac{\partial \vartheta(r, s)}{\partial s} \right) = \frac{\partial G'(s)}{\partial r} = 0$$

∴ se  $u(x, t)$  é solução de  $u_{tt} = c^2 u_{xx}$ , vem que

$$u(x, t) = \vartheta(x+ct, x-ct) = F(x+ct) + G(x-ct)$$

19.  $\vartheta(r, \theta) = u(r \cos \theta, r \sin \theta)$

$$\frac{\partial \vartheta}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = U_x \cos \theta + U_y \sin \theta \quad (I)$$

$$\frac{\partial^2 \vartheta}{\partial r^2} = \frac{\partial x}{\partial r} \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right) + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial r^2} + \frac{\partial y}{\partial r} \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2}$$

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial r} = \cos \theta \\ \frac{\partial^2 x}{\partial r^2} = 0 \\ \frac{\partial y}{\partial r} = \sin \theta \\ \frac{\partial^2 y}{\partial r^2} = 0 \end{array} \right.$$

$$= \cos \theta (U_{xx} \cos \theta + U_{xy} \sin \theta) + \sin \theta (U_{yx} \cos \theta + U_{yy} \sin \theta)$$

$$\Rightarrow \frac{\partial^2 \vartheta}{\partial r^2} = U_{xx} \cos^2 \theta + 2U_{xy} \sin \theta \cos \theta + U_{yy} \sin^2 \theta \quad (II)$$

$$\frac{\partial \theta}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\frac{\partial^2 \theta}{\partial \theta^2} = \frac{\partial x}{\partial \theta} \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial \theta^2} + \frac{\partial y}{\partial \theta} \left( \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \theta} \right) + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial \theta^2}$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta \quad \frac{\partial^2 x}{\partial \theta^2} = -r \cos \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta \quad \frac{\partial^2 y}{\partial \theta^2} = -r \sin \theta$$

$$= -r \sin \theta (U_{xx}(-r \sin \theta) + U_{xy} \cdot r \cos \theta) - r \cos \theta U_x$$

$$+ r \cos \theta (U_{yy} \cdot r \cos \theta + U_{yx}(-r \sin \theta)) - r \sin \theta U_y$$

$$= r^2 \sin^2 \theta \cdot U_{xx} - 2r^2 \sin \theta \cos \theta \cdot U_{xy} + r^2 \cos^2 \theta \cdot U_{yy} - r \cos \theta U_x - r \sin \theta U_y \quad (\text{III})$$

$$\text{Assim: } (\text{II}) + \frac{(\text{I})}{r} + \frac{(\text{III})}{r^2} =$$

$$= U_{xx} \cos^2 \theta + 2U_{xy} \sin \theta \cos \theta + U_{yy} \sin^2 \theta + U_x \cos \theta / r + U_y \sin \theta / r$$

$$+ \frac{U_{xx} r^2 \sin^2 \theta}{r^2} - \frac{2r^2 U_{xy} \sin \theta \cos \theta}{r^2} + \frac{U_{yy} r^2 \cos^2 \theta}{r^2} - \frac{U_x \cdot r \cos \theta - U_y \sin \theta}{r^2}$$

$$= U_{xx} + U_{yy} = 4U(r \cos \theta, r \sin \theta)$$

$$\therefore \frac{\partial^2 \theta}{\partial r^2} (r, \theta) + \frac{1}{r} \frac{\partial \theta}{\partial r} (r, \theta) + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \theta^2} (r, \theta) = 4U(r \cos \theta, r \sin \theta)$$

20.  $U(s, t) = f(e^s \cos t, e^s \sin t)$

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} \cdot e^s \cos t + \frac{\partial f}{\partial y} \cdot e^s \sin t$$

$$\Rightarrow \frac{\partial u}{\partial s} = e^s \left( \cos t \frac{\partial f}{\partial x} + \sin t \frac{\partial f}{\partial y} \right)$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} e^s (-\sin t) + \frac{\partial f}{\partial y} e^s (\cos t)$$

$$\Rightarrow \frac{\partial u}{\partial t} = e^s \left( -\sin t \frac{\partial f}{\partial x} + \cos t \frac{\partial f}{\partial y} \right)$$

$$\left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 = e^{2s} \left[ \left( \frac{\partial f}{\partial x} \right)^2 \cos^2 t + \left( \frac{\partial f}{\partial y} \right)^2 \sin^2 t + 2 \sin t \cos t \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right]$$

$$+ e^{2s} \left[ \left( \frac{\partial f}{\partial x} \right)^2 \sin^2 t + \left( \frac{\partial f}{\partial y} \right)^2 \cos^2 t - 2 \cos t \sin t \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right]$$

$$\Rightarrow \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 = e^{2s} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right]$$

$$\Rightarrow \left[ \frac{\partial f}{\partial x} (e^s \cos t, e^s \sin t) \right]^2 + \left[ \frac{\partial f}{\partial y} (e^s \cos t, e^s \sin t) \right]^2 = e^{-2s} \left[ \left( \frac{\partial u(s,t)}{\partial s} \right)^2 + \left( \frac{\partial u(s,t)}{\partial t} \right)^2 \right]$$

•  $\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$ , cálculo das segundas derivadas  
 (aplicação regra do produto:  $(f \cdot g)' = f'g + fg'$ )

•  $\frac{\partial^2 u}{\partial s^2} = \frac{\partial x}{\partial s} \cdot \frac{\partial}{\partial s} \left( \frac{\partial f}{\partial x} \right) + \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial s} \left( \frac{\partial x}{\partial s} \right) + \frac{\partial y}{\partial s} \cdot \frac{\partial}{\partial s} \left( \frac{\partial f}{\partial y} \right) + \frac{\partial f}{\partial y} \cdot \frac{\partial}{\partial s} \left( \frac{\partial y}{\partial s} \right)$

agora aplicando as regras da cadeia novamente:

$$= \frac{\partial x}{\partial s} \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial s} \right) + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial s^2} + \frac{\partial y}{\partial s} \left( \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial s} \right) + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial s^2}$$

$$\left( \frac{\partial x}{\partial s} = e^s \cos t ; \frac{\partial^2 x}{\partial s^2} = e^s \cos t ; \frac{\partial y}{\partial s} = e^s \sin t = \frac{\partial^2 y}{\partial s^2} \right)$$

substituindo e simplificando notação:

$$= e^s \cos t (f_{xx} \cdot e^s \cos t + f_{xy} \cdot e^s \sin t) + e^s \cos t \cdot f_x \\ + e^s \sin t (f_{yx} \cdot e^s \cos t + f_{yy} \cdot e^s \sin t) + e^s \sin t \cdot f_y$$

$$\Rightarrow U_{ss} = e^{2s} (f_{xx} \cos^2 t + 2f_{xy} \sin t \cos t + f_{yy} \sin^2 t) + e^s (f_x \cos t + f_y \sin t)$$

Analogamente:

$$\cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial x}{\partial t} \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t} \right) + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial y}{\partial t} \left( \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t} \right) + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2}$$

$$\left( \frac{\partial x}{\partial t} = -e^s \sin t ; \frac{\partial^2 x}{\partial t^2} = -e^s \cos t ; \frac{\partial y}{\partial t} = e^s \cos t ; \frac{\partial^2 y}{\partial t^2} = -e^s \sin t \right)$$

substituindo e trocando a notação:

$$= -e^s \sin t (f_{xx} (-e^s \sin t) + f_{xy} e^s \cos t) - e^s \cos t \cdot f_x \\ + e^s \cos t (f_{yx} (-e^s \sin t) + f_{yy} e^s \cos t) - e^s \sin t \cdot f_y$$

$$\Rightarrow U_{tt} = e^{2s} (f_{xx} \sin^2 t - 2f_{xy} \sin t \cos t + f_{yy} \cos^2 t) - e^s (f_x \cos t + f_y \sin t)$$

$$\Rightarrow U_{ss} + U_{tt} = e^{2s} (f_{xx} + f_{yy})$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} (e^s \cos t, e^s \sin t) + \frac{\partial^2 f}{\partial y^2} (e^s \cos t, e^s \sin t) = e^{-2s} \left( \frac{\partial u(s,t)}{\partial s} + \frac{\partial u(s,t)}{\partial t} \right)$$

$$21 \quad g(u, v) = u \cdot f(\underbrace{u^2 - v}_{x}, \underbrace{u + 2v}_{y}) \quad 3x + 5y = 3 + 36, (x_0, y_0, z_0) = (1, 4, f(1, 4))$$

$$\frac{\partial^2 f}{\partial x \partial y}(1, 4) = \frac{\partial^2 f}{\partial x^2}(1, 4) = 1 \quad e \quad \frac{\partial^2 f}{\partial y^2}(1, 4) = -1$$

A equação do plano tangente ao gráfico de  $f$  em  $(1, 4)$  pode ser escrita como:

$$z = f(1, 4) + \frac{\partial f}{\partial x}(1, 4)(x - 1) + \frac{\partial f}{\partial y}(1, 4)(y - 4)$$

Comparando com a equação dada, vem que:

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x}(1, 4) \cdot x + \frac{\partial f}{\partial y}(1, 4) \cdot y = z - f(1, 4) + \frac{\partial f}{\partial x}(1, 4) + 4 \frac{\partial f}{\partial y}(1, 4) \\ 3x + 5y = z + 36 \end{array} \right.$$

$$\Rightarrow \frac{\partial f}{\partial x}(1, 4) = 3; \quad \frac{\partial f}{\partial y}(1, 4) = 5; \quad f(1, 4) = f_x(1, 4) + 4f_y(1, 4) - 36$$

$$f(1, 4) = 3 + 4 \cdot 5 - 36 = -13$$

Agora vamos calcular o que o exercício pede:

$$\frac{\partial g}{\partial v} = u \cdot \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \right)$$

$$\begin{aligned} \frac{\partial}{\partial u} \left( \frac{\partial g}{\partial v} \right) &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} \frac{\partial}{\partial u} \frac{\partial v}{\partial u} + u \frac{\partial}{\partial u} \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \right) \\ &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} \frac{\partial}{\partial u} \frac{\partial v}{\partial u} + u \left( \frac{\partial x}{\partial v} \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial u} \right) + \frac{\partial^2 x}{\partial u \partial v} \frac{\partial f}{\partial x} \right) \\ &\quad + u \left( \frac{\partial y}{\partial v} \left( \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial u} \right) + \frac{\partial^2 y}{\partial u \partial v} \frac{\partial f}{\partial y} \right) \end{aligned}$$

Podemos calcular todos as derivadas parciais que faltam:

$$\frac{\partial x}{\partial v} = \frac{\partial(u^2 - v)}{\partial v} = -1 \Rightarrow \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial(-1)}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial(u + 2v)}{\partial v} = 2 \Rightarrow \frac{\partial^2 y}{\partial u \partial v} = \frac{\partial(2)}{\partial u} = 0$$

$$\frac{\partial x}{\partial u} = \frac{\partial(u^2 - v)}{\partial u} = 2u, \text{ no ponto } (-2, 3) \Rightarrow \frac{\partial x}{\partial u}(-2, 3) = -4$$

$$\frac{\partial y}{\partial u} = \frac{\partial (u+2v)}{\partial u} = 1$$

$$\text{Note que } g(-2, 3) = -2 \cdot f(4-3, -2+2 \cdot 3) = -2 \underbrace{f(1, 4)}$$

ponto em que se tem  
vários dados sobre  $f$ .

Assim, substituindo todos os valores encontrados em (I):

$$\begin{aligned}\frac{\partial^2 g}{\partial u \partial v}(-2, 3) &= 3 \cdot (-1) + 5 \cdot 2 - 2((-1)(1(-4)) + 1 \cdot 1) + 0 \cdot 3 \\ &\quad - 2(2(1(-4)) + (-1) \cdot 1) + 0 \cdot 5 \\ &= -3 + 10 - 2(3 - 10) = 21\end{aligned}$$

$$\Rightarrow \frac{\partial^2 g}{\partial u \partial v}(-2, 3) = 21$$

$$22. F(r, s) = G(e^{rs}, r^3 \cos(s)) \quad x = e^{rs} \quad y = r^3 \cos(s)$$

$$\frac{\partial F}{\partial r} = \frac{\partial G}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial r} \quad \downarrow \text{regras do produto}$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial x}{\partial r} \cdot \frac{\partial}{\partial r} \left( \frac{\partial G}{\partial x} \right) + \frac{\partial G}{\partial x} \cdot \frac{\partial^2 x}{\partial r^2} + \frac{\partial y}{\partial r} \cdot \frac{\partial}{\partial r} \left( \frac{\partial G}{\partial y} \right) + \frac{\partial G}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2}$$

$$= \frac{\partial x}{\partial r} \left( \frac{\partial^2 G \cdot \partial x}{\partial x^2 \partial r} + \frac{\partial^2 G \cdot \partial y}{\partial y \partial x \partial r} \right) + \frac{\partial G}{\partial x} \frac{\partial^2 x}{\partial r^2} + \frac{\partial y}{\partial r} \left( \frac{\partial^2 G \cdot \partial x}{\partial x \partial y \partial r} + \frac{\partial^2 G \cdot \partial y}{\partial y^2 \partial r} \right) + \frac{\partial G}{\partial y} \frac{\partial^2 y}{\partial r^2}$$

$$\frac{\partial x}{\partial r} = s e^{rs} \stackrel{(1, 0)}{\Rightarrow} 0 \cdot e^{1 \cdot 0} = 0 \quad \frac{\partial y}{\partial r} = 3r^2 \cos(s) \stackrel{(1, 0)}{\Rightarrow} 3 \cdot 1^2 \cos 0 = 3$$

$$\frac{\partial^2 x}{\partial r^2} = s^2 e^{rs} \stackrel{(1, 0)}{\Rightarrow} 0^2 \cdot e^{1 \cdot 0} = 0 \quad \frac{\partial^2 y}{\partial r^2} = 6r \cos(s) \stackrel{(1, 0)}{\Rightarrow} 6 \cdot 1 \cdot \cos 0 = 6$$

$$\Rightarrow \frac{\partial^2 F}{\partial r^2}(1, 0) = \frac{\partial^2 G}{\partial y^2} \left( \frac{\partial y}{\partial r}(1, 0) \right)^2 + \frac{\partial G}{\partial y} \cdot \frac{\partial^2 y}{\partial r^2}(1, 0)$$

$$= \frac{\partial^2 G}{\partial y^2} \cdot 3^2 + \frac{\partial G}{\partial y} \cdot 6 = 9 \frac{\partial^2 G}{\partial y^2} + 6 \frac{\partial G}{\partial y} \quad (\text{I})$$

$$\text{Como } \frac{\partial G}{\partial y} = t^2 - 2t + 3 \Rightarrow \frac{\partial^2 G}{\partial y^2} = 2t - 2 + 0$$

$$\text{Então } G(t^2+1, t+1) = G(e^{rs}, r^3 \cos(s))$$

$$\text{Para } (r, s) = (1, 0) \Rightarrow G(1, 1) \Rightarrow t = 0 \Rightarrow \frac{\partial G}{\partial y} = 3 \in \frac{\partial^2 G}{\partial y^2} = -2$$

Voltando ao (I)

$$\frac{\partial^2 F}{\partial r^2} = 9 \cdot (-2) + 6 \cdot 3 = 0$$

$$23. f(x, y) = x^2 + 4y^2 \Rightarrow \nabla f(x, y) = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}) = (2x, 8y)$$

$$\text{Para } (x, y) = (2, 1) \Rightarrow \nabla f(2, 1) = (4, 8)$$

O vetor gradiente tem a propriedade de ser perpendicular ao vetor velocidade ( $\gamma'(t)$ ) de qualquer curva de nível ( $\gamma(t)$ ) da função:  $f(x, y) = \text{cte} \Rightarrow \underbrace{f'(x, y)}_{\text{curva de nível}} = \underbrace{\nabla f(x, y) \cdot \gamma'(t)}_{1^{\text{a}} \text{ regra da cadeia}} = 0 \quad \frac{d \text{cte}}{dt}$

Assim, a reta tangente pode ser escrita como:

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

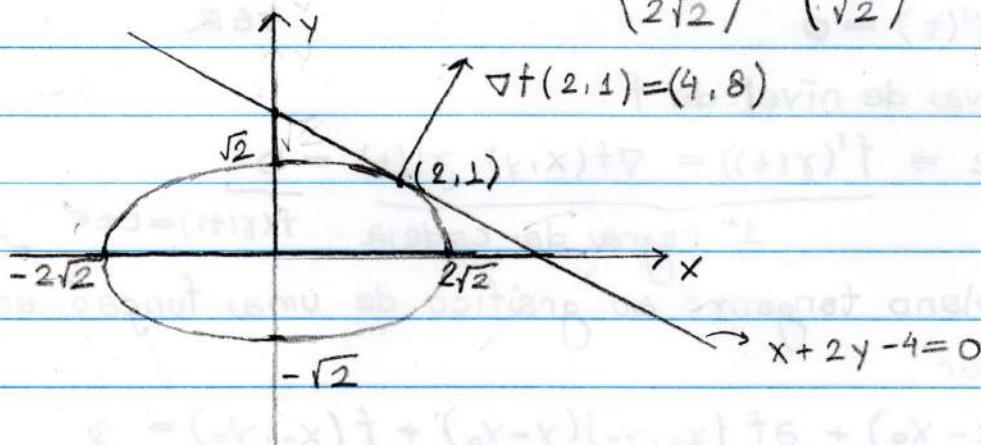
$$\text{Para } (x_0, y_0) = (2, 1) \Rightarrow (4, 8)(x - 2, y - 1) = 0$$

$$\Rightarrow 4x + 8y - 8 - 8 = 0$$

$$\Rightarrow x + 2y - 4 = 0 \leftarrow \text{reta tangente}$$

Vamos esboçar a curva de nível 8 de  $f$ :

$$f(x, y) = 8 = x^2 + 4y^2 \Rightarrow \left(\frac{x}{2\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1 \Rightarrow \text{elipse}$$



$$24. \text{ Seja } f(x, y) = x^3 + 3xy + y^3 + 3x \Rightarrow \nabla f(x, y) = (3x^2 + 3y^2 + 3, 3x + 3y^2)$$

Uma reta tangente ao gráfico de  $f$  em  $(x_0, y_0)$  é dada por:

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

$$\text{Para } (x_0, y_0) = (1, 2) \Rightarrow (3 + 6 + 3, 3 + 12)(x - 1, y - 2) = 0$$

$$\Rightarrow 12x + 15y - 36 = 0$$

$$\Rightarrow 4x + 5y - 12 = 0 \quad (\pi)$$

$$\text{Seja } g(x, y) = x^2 + xy + y^2 \Rightarrow \nabla g(x, y) = (2x + y, x + 2y)$$

Analogamente ao procedimento feito com  $f(x, y)$

$$\nabla g(x_0, y_0)(x - x_0, y - y_0) = 0 : \text{reta tangente} (\alpha)$$

$$\text{Mais } \pi/\alpha \Rightarrow \nabla g(x_0, y_0) = \lambda \nabla f(1, 2) = \underbrace{\lambda \cdot (12, 15)}_{\lambda \in \mathbb{R}} = \underbrace{(4, 5)\lambda_0}_{\lambda_0 \in \mathbb{R}}$$

$$\Rightarrow \begin{cases} 2x_0 + y_0 = 4\lambda_0 \\ x_0 + 2y_0 = 5\lambda_0 \end{cases} \quad \begin{cases} x_0 = \lambda_0 \\ y_0 = 2\lambda_0 \end{cases}$$

E ainda temos que  $(x_0, y_0)$  pertence à curva de nível 7 de  $g(x, y)$ :

$$7 = x_0^2 + x_0 y_0 + y_0^2 \Rightarrow \lambda_0^2 + 2\lambda_0^2 + 4\lambda_0^2 = 7$$

$$\Rightarrow \lambda_0 = 1 \text{ ou } \lambda_0 = -1$$

Assim:  $(x_0, y_0) = (1, 2)$  ou  $(x_0, y_0) = (-1, -2)$  e  $\alpha$  pode ser escrita como:  $\nabla g(x_0, y_0)(x - x_0, y - y_0) = 0$

$$\text{ou: } 4(x - 1) + 5(y - 2) = 0 \quad (x_0, y_0) = (1, 2)$$

$$- 4(x + 1) + 5(y + 2) = 0 \quad (x_0, y_0) = (-1, -2)$$

25. Devemos testar se para cada curva existe  $t$  que satisfaça  $\nabla f(x_0, y_0) \cdot \gamma'(t) = 0$   $t \in \mathbb{R}$

Se  $\gamma(t)$  é curva de nível de  $f$ :

$$f(f(\gamma(t))) = \text{cte} \Rightarrow \underbrace{f'(\gamma(t))}_{\text{1ª regra da cadeia}} \cdot \underbrace{\nabla f(x, y) \cdot \gamma'(t)}_{f(\gamma(t)) = \text{cte}} = 0$$

A equação do plano tangente ao gráfico de uma função em  $(x_0, y_0)$  é dada por:

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + f(x_0, y_0) = g$$

Comparando com a equação do plano  $\Rightarrow \nabla f(x_0, y_0) = (-2, 2)$   
 $(-2x + 2y - 3 + 3 = 0)$

$$\text{ou) } \gamma(t) = \left(-\frac{1}{t}, t\right) \Rightarrow \gamma'(t) = \left(\frac{1}{t^2}, 1\right)$$

$$\Rightarrow \nabla f(x_0, y_0) \cdot \gamma'(t) = (-2, 2) \left(\frac{1}{t^2}, 1\right) = 0 \Rightarrow -2/t^2 + 2 = 0$$

$$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1 \Rightarrow \exists t \in \mathbb{R} \text{ que satisfaçõ}$$

b)  $\gamma(t) = (t^5/5, -2t^3/3 + 3t) \Rightarrow \gamma'(t) = (t^4, -2t^2 + 3)$

 $\Rightarrow \nabla f(x_0, y_0) \cdot \gamma'(t) = (-2, 2)(t^4, -2t^2 + 3) = 0$ 
 $\Rightarrow t^4 + 2t^2 - 3 = 0 \Rightarrow t^2 + 2t - 3 = 0 \quad (\lambda = t^2)$ 
 $\Rightarrow \lambda = -3 \text{ ou } \lambda = 1 \Rightarrow t = \pm 1 \Rightarrow \exists t \in \mathbb{R} \text{ que satisfaz}$

c)  $\gamma(t) = (t^2, t^3 + t) \Rightarrow \gamma'(t) = (2t, 3t^2 + 1)$

 $\Rightarrow \nabla f(x_0, y_0) \cdot \gamma'(t) = (-2, 2)(2t, 3t^2 + 1) = 0$ 
 $\Rightarrow 3t^2 - 2t + 1 = 0 \Rightarrow \Delta = (-2)^2 - 4 \cdot 3 \cdot 1 < 0$ 
 $\Rightarrow \nexists t \in \mathbb{R} \text{ que satisfaz}$

26 A equação do plano tangente ao gráfico de  $f$  em  $(0, 2)$  é dada por:

$$\frac{\partial f}{\partial x}(0, 2)(x-0) + \frac{\partial f}{\partial y}(0, 2)(y-2) + f(0, 2) = \lambda$$

Comparando com a equação dada:  $2x + y + \lambda = 7$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x}(0, 2) = -2 \quad \leftarrow \\ \frac{\partial f}{\partial y}(0, 2) = -1 \quad \leftarrow \end{array} \right. \quad (\text{cuidado com o sinal de } \lambda !)$$

$$f(0, 2) - 2 \frac{\partial f}{\partial x}(0, 2) = 7 \Rightarrow f(0, 2) = 5$$

Para que o plano tangente ao gráfico de  $g$  seja paralelo a  $(4, 2, \omega)$ , podemos impor que  $\vec{n} \cdot (4, 2, \omega) = 0$ , em que  $\vec{n}$  é o vetor normal ao plano:  $\vec{n} = (\partial g / \partial u, \partial g / \partial v, -1)$

Então precisamos calcular  $\partial g(1, 1) / \partial u$  e  $\partial g(1, 1) / \partial v$ :

$$g(u, v) = u f \left( \underbrace{\sin(v^2 - u^3)}_x, \underbrace{2u^2 v}_y \right)$$

$$\cdot \frac{\partial g}{\partial u} = 1 \cdot f(x, y) + u \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \right)$$

$$= 1 \cdot f(x, y) + u \left( \frac{\partial f}{\partial x} 2u \cos(v^2 - u^3) + \frac{\partial f}{\partial y} \cdot 4u \cdot v \right)$$

$$\begin{aligned} \frac{\partial g}{\partial \theta} &= v \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \right) \\ &= v \left( \frac{\partial f}{\partial x} (-3v^2 \cos(v^2 - \theta^3)) + \frac{\partial f}{\partial y} \cdot 2v^2 \right) \end{aligned}$$

Para  $(v, \theta) = (1, 1) \Rightarrow (x, y) = (\sin(v^2 - \theta^3), 2v^2\theta) = (0, 2)$

Assim:

$$\begin{aligned} \frac{\partial g}{\partial v}(1, 1) &= f(0, 2) + 1 \left( \frac{\partial f}{\partial x}(0, 2) \cdot 2 \cdot 1 \cdot \cos 0^\circ + \frac{\partial f}{\partial y}(0, 2) \cdot 4 \cdot 1 \cdot 1 \right) \\ &= 5 + (-4 - 4) = -3 \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial \theta}(1, 1) &= 1 \left( \frac{\partial f}{\partial x}(0, 2) (-3 \cdot 1 \cdot \cos 0^\circ) + \frac{\partial f}{\partial y}(0, 2) \cdot 2 \cdot 1 \right) \\ &= 1 (6 - 2) = 4 \end{aligned}$$

Voltando as imposições:  $\vec{n} \cdot (4, 2, \omega) = 0$

$$\Rightarrow (-3, 4, -1)(4, 2, \omega) = 0 \Rightarrow \omega = -3 \cdot 4 + 4 \cdot 2$$

$$\Rightarrow \omega = -4$$

27. Definindo  $\gamma_1(t) = (2, t)$  e  $\mu_1(t) = (2t^2, t)$ , como essas curvas estão contidas em  $f$ :

$$\begin{cases} f(\gamma_1(t)) = 2t^2 \Leftrightarrow \gamma_1(t) = (2, t, 2t^2) \\ f(\mu_1(t)) = 2t^4 \Leftrightarrow \mu_1(t) = (2t^2, t, 2t^4) \end{cases}$$

Aplicando a 1ª regra da cadeia:

$$\begin{cases} f'(\gamma_1(t)) = \nabla f(x, y) \cdot \gamma_1'(t) = 4t \\ f'(\mu_1(t)) = \nabla f(x, y) \cdot \mu_1'(t) = 8t^3 \end{cases}$$

Para  $(x, y) = (2, 1) \Rightarrow t = 1$ . Se  $\nabla f(2, 1) = (\omega, b)$ , vem que:

$$\begin{cases} f'(\gamma_1(1)) = (\omega, b)(0, 1) = 4 \cdot 1 \Rightarrow b = 4 \\ f'(\mu_1(1)) = (\omega, b)(4 \cdot 1, 1) = 8 \cdot 1^3 \Rightarrow 4\omega + b = 8 \Rightarrow \omega = 1 \end{cases}$$

$$\therefore \nabla f(2, 1) = (1, 4)$$

28. O ponto P pode ser escrito como  $(x_0, y_0) = \gamma(t_0) = (t_0^2, t_0)$

$$\Rightarrow \begin{cases} x_0 = t_0^2 \quad (I) \\ y_0 = t_0 \quad (II) \end{cases}$$

No ponto  $P_0$  o gradiente de  $f(x,y)$  é tangente à  $\gamma(t) \Rightarrow$   
 $\Rightarrow \nabla f(x_0, y_0) \parallel \gamma'(t_0) \Rightarrow (2x_0, 4y_0^3) = \lambda(2t_0, 1)$   
 $\Rightarrow \begin{cases} 2x_0 = 2\lambda t_0 \\ 4y_0^3 = \lambda \end{cases} \Rightarrow \frac{x_0}{4y_0^3} = \frac{t_0}{2} \Rightarrow x_0 = t_0 \quad (\text{III})$

Substituindo (I) e (II) em (III):

$$\frac{t_0^2}{4t_0^3} = t_0 \Rightarrow t_0^2 = \frac{1}{4} \Rightarrow t_0 = \frac{1}{2} \quad (t_0 > 0) \Rightarrow \begin{cases} x_0 = 1/4 \\ y_0 = 1/2 \end{cases}$$

E ainda:  $\nabla f(\frac{1}{4}, \frac{1}{2}) = (2 \cdot \frac{1}{4}, 4 \cdot (\frac{1}{2})^3) = (\frac{1}{2}, \frac{1}{2})$

A reta tangente à  $f$  a uma curva de nível  $\gamma(t)$  de  $f$  em  $(\frac{1}{4}, \frac{1}{2})$  pode ser descrita por:

$$X: (x_0, y_0) + \nabla f(x_0, y_0) \cdot \lambda, \lambda \in \mathbb{R}$$

$$\Rightarrow X: (\frac{1}{4}, \frac{1}{2}) + \lambda(\frac{1}{2}, \frac{1}{2}), \lambda \in \mathbb{R}$$

29.  $\text{Im } \gamma(t) \subset \text{Gr } f(x, y) \Rightarrow \begin{cases} \gamma(t) = (t, 2t^2, t^2) \\ \gamma_1(t) = (t, 2t^2) \\ f(\gamma_1(t)) = t^2 \end{cases}$

Pela regra da cadeia:  $f'(\gamma_1(t)) = \nabla f(x, y) \cdot \gamma_1'(t) = 2t$

Para  $t = 2 \Rightarrow (x, y) = (2, 8) \Rightarrow \nabla f(2, 8) \cdot (1, 4 \cdot 2) = 2 \cdot 2$

$$\Rightarrow \nabla f(2, 8) \cdot (1, 8) = 4 \quad (\text{I})$$

A reta tangente a uma curva de nível de  $f$  pode ser escrita como:  $\nabla f(x_0, y_0)(x - x_0, y - y_0) = 0$ . Como  $(2, 8)$  é o ponto de tangência:  $\nabla f(2, 8)(x - 2, y - 8) = 0$

Como  $(1, -4)$  é reta:  $\nabla f(2, 8)(1 - 2, -4 - 8) = 0$

$$\Rightarrow \nabla f(2, 8)(-1, -12) = 0 \quad (\text{II})$$

Substituindo  $\nabla f(2, 8)$  em (I) e (II):

$$\begin{cases} a + 8b = 4 \\ -a - 12b = 0 \end{cases} \Rightarrow b = -1 \Rightarrow a = 12 \Rightarrow \nabla f(2, 8) = (12, -1)$$

Note que  $f(\gamma_1(2)) = f(2, 8) = 2^2 = 4$  e, assim podemos escrever a equação do plano tangente ao gráfico de  $f$  em  $(2, 8, f(2, 8))$ :

$$g = f(2, 8) + \nabla f(2, 8)(x - 2, y - 8) \Rightarrow 12x - y - g = 12$$

30 Como  $\gamma(t) = (1 + \frac{1}{t}, t)$  é curva de nível 1 de  $f$

$$f(\gamma(t)) = 1 \Rightarrow f'(\gamma(t)) = 0 = \nabla f(x(t), y(t)) \cdot \gamma'(t) = 0$$

$$\text{Se } \nabla f(x_0, y_0) = (\omega, b) \Rightarrow (\omega, b)(-\frac{1}{t_0^2}, 1) = 0$$

$$\Rightarrow b = \omega/t_0^2$$

O plano  $\Pi$  é descrito por:

$$\Pi: \begin{aligned} z &= f(x_0, y_0) + \nabla f(x_0, y_0) \left( x - \underbrace{x_0}_{1 + \frac{1}{t_0}}, y - \underbrace{y_0}_{t_0} \right) \\ &\quad + \underbrace{f(\gamma(t_0))}_{1} = 1 \end{aligned}$$

$$\Rightarrow z = 1 + (\omega, b)(x - (1 + \frac{1}{t_0}), y - t_0) \quad (\text{II})$$

Substituindo (I) em (II):

$$z - 1 = \omega \cdot x - \omega - \omega/t_0 + \omega/t_0^2 \cdot y - \omega/t_0$$

$$\Rightarrow z - 1 = \omega(x - 1 - 2/t_0 + y/t_0^2) \quad (\text{III})$$

Como o plano contém  $(1, 1, 1/2)$  e  $(4, 1, 2)$ , vem que:

$$\frac{1}{2} - 1 = \omega(1 - 1 - 2/t_0 + 1/t_0^2) \Rightarrow -\frac{1}{2}\omega = 2(-2/t_0 + 1/t_0^2) \quad (\text{I})$$

$$2 - 1 = \omega(4 - 1 - 2/t_0 + 1/t_0^2) \Rightarrow \frac{1}{\omega} = (3 - 2/t_0 + 1/t_0^2)$$

$$\Rightarrow 1 = 3 - 2/t_0 + 1/t_0^2 \Rightarrow 2t_0^2 - 2t_0 + 1 = 0$$

$$\Rightarrow (t_0 - 1)^2 = 0 \Rightarrow t_0 = 1 \Rightarrow \omega = 1/2 \text{ e } b = 1/2$$

$$(x_0, y_0) = (2, 1)$$

Voltando à equação do plano  $\Pi$ :

$$z = 1 + (\frac{1}{2}, \frac{1}{2})(x - 2, y - 1)$$

$$z = 1 + \frac{1}{2}x + \frac{1}{2}y - 1 - \frac{1}{2} \Rightarrow x + y - 2y = 1$$