Questão A1) Calcule as integrais indefinidas abaixo:

(a) (1,0) $\int \cos(2x)\ln(\operatorname{tg} x)\,dx$

(b) (1,5) $\int \dfrac{3}{e^x(e^x-3)}\,dx$

(c) (1,5) $\int \dfrac{3-\sqrt{x}}{\sqrt{9-x}}\,dx$

(a) $\int \cos(2x)\ln(\operatorname{tg} x)\,dx =$

$\left[\begin{array}{ll} u = \ln(\operatorname{tg} x) & v = \dfrac{\operatorname{sen} 2x}{2} \\ du = \dfrac{1}{\operatorname{tg} x}\cdot \sec^2 x\,dx = \dfrac{1}{\operatorname{sen} x\cdot\cos x}\,dx & dv = \cos 2x\,dx \end{array}\right]$ (por partes)

$= \dfrac{\operatorname{sen} 2x\cdot\ln(\operatorname{tg} x)}{2} - \int \dfrac{\operatorname{sen} 2x}{2\operatorname{sen} x\cdot\cos x}\,dx = \dfrac{\operatorname{sen} 2x\cdot\ln(\operatorname{tg} x)}{2} - x + C$

(b) $\int \dfrac{3}{e^x(e^x-3)}\,dx =$

$[u = e^x,\ du = e^x\,dx]$ frações parciais

$= \int \dfrac{3}{u(u-3)}\cdot\dfrac{du}{u} = \int \dfrac{3}{u^2(u-3)}\,du = \int \dfrac{-1/3}{u}\,du + \int \dfrac{-1}{u^2}\,du + \int \dfrac{1/3}{u-3}\,du$

$= -\dfrac{1}{3}\ln|u| + \dfrac{1}{u} + \dfrac{1}{3}\ln|u-3| + C = -\dfrac{1}{3}x + \dfrac{1}{e^x} + \dfrac{1}{3}\ln|e^x-3| + C$

(c) $\int \dfrac{3-\sqrt{x}}{\sqrt{9-x}}\,dx =$

$[u = \sqrt{9-x} \Rightarrow x = 9-u^2;\ dx = -2u\,du]$

$= -6\int 1 - \sqrt{1-\left(\dfrac{u}{3}\right)^2}\,du = \left[\begin{array}{l} \dfrac{u}{3} = \operatorname{sen} t; \\ -\frac{\pi}{2}\le t\le\frac{\pi}{2} \end{array}\ du = 3\cos t\,dt\right]$

$= -18\left[\int \cos t\,dt - \int \cos^2 t\,dt\right] = -18\operatorname{sen} t + 9t + 9\operatorname{sen} t\cos t + C$

$= -18\cdot\dfrac{u}{3} + 9\arcsen\dfrac{u}{3} + 9\cdot\dfrac{u}{3}\cdot\cos\left(\arcsen\dfrac{u}{3}\right) + C$

$= -6\sqrt{9-x} + 9\arcsen\dfrac{\sqrt{9-x}}{3} + \sqrt{9x-x^2} + C$

Questão B1) Calcule as integrais indefinidas abaixo:

(a) (1,0) $\int \cos(2x)\ln(\operatorname{cotg} x)\,dx$

(b) (1,5) $\int \dfrac{2}{e^x(e^x-2)}\,dx$

(c) (1,5) $\int \dfrac{2-\sqrt{x}}{\sqrt{4-x}}\,dx$

(a) $\int \cos(2x)\ln(\operatorname{cotg} x)\,dx =$

$\left[\begin{array}{ll} u=\ln(\operatorname{cotg} x) & v=\dfrac{\operatorname{sen} 2x}{2} \\ du=\dfrac{1}{\operatorname{cotg} x}\cdot -\operatorname{cossec}^2 x\,dx = -\dfrac{1}{\operatorname{sen} x\cdot\cos x}\,dx & dv=\cos 2x\,dx \end{array}\right]$ (por partes)

$= \dfrac{\operatorname{sen} 2x\cdot\ln(\operatorname{cotg} x)}{2} + \int \dfrac{\operatorname{sen} 2x}{2\operatorname{sen} x\cdot\cos x}\,dx = \dfrac{\operatorname{sen} 2x\cdot\ln(\operatorname{cotg} x)}{2} + x + C.$

(b) $\int \dfrac{2}{e^x(e^x-2)}\,dx =$

$[u=e^x\ ;\ du=e^x dx]$

$= \int \dfrac{2}{u(u-2)}\cdot\dfrac{du}{u} = \int \dfrac{2}{u^2(u-2)}\,du =$ (frações parciais) $\int \dfrac{-1/2}{u}\,du + \int \dfrac{-1}{u^2}\,du + \int \dfrac{1/2}{u-2}\,du =$

$= -\dfrac{1}{2}\ln|u| + \dfrac{1}{u} + \dfrac{1}{2}\ln|u-2| + C = -\dfrac{1}{2}x + \dfrac{1}{e^x} + \dfrac{1}{2}\ln|e^x-2| + C$

(c) $\int \dfrac{2-\sqrt{x}}{\sqrt{4-x}}\,dx =$

$[u=\sqrt{4-x} \rightarrow x=4-u^2;\ dx=-2u\,du]$

$= -4\int 1-\sqrt{1-\left(\dfrac{u}{2}\right)^2}\,du = \left[\begin{array}{l}\dfrac{u}{2}=\operatorname{sen} t;\ du=2\cos t\,dt \\ -\frac{\pi}{2}\le t\le\frac{\pi}{2}\end{array}\right]$

$= -8\left[\int \cos t\,dt - \int \cos^2 t\,dt\right] = -8\operatorname{sen} t + 4t + 4\operatorname{sen} t\cos t + C$

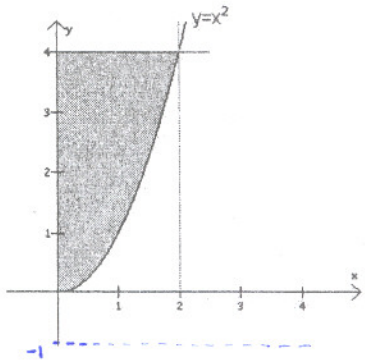
$= -8\dfrac{u}{2} + 4\operatorname{arcsen}\dfrac{u}{2} + 4\dfrac{u}{2}\cos\left(\operatorname{arcsen}\dfrac{u}{2}\right) + C$

$= -4\sqrt{4-x} + 4\operatorname{arcsen}\dfrac{\sqrt{4-x}}{2} + \sqrt{4x-x^2} + C$

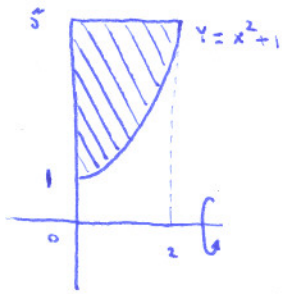
Questão A2) Considere a região do plano limitada pela parábola $y = x^2$ e pelas retas $y = 4$ e $x = 0$, conforme a figura abaixo.

(a) (1,5) Calcule o volume do sólido obtido pela rotação dessa região em torno da reta $y = -1$.

(b) (1,5) Determine $b \in]0, 4[$ tal que a reta $y = b$ divida a região em duas regiões de mesma área.



(a) O VOLUME DO SÓLIDO É EQUIVALENTE AO OBTIDO PELA ROTAÇÃO (EM TORNO DO EIXO DOS x) DA FIGURA A SEGUIR:

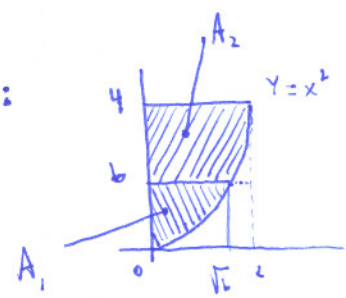


Assim, $V = \pi(5)^2 \cdot 2 - \pi \int_0^2 (x^2+1)^2 \, dx$

$= 50\pi - \pi \int_0^2 (x^4 + 2x^2 + 1) \, dx$

$= 50\pi - \pi \left(\frac{x^5}{5} + \frac{2}{3}x^3 + x \right) \Big|_0^2 = \boxed{\pi \cdot \left(50 - \left[\frac{32}{5} + \frac{16}{3} + 2 \right] \right)}$

(b) DEVEMOS TER:



$A_1 = b\sqrt{b} - \int_0^{\sqrt{b}} x^2 \, dx = \frac{2}{3} b^{3/2}$

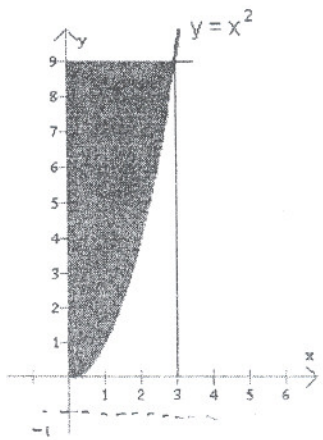
$A_2 = 2(4-b) - \int_{\sqrt{b}}^2 (x^2 - b) \, dx = \frac{16}{3} - \frac{2b}{3} - \frac{b^{3/2}}{3}$

$\therefore A_1 = A_2 \iff \frac{2}{3} b^{3/2} = \frac{16}{3} - \frac{2b}{3} \iff \boxed{b = 4^{2/3}}$

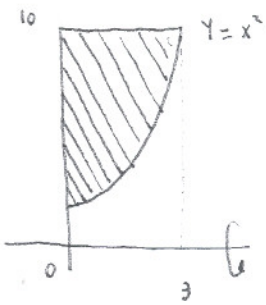
Questão B2) Considere a região do plano limitada pela parábola $y = x^2$ e pelas retas $y = 9$ e $x = 0$, conforme a figura abaixo.

(a) (1,5) Calcule o volume do sólido obtido pela rotação dessa região em torno da reta $y = -1$.

(b) (1,5) Determine $b \in]0, 9[$ tal que a reta $y = b$ divida a região em duas regiões de mesma área.



(a) O VOLUME DO SÓLIDO É EQUIVALENTE AO OBTIDO PELA ROTAÇÃO (EM TORNO DO EIXO x) DA FIGURA A SEGUIR:



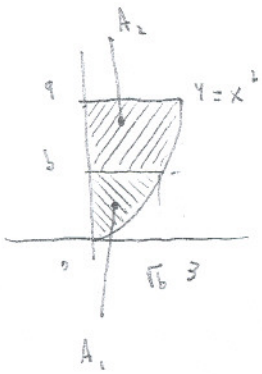
ASSIM;

$$V = \pi(10)^2 \cdot 3 - \pi \int_0^3 (x^2+1)^2 \, dx$$

$$= 300\pi - \pi \int_0^3 x^4 + 2x^2 + 1 \, dx = 300\pi - \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right)_0^3 =$$

$$= \boxed{\pi \left(300 - \left(\frac{243}{5} + 18 + 3 \right) \right)}$$

(b)



$$A_1 = b\sqrt{b} - \int_0^{\sqrt{b}} x^2 \, dx = \frac{2}{3} b^{3/2}$$

$$A_2 = (9-b)3 - \int_{\sqrt{b}}^3 (x^2 - b) \, dx = 18 - \frac{2}{3} b^{3/2}$$

$$\therefore \quad A_1 = A_2 \iff 18 - \frac{2}{3} b^{3/2} = \frac{2}{3} b^{3/2} \iff 18 = \frac{4}{3} b^{3/2} \iff \boxed{b = \left(\frac{27}{2} \right)^{2/3}}$$

Questão A3) Seja $g(x) = \int_1^{\operatorname{sen} x} e^{t^2} dt$

(a) (0,5) Calcule $g'(x)$

(b) (1,0) Mostre que $\int_a^b \cos x \; g(x) dx \;=\; \operatorname{sen} b \; g(b) - \operatorname{sen} a \; g(a) - \int_a^b \operatorname{sen} x \;\cos x \; e^{\operatorname{sen}^2 x} dx$

(c) (1,5) Calcule $\int_0^{\pi/2} \cos x \left(\int_1^{\operatorname{sen} x} e^{t^2} dt \right) dx$

a) Seja $F(x)$ uma primitiva de $f(x) = e^{x^2}$. Pelo TFC,

$$\int_1^{\operatorname{sen} x} e^{t^2} dt = F(t)\Big|_1^{\operatorname{sen} x} = F(\operatorname{sen} x) - F(1)$$

logo, $g'(x) = F'(\operatorname{sen} x) \cdot \cos x = \boxed{\cos x \, e^{\operatorname{sen}^2 x}}$

b) $\int_a^b \cos x \, g(x) dx \underset{\text{partes}}{=} \operatorname{sen} x \, g(x)\Big|_a^b - \int_a^b \operatorname{sen} x \cos x \, e^{\operatorname{sen}^2 x} dx$

$\begin{pmatrix} u = g(x) & du = \cos x \, e^{\operatorname{sen}^2 x} dx \\ dv = \cos x dx & v = \operatorname{sen} x \end{pmatrix}$

$$= \operatorname{sen} b \, g(b) - \operatorname{sen} a \, g(a) - \int_a^b \operatorname{sen} x \cos x \, e^{\operatorname{sen}^2 x} dx$$

c) $g\left(\frac{\pi}{2}\right) = \int_1^1 e^{t^2} dt = 0$ e $\operatorname{sen} 0 = 0$.

Portanto,

$$\int_0^{\pi/2} \cos x \left(\int_1^{\operatorname{sen} x} e^{t^2} dt \right) dx = \operatorname{sen}\frac{\pi}{2} \, g\left(\frac{\pi}{2}\right) - \operatorname{sen} 0 \, g(0) - \int_0^{\pi/2} \operatorname{sen} x \cos x \, e^{\operatorname{sen}^2 x} dx$$

$$= -\int_0^{\pi/2} \operatorname{sen} x \cos x \, e^{\operatorname{sen}^2 x} dx = -\frac{1}{2} \int_0^1 e^t dt = \boxed{-\frac{1}{2}(e-1)}$$

$\begin{pmatrix} u = \operatorname{sen}^2 x \Rightarrow du = 2 \operatorname{sen} x \cos x dx \\ x = 0 \Rightarrow u = 0 \\ x = \pi/2 \Rightarrow u = 1 \end{pmatrix}$

Questão B3) Seja $g(x) = \int_{-1}^{\cos x} e^{t^2}\,dt$

(a) (0,5) Calcule $g'(x)$

(b) (1,0) Mostre que $\int_a^b \operatorname{sen} x\; g(x)dx \;=\; -\cos b\; g(b) + \cos a\; g(a) - \int_a^b \operatorname{sen} x\;\cos x\; e^{\cos^2 x}\;dx$

(c) (1,5) Calcule $\int_{\pi/2}^{\pi} \operatorname{sen} x\;\left(\int_{-1}^{\cos x} e^{t^2}\,dt\right)\;dx$

a) Seja $F(x)$ uma primitiva de $f(x) = e^{x^2}$. Pelo TFC,

$$\int_{-1}^{\cos x} e^{t^2}dt = F(t)\Big|_{-1}^{\cos x} = F(\cos x) - F(-1).$$

Logo, $g'(x) = F'(\cos x).(-\operatorname{sen} x) = \boxed{-\operatorname{sen} x\, e^{\cos^2 x}}$

b) $\int_a^b \operatorname{sen} x\, g(x)dx = -\cos x g(x)\Big|_a^b - \int_a^b \operatorname{sen} x \cos x e^{\cos^2 x}dx$

(por partes)

$\left(\begin{array}{ll} u = g(x) & du = -\operatorname{sen} x\, e^{\cos^2 x}dx \\ dv = \operatorname{sen} x dx & v = -\cos x \end{array}\right)$

$$= -\cos b\, g(b) + \cos a\, g(a) - \int_a^b \operatorname{sen} x \cos x e^{\cos^2 x}dx$$

c) $g(\pi) = \int_{-1}^{-1} e^{t^2}dt = 0$ e $\cos\frac{\pi}{2} = 0$.

Portanto,

$$\int_{\pi/2}^{\pi} \operatorname{sen} x \left(\int_{-1}^{\cos x} e^{t^2}dt\right)dx = \cos\pi\, g(\pi) + \cos\frac{\pi}{2}\, g(\pi/2) - \int_{\pi/2}^{\pi} \operatorname{sen} x \cos x e^{\cos^2 x}dx$$

$$= -\int_{\pi/2}^{\pi} \operatorname{sen} x \cos x e^{\cos^2 x}dx = \frac{1}{2}\int_0^1 e^u du = \boxed{\frac{1}{2}(e-1)}$$

$\left(\begin{array}{l} u = \cos^2 x \Rightarrow du = -2\cos x \operatorname{sen} x dx \\ x = \pi/2 \Rightarrow u = 0 \\ x = \pi \Rightarrow u = 1 \end{array}\right)$