Lista 1 Paulo Akira 2016.2

1) Seja $u = (x_1, y_1)$. Basta verificar que $\langle u,u\rangle > 0$:

$\langle u,u\rangle = x_1 \cdot x_1 + t\, y_1 \cdot y_1 = x_1^2 + t\, y_1^2 > 0 \iff$

$\iff t > 0$

2) (1) Simetria:

$\langle u,v\rangle = \langle v,u\rangle \Rightarrow 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 =$

$= 2y_1x_1 - y_2x_1 - y_1x_2 + 2x_2y_2$ (verdadeiro)

(2) Distributiva:

Seja $w = (w_1, w_2)$ e $u + w = (x_1 + w_1, x_2 + w_2)$:

$\langle u+w, v\rangle = \langle u,v\rangle + \langle v,w\rangle$

$\langle u+w, v\rangle = 2(x_1+w_1)y_1 - (x_1+w_1)y_2 - (x_2+w_2)y_1 +$

$+ 2(x_2+w_2)y_2 = 2x_1y_1 + 2w_1y_1 - x_1y_2 - w_1y_2 -$

$- x_2y_1 - w_2y_1 + 2x_2y_2 + 2w_2y_2$

$\langle u,v\rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$

$\langle v,w\rangle = 2y_1w_1 - y_1w_2 - y_2w_1 + 2y_2w_2$

$\therefore \langle u+w, v\rangle = \langle u,v\rangle + \langle v,w\rangle$ (verdadeiro)

(3) $\langle \lambda u, v\rangle = \langle u, \lambda v\rangle = \lambda\langle u,v\rangle$

$\langle \lambda u, v\rangle = 2\lambda x_1y_1 - \lambda x_1y_2 - \lambda x_2y_1 + 2\lambda x_2y_2$

$\langle u, \lambda v\rangle = 2\lambda x_1y_1 - 2\lambda x_1y_2 - \lambda x_2y_1 + 2\lambda x_2y_2$

$\lambda\langle u,v\rangle = \lambda(2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2)$

$\therefore \langle \lambda u, v\rangle = \langle u, \lambda v\rangle = \lambda\langle u,v\rangle$ (verdadeiro)

(4) $\langle u,u\rangle \geq 0$ e $\langle u,u\rangle = 0 \iff u = \vec{0}$

$\langle u,u\rangle = 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 =$

$= 2x_1^2 - 2x_1x_2 + 2x_2^2 = 2\left(x_1^2 - x_1x_2 + \frac{x_2^2}{4} + \frac{3}{4}x_2^2\right)$

$= 2\left[(x_1 - x_2/2)^2 + \frac{3}{4}x_2^2\right] \geq 0$ (verdadeiro)

$\langle u,u\rangle = 0 \iff (x_1 - x_2/2)^2 + \frac{3}{4}x_2^2 = 0 \iff$

$\iff x_1 = x_2 = 0$ (verdadeiro)

Dois vetores são ortogonais se $\langle u,v\rangle = 0$

$\langle u,(1,0)\rangle = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow x_2 = 2x_1 \Rightarrow$

$\Rightarrow x_1 = \frac{x_2}{2}$.

Seja $x_1 = \alpha$ temos que os vetores ortogonais a $(1,0)$ serão dados por $\{(\alpha, 2\alpha) : \alpha \in \mathbb{R}\}$

Pela definição de norma temos que:

$\|(1,0)\| = \sqrt{\langle (1,0),(1,0)\rangle} = \sqrt{2}$

3) a) $\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx = 0 \Rightarrow$

$\Rightarrow \int_{-1}^{1} (x^2-1)(\lambda x - 2)\,dx = 0 \Rightarrow$

$\Rightarrow \int_{-1}^{1} \lambda x^3 - 2x^2 - \lambda x + 2\,dx = 0 \Rightarrow$

$\Rightarrow \left[\frac{\lambda x^4}{4} - \frac{2x^3}{3} - \frac{\lambda x^2}{2} + 2x\right]_{-1}^{1} =$

$= \left[\cancel{\frac{\lambda}{4}} - \frac{2}{3} - \cancel{\frac{\lambda}{2}} + 2 - \cancel{\frac{\lambda}{4}} - \frac{2}{3} + \cancel{\frac{\lambda}{2}} - 2\right] = 0 \Rightarrow$

$\Rightarrow 4 - \frac{4}{3} = 0 \therefore \nexists\, \lambda \in \mathbb{R}$

b) $\langle p,q\rangle = \underbrace{\cancel{p(-1)}}_{0}q(-1) + p(0)q(0) + \underbrace{\cancel{p(1)}}_{0}q(1) +$

$p(2)q(2) = 0 \Rightarrow 2 + 3(2\lambda + 2) = 0 \Rightarrow$

$\Rightarrow 2 + 6\lambda - 6 = 0 \Rightarrow 6\lambda = 4 \Rightarrow \lambda = \frac{2}{3}$

5) Seja $u = (u_1, u_2)$ e $v = (v_1, v_2)$, com $u, v \in$

$u + v = (u_1 + v_1, u_2 + v_2)$

$u - v = (u_1 - v_1, u_2 - v_2)$

$\langle u+v, u-v\rangle = (u_1+v_1)(u_1-v_1) + (u_2+v_2)(u_2-v_2)$

$= u_1^2 - v_1^2 + u_2^2 - v_2^2 = 0 \iff u_1^2 + u_2^2 = v_1^2 + v_2^2$

$\iff \sqrt{u_1^2 + u_2^2} = \sqrt{v_1^2 + v_2^2} \iff \|u\| = \|v\|$

$\langle u,v\rangle = 0 \iff u_1v_1 = -u_2v_2$

$\|u+v\|^2 = \left(\sqrt{(u_1+v_1)^2 + (u_2+v_2)^2}\right)^2 =$

$= u_1^2 + \cancel{2u_1v_1} + v_1^2 + u_2^2 + \cancel{2u_2v_2} + v_2^2 =$

$= u_1^2 + v_1^2 + u_2^2 + v_2^2 = \|u\|^2 + \|v\|^2$

6) Vamos verificar que se trata de um produto interno:

(1) $\langle p,q\rangle = \int_a^b p(t)q(t)\,dt = \int_a^b p(t)q(t)\,dt$

(2) Seja $r \in P(\mathbb{R})$:

$\langle p+r, q\rangle = \int_a^b (p+r)(t)\cdot q(t)\,dt = \int_a^b [p(t)+r(t)]q(t)\,d$

$$= \int_a^b p(t)r(t) + q(t)r(t)\,dt = \int_a^b p(t)r(t)\,dt + \int_a^b q(t)r(t)\,dt$$

$$= \langle p,r\rangle + \langle q,r\rangle$$

(3) seja $\lambda \in \mathbb{R}$:

$$\langle \lambda p, q\rangle = \int_a^b \lambda p(t)q(t)\,dt = \lambda \int_a^b p(t)q(t)\,dt = \lambda\langle p,q\rangle$$

(4) $\langle p,p\rangle = \int_a^b \underbrace{p(t)^2}_{\geq 0}\,dt \geq 0$

$$\langle p,p\rangle = 0 \iff \int_a^b p(t)^2\,dt = 0 \iff p(t)^2 = 0 \iff$$

$\iff p(t) = 0,\ \forall t \in [a,b]$, já que p é contínuo.

Para as funções contínuas pertencentes a um espaço $\mathcal{C}([a,b])$, o produto interno $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$, com $f,g \in \mathcal{C}([a,b])$ funciona pelos mesmos argumentos dados para os polinômios pertencentes a $P(\mathbb{R})$. Já para as funções pertencentes a $\mathcal{C}(\mathbb{R})$ temos que

$$\langle f,f\rangle = \int_{-\infty}^{+\infty} f^2(t)\,dt = \int_k^{+\infty} f(t)^2\,dt + \int_{-\infty}^{k} f(t)^2\,dt =$$

$$= \lim_{c\to+\infty} \int_k^c f(t)^2\,dt + \lim_{c'\to-\infty} \int_{c'}^k f(t)^2\,dt = 0, \text{ com}$$

$f(t) \neq 0$. Assim esse produto interno não funciona para funções definidas em $\mathbb{R} \longrightarrow \mathbb{R}$.

11) Seja $u = (\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ e $v = \left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_3}}\right)$ com $u, v \in \mathbb{R}^3$. Então,

$$\langle u,v\rangle = \frac{\sqrt{a_1}}{\sqrt{a_1}} + \frac{\sqrt{a_2}}{\sqrt{a_2}} + \frac{\sqrt{a_3}}{\sqrt{a_3}} = 3$$

Pela Desigualdade Cauchy-Schwarz:

$$|\langle u,v\rangle|^2 = 9 \leq \|u\|^2\|v\|^2 = (a_1+a_2+a_3)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$$

$$\therefore 9 \leq (a_1+a_2+a_3)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$$

12) Seja $f, g \in \mathcal{C}([0,1])$. Temos que:

$$\left(\int_0^1 (f(t)+g(t))^2\,dt\right)^{1/2} = \|f+g\|$$

$$\|f+g\|^2 = \int_0^1 f(t)^2\,dt + \int_0^1 2f(t)g(t)\,dt + \int_0^1 g(t)^2\,dt =$$

$$= \langle f,f\rangle + 2\langle f,g\rangle + \langle g,g\rangle =$$

$$= \|f\|^2 + 2\langle f,g\rangle + \|g\|^2$$

Pela Desigualdade de Cauchy-Schwarz:

$|\langle f,g\rangle| \leq \|f\|\cdot\|g\| \therefore 2\cdot|\langle f,g\rangle| \leq 2\|f\|\cdot\|g\| \therefore$

$\|f\|^2 + 2|\langle f,g\rangle| + \|g\|^2 \leq \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2 =$

$= (\|f\| + \|g\|)^2 \therefore \|f+g\|^2 \leq (\|f\|+\|g\|)^2 \iff$

$\iff \|f+g\| \leq \|f\| + \|g\|$, ou seja,

$$\left(\int_0^1 (f(t)+g(t))^2\,dt\right)^{1/2} \leq \left(\int_0^1 f(t)^2\,dt\right)^{1/2} + \left(\int_0^1 g(t)^2\,dt\right)^{1/2}$$

13) Pela propriedade da desigualdade triangular da norma, temos que:

$\|u+v\| \leq \|u+v\| + \|w\| \leq \|u\| + \|v\| + \|w\| \therefore$

$\therefore \|u+v\| \leq \|u\| + \|v\| + \|w\|$

16) Escolhendo uma base de $P_2(\mathbb{R})$ que não é ortogonal: $\mathcal{B} = \{1, t, t^2\}$

$f_1 = 1 \therefore \|f_1\| = \sqrt{1+1+1+1} = 2$

$f_2 = t - \text{proj}_{f_1} t = t - \frac{\langle f_1, t\rangle f_1}{\|f_1\|^2} = t - \frac{1}{2}$

$\langle f_1, t\rangle = -1 + 0 + 1 + 2 = 2$

$\|f_2\| = \sqrt{9/4 + 1/4 + 1/4 + 9/4} = \sqrt{5}$

$f_3 = t^2 - \text{proj}_{f_1} t^2 - \text{proj}_{f_2} t^2 = t^2 - \frac{\langle f_1, t^2\rangle f_1}{\|f_1\|^2} - \frac{\langle f_2, t^2\rangle f_2}{\|f_2\|^2}$

$\langle 1, t^2\rangle = 1 + 0 + 1 + 4 = 6$

$\left\langle t - 1/2, t^2\right\rangle = -\frac{3}{2} + \frac{1}{2} + 6 = 5$

$f_3 = t^2 - \frac{3}{2} - t + \frac{1}{2} = t^2 - t - 1$

$\|f_3\| = \sqrt{1+1+1+1} = 2$

A base $\left\{1, t - \frac{1}{2}, t^2 - t - 1\right\}$ é ortogonal, para que seja ortonormal devemos ter $\left\{\frac{1}{2}, \left(t - \frac{1}{2}\right)\cdot\frac{1}{\sqrt{5}}, \frac{t^2-t-1}{2}\right\}$

17) $e_1 = (1,1,1,1) \therefore \|e_1\| = \sqrt{1+1+1+1} = 2$

$e_2 = (1,1,2,4) - proj_{e_1}(1,1,2,4)$

$\langle (1,1,1,1),(1,1,2,4)\rangle = 1+1+2+4 = 8$

$e_2 = (1,1,2,4) - \frac{8}{4}\cdot(1,1,1,1) = (-1,-1,0,2)$

$\|e_2\| = \sqrt{1+1+4} = \sqrt{6}$

$e_3 = (1,2,-4,-3) - proj_{e_1}(1,2,-4,-3) - proj_{e_2}(1,2,-4,-3)$

$\langle (1,1,1,1),(1,2,-4,-3)\rangle = 1+2-4-3 = -4$

$\langle (-1,-1,0,2),(1,2,-4,-3)\rangle = -1-2-6 = -9$

$e_3 = (1,2,-4,-3) + (1,1,1,1) + \frac{3}{2}(-1,-1,0,2) =$

$= (1,2,-4,-3) + (1,1,1,1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right)$

$= (2,3,-3,-2) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right) =$

$= \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$

$\|e_3\| = \sqrt{\frac{1}{4} + \frac{9}{4} + 9 + 1} = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{36}{4} + \frac{4}{4}} = \frac{5\sqrt{2}}{2}$

A base ortonormal será $\left\{\frac{1}{2}(1,1,1,1), \frac{1}{\sqrt{6}}(-1,-1,0,-2), \frac{5\sqrt{2}}{2}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)\right\}$

18) a) $f_1 = 1 \therefore \|f_1\| = \sqrt{\int_0^1 1\,dx} = 1$

$f_2 = x - \frac{\langle f_1, x\rangle f_1}{\|f_1\|^2} = x - \frac{1}{2}$

$\langle 1, x\rangle = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$\|f_2\| = \sqrt{\int_0^1 x^2 - x + \frac{1}{4}\,dx} = \sqrt{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1} = \sqrt{\frac{1}{12}}$

$f_3 = x^2 - \langle 1, x^2\rangle - 12\left(x - \frac{1}{2}\right)\langle (x-1/2), x^2\rangle$

$\langle 1, x^2\rangle = \int_0^1 x^2\,dx = \frac{1}{3}$

$\langle (x-1/2), x^2\rangle = \int_0^1 x^3 - \frac{x^2}{2}\,dx = \left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1 = \frac{1}{12}$

$f_3 = x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$

$\|f_3\| = \sqrt{\int_0^1 x^4 - 2x^3 + \frac{4x^2}{3} - \frac{x}{3} + \frac{1}{36}\,dx} =$

$= \sqrt{\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\right]_0^1} =$

$= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = \sqrt{\frac{1}{180}}$

Uma base ortogonal de coeficientes inteiros seria:

$B = \{1, 2x-1, 6x^2-6x+1\}$

b) Uma base ortonormal será:

$B' = \left\{1, \frac{x - 1/2}{1/\sqrt{12}}, \frac{x^2 - x + 1/6}{\sqrt{1/180}}\right\}$

19) $f_1 = x - 1 \therefore \|f_1\| = \sqrt{6}$

$\langle x-1, x-2\rangle = (-2)(-3) + 2 = 8$

$f_2 = x - 2 - \frac{\cancel{8}^{4}}{\cancel{6}_{3}}\cdot(x-1) = x - 2 - \frac{4x}{3} + \frac{4}{3} =$

$= -\frac{x}{3} - \frac{2}{3}$

20) $\|f_1\| = \sqrt{1+1+1+1} = 2$

$\|f_2\| = \sqrt{1+1+4} = \sqrt{6}$

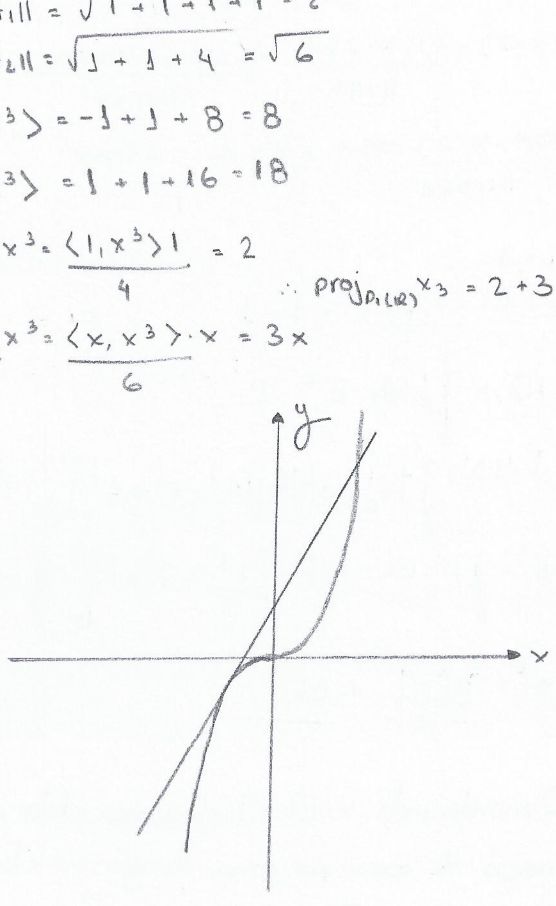
$\langle 1, x^3\rangle = -1 + 1 + 8 = 8$

$\langle x, x^3\rangle = 1 + 1 + 16 = 18$

$proj_{f_1} x^3 = \frac{\langle 1, x^3\rangle 1}{4} = 2$

$\therefore proj_{P_1(\mathbb{R})} x^3 = 2 + 3x$

$proj_{f_2} x^3 = \frac{\langle x, x^3\rangle \cdot x}{6} = 3x$



21) $\langle x-2, 1\rangle = \int_0^{2\pi} x-2\,dx = \left[\frac{x^2}{2} - 2x\right]_0^{2\pi} = \frac{4\pi^2}{2} - 4\pi =$
$= 2\pi^2 - 4\pi$

$\langle x-2, \operatorname{sen} x\rangle = \int_0^{2\pi} x\operatorname{sen} x - 2\operatorname{sen} x\, dx =$

Note que: $\int x \operatorname{sen} x\, dx = -x\cos x + \int \cos x\, dx =$
$= \operatorname{sen} x - x\cos x + k$

$= \left[\operatorname{sen} x - x\cos x + 2\cos x\right]_0^{2\pi} = -2\pi + 2 - 2 = -2\pi$

$\langle x-2, \cos x\rangle = \int_0^{2\pi} x\cos x - 2\cos x\, dx =$

$= \left[x\operatorname{sen} x + \cos x - 2\operatorname{sen} x\right]_0^{2\pi} = 0$

Note que: $\cos 2x = \cos^2 x - \operatorname{sen}^2 x \Rightarrow$
$\Rightarrow \cos 2x = 1 - 2\operatorname{sen}^2 x \Rightarrow \operatorname{sen}^2 x = \frac{1-\cos 2x}{2}$

$\int \operatorname{sen}^2 x\, dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\operatorname{sen} 2x}{4}$

$\|1\|^2 = \int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi$

$\|\operatorname{sen} x\|^2 = \int_0^{2\pi} \operatorname{sen}^2 x\, dx = \left[\frac{x}{2} - \frac{\operatorname{sen} 2x}{4}\right]_0^{2\pi} = \pi$

$\operatorname{proj}_S (x-2) = \frac{\langle 1, x-2\rangle 1}{\|1\|^2} + \frac{\langle \operatorname{sen} x, x-2\rangle \operatorname{sen} x}{\|\operatorname{sen} x\|^2} +$
$+ \frac{\langle \cos x, x-2\rangle \cos x}{\|\cos x\|} = \pi - 2 - 2\operatorname{sen} x$

22) $f_1 = 1$
$f_2 = t - \frac{\langle 1, t\rangle 1}{2} = t - 1$

$\langle e^t, 1\rangle = \int_0^2 e^t dt = e^2 - 1$

$\langle e^t, t-1\rangle = \int_0^2 te^t - e^t dt = \left[te^t - e^t - e^t\right]_0^2 = 2$

$\|t-1\|^2 = \int_0^2 t^2 - 2t + 1\, dt = \left[\frac{t^3}{3} - t^2 + t\right]_0^2 = \frac{8}{3} - 4 + 2 = \frac{2}{3}$

$\operatorname{proj}_U e^t = \frac{e^2-1}{2} + 3t$

23) Considerando $Y = [1, x]$, devemos obter a melhor aproximação de $\cos x$ por uma função $ax+b$ no intervalo $[-\pi/2, \pi/2]$.

$\begin{cases} \cos x - (ax+b) \perp 1 \\ \cos x - (ax+b) \perp x \end{cases} \Leftrightarrow \begin{cases} \langle \cos x - (ax+b), 1\rangle = 0 \\ \langle \cos x - (ax+b), x\rangle = 0 \end{cases}$

$\langle \cos x - ax - b, 1\rangle = \int_{-\pi/2}^{\pi/2} \cos x - ax - b\, dx = \left[\operatorname{sen} x - \frac{ax^2}{2} - bx\right]_{-\pi/2}^{\pi/2}$

$= 1 - \frac{a\pi^2}{8} - \frac{b\pi}{2} + 1 + \frac{a\pi^2}{8} - \frac{b\pi}{2} = 0 \Leftrightarrow b\pi = 2 \Leftrightarrow$

$\Leftrightarrow b = \frac{2}{\pi}$

24) Considerando $Y = [1, x, x^2]$.

$\begin{cases} e^x - (ax^2+bx+c) \perp 1 \\ e^x - (ax^2+bx+c) \perp x \\ e^x - (ax^2+bx+c) \perp x^2 \end{cases} \Leftrightarrow \begin{cases} \langle e^x - ax^2 - bx - c, 1\rangle = 0 \\ \langle e^x - ax^2 - bx - c, x\rangle = 0 \\ \langle e^x - ax^2 - bx - c, x^2\rangle = 0 \end{cases}$

$\langle e^x - ax^2 - bx - c, 1\rangle = \int_0^1 e^x - ax^2 - bx - c\, dx =$

$= \left[e^x - \frac{ax^3}{3} - \frac{bx^2}{2} - cx\right]_0^1 = e - \frac{a}{3} - \frac{b}{2} - c - 0$

$\langle e^x - ax^2 - bx - c, x\rangle = \int_0^1 xe^x - ax^3 - bx^2 - cx\, dx =$

$= \left[xe^x - e^x - \frac{ax^4}{4} - \frac{bx^3}{3} - \frac{cx^2}{2}\right]_0^1 = e - e - \frac{a}{4} - \frac{b}{3} - \frac{c}{2} + 1 =$

$\Rightarrow$

$\langle e^x - ax^2 - bx - c, x^2\rangle = \int_0^1 x^2e^x - ax^4 - bx^3 - cx^2 dx = 0 \Rightarrow$

$\Rightarrow \left[(x^2 - 2x + 2)e^x - \frac{ax^5}{5} - \frac{bx^4}{4} - \frac{cx^3}{3}\right]_0^1 = 0 \Rightarrow$

$\Rightarrow e - \frac{a}{5} - \frac{b}{4} - \frac{c}{3} - 2 = 0$

$\begin{cases} \frac{a}{3} + \frac{b}{2} + c = e \\ \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = 1 \\ \frac{a}{5} + \frac{b}{4} + \frac{c}{3} = e - 2 \end{cases}$

25) a) $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \therefore B^t = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$

$B^t \cdot A = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{21}a_{21} & b_{11}a_{12} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{21} & b_{12}a_{12} + b_{22}a_{22} \end{pmatrix}$

$tr(B^t A) = b_{11}a_{11} + b_{21}a_{21} + b_{12}a_{12} + b_{22}a_{22} = \langle A, B\rangle$

26) $\langle 1, t\rangle = \int_{-\pi}^{\pi} t\,dt = [t^2]_{-\pi}^{\pi} = 0$

$\langle \text{sen}t, t\rangle = \int_{-\pi}^{\pi} t\,\text{sen}t\,dt = [\text{sen}t - t\cos t]_{-\pi}^{\pi} = 2\pi$

$\langle \cos t, t\rangle = \int_{-\pi}^{\pi} t\cos t\,dt = [t\,\text{sen}t + \cos t]_{-\pi}^{\pi} = 0$

$\|\text{sen}t\|^2 = \int_{-\pi}^{\pi} \text{sen}^2 t\,dt = \left[\frac{t}{2} - \frac{\text{sen}2t}{4}\right]_{-\pi}^{\pi} = \pi$

$\text{proj}_W t = \dfrac{\langle \text{sen}t, t\rangle \cdot \text{sen}t}{\|\text{sen}t\|^2} = 2\,\text{sen}t$

28) $x = y$

$-2y + z + 3w = 0 \Rightarrow z = 2y - 3w$

$(y, y, 2y - 3w, w) = y(1,1,2,0) + w(0,0,-3,1)$

$\langle (x', y', z', w'), (1,1,2,0)\rangle = 0 \Rightarrow x' + y' + 2z' = 0$

$\langle (x', y', z', w'), (0,0,-3,1)\rangle = 0 \Rightarrow -3z' + w' = 0$

$w' = 3z'$
$x' = -y' - 2z'$ $\therefore (-y' - 2z', y', z', 3z') =$

$= y'(-1,1,0,0) + z'(-2,0,1,3)$

29) a) $(x, y, z, w) = (2y - z - w, y, z, w) =$

$= y(2,1,0,0) + z(-1,0,1,0) + w(-1,0,0,1)$

$B = \{(2,1,0,0), (-1,0,1,0), (-1,0,0,1)\}$

Calculando uma base ortogonal temos:

$f_1 = (-1,0,0,1)$

$f_2 = (2,1,0,0) + (-1,0,0,1) = (1,1,0,1)$

$f_3 = (-1,0,1,0) - \frac{1}{2}(-1,0,0,1) + \frac{1}{3}(1,1,0,1) =$

$= (-1,0,1,0) + (1/2, 0, 0, -1/2) + (1/3, 1/3, 0, 1/3) =$

$= (-1/2, 0, 1, -1/2) + (1/3, 1/3, 0, 1/3) =$

$= (-1/6, 1/3, 1, 1/6)$

A base ortonormal será dada por

$B_{ort} = \left\{\frac{1}{\sqrt{2}}(-1,0,0,1); \frac{1}{\sqrt{3}}(1,1,0,1); \frac{\sqrt{18}}{5}(-1/6, 1/3, 1, 1/6)\right\}$

b) Seja $(a,b,c,d) \in S^{\perp}$. Temos:

$\langle (a,b,c,d), (2,1,0,0)\rangle = 0$

$\langle (a,b,c,d), (-1,0,1,0)\rangle = 0$

$\langle (a,b,c,d), (-1,0,0,1)\rangle = 0$

$\begin{cases} 2a + b = 0 \\ -a + c = 0 \\ -a + d = 0 \end{cases} \Rightarrow \begin{cases} b = -2a \\ c = a \\ d = a \end{cases} \Rightarrow a(1,-2,1,1)$

Assim: $v = (2,1,0,0) + (1,-2,1,1) = (3,-1,1,1)$

$v = (-1,0,1,0) + (1,-2,1,1) = (0,-2,2,0)$

$v = (-1,0,0,1) + (1,-2,1,1) = (0,-2,1,2)$

34) a) (1) $T(f+g) = f(a) + g(a) = T(f) + T(g)$

(2) $T(\lambda f) = \lambda \cdot f(a) = \lambda \cdot T(f)$

T é uma transformação linear

b) (1) $T(f+g) = f' + g' = T(f) + T(g)$

(2) $T(\lambda f) = \lambda \cdot f' = \lambda T(f)$

T é uma transformação linear

c) (1) $T(f+g) = a(f'' + g'') + b(f' + g') + c(f + g) =$

$= af'' + ag'' + bf' + bg' + cf + cg = T(f) + T(g)$

(2) $T(\lambda f) = \lambda a f'' + \lambda b f' + \lambda c f =$

$= \lambda(af'' + bf' + cf) = \lambda T(f)$

T é uma transformação linear

d) (1) $T(f+g)(x) = \int_a^x f(t) + g(t)\,dt =$

$= \int_a^x f(t)\,dt + \int_a^x g(t)\,dt = T(f)(x) + T(g)(x)$

(2) $T(\lambda f)(x) = \int_a^x \lambda f(t)\,dt = \lambda \int_a^x f(t)\,dt = \lambda T(f)(x)$

T é uma transformação linear

35) $p(x) = \alpha + \beta(x + x^2) + \gamma(x - x^2)$

$T(p(x)) = \alpha T(1) + \beta T(x) + \beta T(x^2) + \gamma T(x) - \gamma T(x^2)$

$= \alpha x^4 + \beta + \gamma x + \gamma x^3 = \alpha x^4 + \gamma x^3 + \gamma x + \beta$,

com $\alpha, \beta, \gamma \in \mathbb{R}$.

Vamos agora determinar $q(t) = a + bx + cx^2$

$\begin{cases} T(x) + T(x^2) = 1 \\ T(x) - T(x^2) = x + x^3 \end{cases} \Rightarrow T(x) = \frac{1}{2} + \frac{x}{2} + \frac{x^3}{2}$

$T(x^2) = 1 - T(x) = \frac{1}{2} - \frac{x}{2} - \frac{x^3}{2}$

$T(q(t)) = aT(1) + bT(x) + cT(x^2) =$

$= ax^4 + \frac{b}{2} + \frac{bx}{2} + \frac{bx^3}{2} + \frac{c}{2} - \frac{cx}{2} - \frac{cx^3}{2} =$

$= ax^4 + \frac{b+c}{2} + \frac{bx-cx}{2} + \frac{bx^3-cx^3}{2}$

36) $\begin{cases} T(v) - T(w) = 2v - w \\ 2T(w) - T(v) = v + w \end{cases} \Rightarrow T(w) = 3v$

$T(v) = 2v - w + T(w) = 2v - w + 3v = 5v - w$

$T(3v+w) = 3T(v) + T(w) = 15v - 3w + 3v = 18v - 3w$

39) (1) $T(M+N) = A(M+N) - (M+N) \cdot A =$
$= AM + AN - MA - NA = AM - MA + AN - NA = T(M) - T(N)$

(2) $T(\lambda M) = A\lambda M - \lambda MA = \lambda(AM - MA) = \lambda T(M)$.

T é uma transformação linear

O núcleo de T pode ser encontrado fazendo $T(M) = 0$.

$T(M) = 0 \Leftrightarrow AM - MA = 0 \Leftrightarrow AM = MA \therefore$

$\ker(T) = \{A, M \in M_n(\mathbb{R}) : AM = MA\}$

Pode se observar que a identidade não pertence à imagem de T.

43) a) Falsa.

$T(x+y) = (x+y)^2 = x^2 + 2xy + y^2 \neq T(x) + T(y)$

b) Falsa.

$T(x+y) = |(x+y)| \leq |x| + |y| \therefore$ pode ser que $T(x+y) \neq T(x) + T(y)$

c) Verdadeiro.

$T(a_0 + a_1x + \ldots + a_nx^n + b_0 + b_1x + \ldots + b_nx^n) = a_n + b_n =$
$= T(a_0 + a_1x + \ldots + a_nx^n) + T(b_0 + b_1x + \ldots + b_nx^n)$

$T(\lambda a_0 + \lambda a_1x + \ldots + \lambda a_nx^n) = \lambda a_n = \lambda T(a_0 + a_1x + \ldots + a_nx^n)$.

d) Verdadeira.

Sabemos que qualquer transformação matricial de $\mathbb{R}^p$ em $\mathbb{R}^k$ é linear.

e) Verdadeiro.

Como $\dim U > \dim V$ e $\dim \operatorname{Im} T = \dim W$ então a transformação é injetora.

f) Verdadeira.

Basta verificar a definição de imagem para a transformação linear T.

g) Falsa.

A condição $\dim V \leq \dim W$ não garante que a função é injetiva. Basta tomar a transformação linear

$T: P_n(\mathbb{R}) \longrightarrow P_{n+1}(\mathbb{R})$
$p \longmapsto p'$

h) Verdadeiro.

Caso contrário teríamos elementos distintos no domínio com a mesma imagem.

44) Podemos ter: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x,y) \longmapsto (x-y, 2x-2y)$

45) Podemos ter: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x,y) \longmapsto (y, 0)$

48) a) Verdadeiro.

Basta considerar à seguinte transformação:

$T: P_3(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$
$(a + bx + cx^2 + dx^3) \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

b) Falso.

$\operatorname{Im} T \subset P_{n-1}(\mathbb{R})$ e, portanto, $\dim \operatorname{Im} T < \dim P_n(\mathbb{R})$

c) Verdadeiro.

A existência de uma transformação linear injetora é possível, pois, $\dim \mathbb{R}^3 < \dim M_2(\mathbb{R})$

d) Verdadeiro.

Como o domínio e o contradomínio tem a mesma dimensão então T só será sobrejetora se, e somente se, T for também injetora que, pela definição, decorre do fato de $\ker T = \{0\}$.

e) Verdadeira.

Seja Y um subespaço vetorial de $\mathbb{R}^3$ e $T: \operatorname{proj}_Y v, \forall v \in \mathbb{R}^3$. Então teremos que $\operatorname{Img} T = Y$ e $\ker T = Y^\perp$. Sabemos que $\mathbb{R}^3 = Y + Y^\perp \Rightarrow \mathbb{R}^3 = \operatorname{Im} T + \ker T$.

49) Podemos definir $T: V \longrightarrow W$
$v \longmapsto v$
Assim podemos observar que cada elemento da imagem tem um correspondente único no domínio e, portanto, T é injetora.

Para que seja um produto interno devemos ter que:

$\langle v, v \rangle = \langle T(v), T(v) \rangle = 0 \Leftrightarrow T(v) = 0 \Leftrightarrow$

$\Leftrightarrow T$ é injetora

46) Podemos ter: $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$(x, y, z, w) \longrightarrow (z, w, 0, 0)$

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Q2) $T(f) = 0 \Rightarrow f + f' = 0 \Rightarrow f = -f' \Leftrightarrow f = 0$

$\dim \ker T = 0$ e T é injetora (e)

Q3) (I) Verdadeiro.

Sabemos que $E = Y + Y^{\perp}$ e que $Y \cap Y^{\perp} = \{0\}$, portanto, y e z são únicos

(II) Verdadeiro.

Como E tem dimensão finita, sabemos que pelo Teorema da dimensão $\dim E = \dim \ker T + \dim \operatorname{Im} T$. Logo, pode ocorrer que $\ker T = Y^{\perp}$ e $\operatorname{Im} T = Y$.

(III) Falso.

Pelo Teorema da Dimensão, se $\ker T = \operatorname{Im} T = Y$ então $\dim T = \dim \operatorname{Im} T = \dim Y$ e, portanto, $\dim E = \dim \ker T + \dim \operatorname{Im} T \Rightarrow \dim E = 2$. (e)

Q4) Base de $\ker T = \left\{ \overbrace{\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}^{e_1}, \overbrace{\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}}^{e_2} \right\}$

Base de $\operatorname{Im} T = \left\{ \overbrace{\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}}^{f_1}, \overbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}^{f_2} \right\}$

$\langle e_1, f_1 \rangle = 4 - 2 - 2 + 0 = 0$

$\langle e_1, f_2 \rangle = 1 - 4 + 3 = 0$

$\langle e_2, f_1 \rangle = -4 + 4 = 0 \quad \therefore \ker T = (\operatorname{Im} T)^{\perp}$

$\langle e_2, f_2 \rangle = -1 + 1 = 0$ (a)

Q5) (I) Verdadeiro.

O vetor nulo é o único vetor ortogonal a todos os elementos da base ortogonal.

(II) Falso.

Podemos ter $E = Y \oplus Z$ com Y e Z não sendo ortogonais.

(III) Falso. (d)

Vem como consequência do item (II).

Q6) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ e $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \quad A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

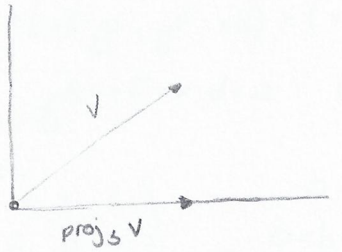
$(A+B)^t = \begin{bmatrix} a+e & c+g \\ b+f & d+h \end{bmatrix} \quad B^t = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$

$T(A+B) = \begin{bmatrix} (a+e)^2 + (b+f)^2 & (a+e)(c+g) + (b+f)(d+h) \\ (a+e)(c+g) + (b+f)(d+h) & (b+f)^2 + (d+h)^2 \end{bmatrix}$

$T(A) = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{bmatrix}$

$T(B) = \begin{bmatrix} e^2 + f^2 & eg + fh \\ ge + hf & g^2 + h^2 \end{bmatrix}$ (a)

Q7)



$\operatorname{Im} T = S$

$\ker T = S^{\perp}$ (e)

Q8) (I) Verdadeiro.

Pela Desigualdade de Cauchy-Schwarz:

$-\|x\| \cdot \|y\| \leq \langle x, y \rangle \leq \|x\| \cdot \|y\|$

(II) Verdadeiro.

Trata-se da Desigualdade de Cauchy-Schwarz

(III) Verdadeiro.

Basta considerar $x = \lambda y$. Então $\langle \lambda y, y \rangle = \lambda \langle y, y \rangle \neq 0$, $\lambda \neq 0 \in \mathbb{R}$ (b)

Q9) $\begin{cases} x = 0 \\ 2x + y - z = 0 \\ -x = 0 \\ x + y + z = 0 \end{cases} \Rightarrow 2y = 0 \Rightarrow y = 0 \quad \therefore z = 0$

$T \begin{cases} B = \{(1,0,0), (0,1,0), (0,0,1)\} \\ \operatorname{Im} T = [(1,2,-1,1), (0,1,0,1), (0,-1,0,1)] \end{cases}$ (c)

Q10) $\langle \operatorname{sen} t - at - b, 1 \rangle = \int_{-\pi}^{\pi} \operatorname{sen} t - at - b \, dt =$

$= \left[-\cos t - \frac{at^2}{2} - bt \right]_{-\pi}^{\pi} = +1 - \frac{a\pi^2}{2} - bt - 1 + \frac{a\pi^2}{2}$

$- bt = -2bt = 0 \Rightarrow b = 0$

$\langle \operatorname{sen} t - at, t \rangle = \int_{-\pi}^{\pi} (\operatorname{sen} t - at) t \, dt =$

$= \int_{-\pi}^{\pi} t \operatorname{sen} t - at^2 \, dt = \left[-t\cos t + \operatorname{sen} t - \frac{at^3}{3} \right]_{-\pi}^{\pi} =$

$= \pi - \frac{a\pi^3}{3} + \pi - \frac{a\pi^3}{3} = 0 \Rightarrow \frac{2a\pi^3}{3} = 2\pi \Rightarrow$ (b)

$\Rightarrow a = 3/\pi^2 \therefore 3t/\pi^2$

Q11) $(x,y) = \alpha(2,3) + \beta(1,1)$ $\begin{cases} 2\alpha + \beta = x \\ 3\alpha + \beta = y \end{cases}$

$\alpha = y - x \therefore \beta = 3x - 2y$

$T(x,y) = T(\alpha(2,3) + \beta(1,1)) = \alpha T(2,3) + \beta T(1,1) =$

$= (y-x)(4,6) + (3x-2y)(2,-1) =$

$= (4y-4x, 6y-6x) + (6x-4y, 2y-3x) =$

$= (2x, 8y-9x) \therefore a+b = -7x+8y$ (e)

Q12) $\begin{cases} 2a+5b-c=0 \\ -a-3b+c+d=0 \\ a+2c+5d=0 \\ 3a+7b-c+d=0 \end{cases} \sim \begin{pmatrix} -1 & -3 & 1 & 1 \\ 1 & 0 & 2 & 5 \\ 2 & 5 & -1 & 0 \\ 3 & 7 & -1 & 1 \end{pmatrix} \sim$

$\sim \begin{pmatrix} -1 & -3 & 1 & 1 \\ 0 & -3 & 3 & 6 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & -3 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4 incógnitas e 2 eqs $\therefore$ 2 variáveis livres $\therefore \dim \ker T = 2$

Pelo Teorema das Dimensões:

$\dim P_3(\mathbb{R}) = \dim \ker T + \dim \operatorname{Im} T \Rightarrow 4 = 2 + \dim \operatorname{Im} T \Rightarrow$

$\Rightarrow \dim \operatorname{Im} T = 2$ (a)

Q13) $\langle(\alpha_1,\alpha_2),(\beta_1,\beta_2)\rangle = \det\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1$

$\langle(\beta_1,\beta_2),(\alpha_1,\alpha_2)\rangle = \det\begin{vmatrix} \beta_1 & \beta_2 \\ \alpha_1 & \alpha_2 \end{vmatrix} = \alpha_2\beta_1 - \alpha_1\beta_2$

$\langle(\alpha_1,\alpha_2),(\beta_1,\beta_2)\rangle \neq \langle(\beta_1,\beta_2),(\alpha_1,\alpha_2)\rangle$

$\langle(\alpha_1,\alpha_2),(\alpha_1,\alpha_2)\rangle = \det\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{vmatrix} = \alpha_1\alpha_2 - \alpha_1\alpha_2 = 0$

com $(\alpha_1,\alpha_2) \neq (0,0)$ (b)

14) $\begin{cases} a + bx + cx^2 = p(x) \\ \cancel{a} + b + \cancel{c} = \cancel{a} - b + \cancel{c} \end{cases} \Rightarrow p(x) = a + cx^2$

Uma base para V seria $\{1, x^2\}$

$f_1 = 1 \therefore \|f_1\| = \sqrt{\int_0^1 dx} = \sqrt{1} = 1$

$\langle 1, x^2\rangle = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$f_2 = x^2 - \frac{1}{3} \therefore \|f_2\| = \sqrt{\int_0^1 x^4 - \frac{2}{3}x^2 + \frac{1}{9}\,dx} =$

$= \sqrt{\left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]_0^1} = \sqrt{\frac{1}{5} - \frac{2}{9} + \frac{1}{9}} =$

$= \sqrt{\frac{1}{5} - \frac{1}{9}} = \sqrt{\frac{9-5}{45}} = \sqrt{\frac{4}{45}}$

Base = $\left\{1, \frac{2}{\sqrt{45}}\left(x^2 - \frac{1}{3}\right)\right\}$ (a)

Q15) (I) Falso

Para operadores lineares: injetor $\Leftrightarrow$ sobrejetor $\Leftrightarrow$ bijetor

(II) Falso.

$\dim P_4(\mathbb{R}) = 5$

$\dim M_{2\times3}(\mathbb{R}) = 6$ $\therefore$ Não é possível ser sobrejetor

(e)

(III) Falso.

Q16) Base de $P_3(\mathbb{R}) = \{1, t, t^2, t^3\}$

Base de $V = \{1+t^2, t+t^2\}$

Base de $W = \{t, t^3\}$

Na base de $P_3(\mathbb{R})$:

$V: \{(1,0,1,0), (0,1,1,0)\}$

$W = \{(0,1,0,0), (0,0,0,1)\}$

$V+W = \{(1,0,1,0), (0,1,1,0), (0,1,0,0), (0,0,0,1)\} \Rightarrow$ l.i.

$\dim(V+W) = 4$

$\dim(V+W) = \dim V + \dim W - \dim(V \cap W)$

$4 = 4 - \dim(V\cap W) \Rightarrow \dim(V\cap W) = 0$ (b)

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Q1) $\langle 1-t, \alpha-\beta t+t^2\rangle = \alpha+0-\alpha+2\beta-4=0 \Rightarrow \beta=2$ ⓒ

$\langle 1-t^2, \alpha-\beta t+t^2\rangle = \alpha+0-3\alpha+6\beta-12 \Rightarrow \alpha=0$

Q2) $\begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ a & b & 0 & 0\\ 0 & 0 & c & d\end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ (a-1) & (b-1) & 0 & 0\\ 0 & 0 & (c-1) & (d+1)\end{pmatrix}$

$\begin{cases} x+y=0\\ z+w=0\\ (a-1)x+(b-1)y=0\\ (c-1)z+(d+1)w=0\end{cases}$ $\dim \ker T = 0 \Leftrightarrow a\neq b$ e $c\neq -d$ ⓑ

Q4) i) $\|u+v\|^2 = \langle u+v,u+v\rangle = \langle u,u\rangle+2\langle u,v\rangle+\langle v,v\rangle =$
$= \|u\|^2+2\|u\|\cdot\|v\|+\|v\|^2 \Rightarrow \|u+v\| = \|u\|+\|v\|$
$\|u\|\cdot\|v\| = \sqrt{\langle u,u\rangle\cdot\langle v,v\rangle} \neq \|u+v\| = \|u\|+\|v\|$

ii) $\|u+v\| = \sqrt{\langle u,u\rangle+2\langle u,v\rangle+\langle v,v\rangle}$ ∴ ⓒ
$\|u-v\| = \sqrt{\langle u,v\rangle-2\langle u,v\rangle-\langle v,v\rangle}$
$\|u+v\| = \|u-v\| \Leftrightarrow \langle u,v\rangle=0$

iii) $\langle u,v\rangle = \langle u,\lambda u\rangle = \lambda\langle u,u\rangle = \lambda\cdot\|u\|^2$

Q5) $\dim \mathbb{R}^4 = \dim S+\dim S^\perp \Rightarrow \dim S = 3$

$\begin{pmatrix} 1 & a & 0 & 1\\ 0 & 1 & 1 & 0\\ 1 & 0 & b & 1\end{pmatrix} \sim \begin{pmatrix} 1 & a & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & -a & b & 0\end{pmatrix}$ ⓒ

$a\neq -1$ e $b\neq 1$ ∴ $a+b\neq 0$

Q6) i) $T(u+v) = \langle u+v,u+v\rangle =$
$= \langle u,u\rangle+2\langle u,v\rangle+\langle v,v\rangle \neq T(u)+T(v)$

ii) $T(u+v) = \langle u+v,z\rangle = \langle u,z\rangle+\langle v,z\rangle =$
$= T(u)+T(v)$ ⓓ
$T(\lambda u) = \langle \lambda u,z\rangle = \lambda\langle u,z\rangle = \lambda T(u)$

Q7) Advém do caso particular da fórmula geral da projeção ⓒ

Q8) $\begin{pmatrix} 1 & a & 0\\ 0 & 1 & 0\\ 1 & 0 & -a\end{pmatrix} \sim \begin{pmatrix} 1 & a & 0\\ 0 & 1 & 0\\ 0 & -a & -a\end{pmatrix}$ ∴ $a\neq 0$, $a\neq 1$ ⓓ

Q9) $T(-1+t+t^2) = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$ ∴ $\ker T = [(-1+t+t^2)]$

$\mathrm{Im}\, T = \left[\begin{pmatrix} 2 & 0\\ 0 & 4\end{pmatrix}, \begin{pmatrix} 1 & 1\\ -1 & 2\end{pmatrix}\right]$ ⓓ

Q10) i) $7 = 2\dim \ker T \Rightarrow \dim\ker T = \frac{7}{2}$

ii) $8 = 2\dim\ker T \Rightarrow \dim\ker T = 4$ ⓔ

iii) $\dim M_{2\times 3}(\mathbb{R}) = 6 > \dim P_4(\mathbb{R}) = 5$

Q11) $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ ∴ $A^t = \begin{pmatrix} a & c\\ b & d\end{pmatrix}$

$\begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix} a & c\\ b & d\end{pmatrix}$ ∴ $tr = a+d$

$\begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix} -c & a\\ -d & b\end{pmatrix}$ ∴ $tr = b-c$

$S = [a_1, a_2]$

$\|a_1\|^2 = tr\left[\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right] = 2$

$\|a_2\|^2 = tr\left[\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\right] = 2$ ⓒ

$\frac{(a+d)}{2}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} + \frac{(b-c)}{2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1\\ 1 & 1\end{pmatrix}$

$\frac{a+d}{2} = \frac{1}{2} \Rightarrow a+d=1 \Rightarrow a+d-1=0$

$\frac{b-c}{2} = -\frac{1}{2} \Rightarrow b-c=-1 \Rightarrow b-c+1=0$

Q12) $\begin{cases} a+b=0\\ b+c=0\\ b+c=0\\ a+d=0\end{cases} \Rightarrow \begin{cases} a+b=0 \Rightarrow a=-b\\ b+c=0 \Rightarrow c=-b\\ a+d=0 \Rightarrow d=b\end{cases}$

$\dim(\ker T)=1$ ∴ $\dim \mathrm{Im}\, T = 3$ ⓒ

Q13) i) $\mathrm{Im}\, T = S$ e $\ker T = S^\perp$ ∴ $V = S+S^\perp \Rightarrow$
$\Rightarrow V = \mathrm{Im}\, T+\ker T$

ii) $\ker T\cap \mathrm{Im}\, T = \{0_V\}$ ⓑ

iii) $V = \ker T+\mathrm{Im}\, T$

Q15) $(a,b,-a-b,d,e,d+e) =$
$= a\underbrace{(1,0,-1,0,0,0)}_{u_1} + b\underbrace{(0,1,-1,0,0,0)}_{u_2} +$
$+ d\underbrace{(0,0,0,1,0,1)}_{u_3} + e\underbrace{(0,0,0,0,1,1)}_{u_4}$

$\langle u_1,(a,b,c,d,e,f)\rangle = 0 \Rightarrow a-c=0 \Rightarrow a=c$ ⓒ
$\Rightarrow b-c=0 \Rightarrow b=c$
$\Rightarrow d+f=0 \Rightarrow d=-f$
$\Rightarrow e+f=0 \Rightarrow e=-f$

$(c,c,c,-f,-f,f)$

$\langle t^2 - a - bt, 1\rangle = 0 \Rightarrow 1 - a + \cancel{b} - a + 1 - a - \cancel{b} = 0 \Rightarrow$

$\Rightarrow -3a + 2 = 0 \Rightarrow a = \frac{2}{3}$

$\langle t^2 - a - bt, t\rangle = 0 \Rightarrow -1 + \cancel{a} - b + 0 + 1 - \cancel{a} - b = 0 \Rightarrow$

$\Rightarrow -2b = 0 \Rightarrow b = 0$ (a)

P_2 de 2014:

Q1) i) $\dim U = 200 \therefore \begin{cases} \dim \operatorname{Im} T + \dim \ker T = 200 \\ 20 \dim \ker T + 30 \dim \operatorname{Im} T = 3500 \end{cases}$

$\dim \ker T = 200 - \dim \operatorname{Im} T \Rightarrow 4000 - 20 \dim \operatorname{Im} T + 30 \dim \operatorname{Im} T =$
$= 3500 \Rightarrow 10 \dim \operatorname{Im} T = -500$ (e)

ii) $\dim \ker T = 200 - \dim \operatorname{Im} T \Rightarrow \dim \ker T = 50$

iii) $\dim \operatorname{Im} T = 200 - \dim \ker T \Rightarrow \dim \operatorname{Im} T = 15$

Q2) $v = \lambda w \therefore \langle v, w\rangle = \langle \lambda w, w\rangle = \lambda \langle w, w\rangle \Rightarrow$

$\Rightarrow \langle v, w\rangle = \lambda \cdot \|w\| \cdot \|w\| \Rightarrow \langle v, w\rangle = \cancel{\lambda} \cdot \|w\| \cdot \frac{\|v\|}{\cancel{|\lambda|}} \Rightarrow$

$\Rightarrow \langle v, w\rangle = \|w\| \cdot \|v\| \Rightarrow |\langle v, w\rangle| = \|w\| \cdot \|v\|$

Q3) $\|x\| = \|y\| \Rightarrow \|x\|^2 = \|y\|^2 \Rightarrow \langle x, x\rangle = \langle y, y\rangle$

$\langle x - y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle - \langle x, y\rangle - \langle y, y\rangle = 0$ (a)

Q4) i) $\|v - 5w\|^2 = \langle v - 5w, v - 5w\rangle =$

$= \langle v, v\rangle - \langle v, 5w\rangle - \langle v, 5w\rangle + \langle 5w, 5w\rangle =$

$= \|v\|^2 - 5\langle v, w\rangle - 5\langle v, w\rangle + 25\|w\|^2 =$

$= \|v\|^2 - 10\langle v, w\rangle + 5\|w\|^2 = 4 + 20 + 5 = 49 \therefore$

$\therefore \|v - 5w\| = \sqrt{49} = 7$

ii) Se os vetores são ortogonais entre si então devem ser li e, portanto formam uma base de U.

iii) Suponha $n = 3$: (d)

$\|u_1 + u_2 + u_3\| = \sqrt{\langle u_1 + u_2 + u_3, u_1 + u_2 + u_3\rangle} =$

$= \sqrt{\langle u_1, u_1\rangle + \underbrace{\cancel{\langle u_1, u_2\rangle}}_{0} + \underbrace{\cancel{\langle u_1, u_3\rangle}}_{0} + \underbrace{\cancel{\langle u_2, u_1\rangle}}_{0} + \langle u_2, u_2\rangle + \underbrace{\cancel{\langle u_2, u_3\rangle}}_{0} + \underbrace{\cancel{\langle u_3, u_1\rangle}}_{0} + \underbrace{\cancel{\langle u_3, u_2\rangle}}_{0} + \langle u_3, u_3\rangle} = \sqrt{3}$

Q5) O número de icognitas do sistema é o mesmo que o número de equações (c)

Q6) $T(v) = 0 \Leftrightarrow v = 0 \therefore$ T é injetora (b)

$\dim \operatorname{Im} T = \dim \ker T \therefore$ T é bijetora

Q7) i) Isso se dá por consequência de que $V = U + U^{\perp}$

ii) Se, e somente se, a base for ortogonal

iii) Não necessariamente $V = U + U^{\perp}$. (d)

Q8) Sendo $B = \{1, 1+x, 1+x+x^2\}$ base de $P_2(\mathbb{R})$:

$p(x) = \alpha(1) + \beta(1+x) + \gamma(1+x+x^2)$

$T(p(x)) = \alpha T(1) + \beta T(1+x) + \gamma T(1+x+x^2)$

Para que a transformação seja injetora devemos ter $T(p(x)) = 0$ para α, β, γ únicos:

$\alpha(1,2,1) + \beta(1,a,b) + \gamma(1,1,2) = 0$

$\begin{cases} \alpha + \beta + \gamma = 0 \\ 2\alpha + a\beta + \gamma = 0 \\ \alpha + b\beta + 2\gamma = 0 \end{cases} \therefore$ para que o sistema

mais de uma solução devemos ter:

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & a & 1 \\ 1 & b & 2 \end{vmatrix} = 0 \Rightarrow 2a + 2b + 1 - a - 4 - b = 0 \Rightarrow$ (c)

$\Rightarrow a + b = 3$

Q9) $-\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{cases} -a = a \Rightarrow 2a = 0 \Rightarrow a = 0 \\ \left.\begin{matrix} -b = c \\ -c = b \end{matrix}\right\} \Rightarrow b = -c \\ -d = d \Rightarrow 2d = 0 \Rightarrow d = 0 \end{cases}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} = c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\dim W = 1 \Rightarrow \dim M_2(\mathbb{R}) = 4 \Rightarrow$ (c)

$\Rightarrow \dim W^{\perp} = 3$

Q10) $\begin{cases} x = y \\ z = 2x - 3t = 2y - 3t \end{cases}$

$(x, y, z, t) = (y, y, 2y - 3t, t) =$

$= y(1,1,2,0) + t(0,0,-3,1) \therefore \dim U^{\perp} = 2$

$\langle (1,1,2,0), (x,y,z,t)\rangle = 0$

$\langle (0,0,-3,1), (x,y,z,t)\rangle = 0$

$\begin{cases} x + y + 2z = 0 \Rightarrow x = -y - 2z \\ -3z + t = 0 \Rightarrow t = 3z \end{cases}$

$(x,y,z,t) = (-y-2z, y, z, 3z) =$

$= y(-1,1,0,0) + z(-2,0,1,3)$ (c)

Q11) A base U não é ortogonal.

$\langle e^t - a - bt, 1\rangle = 0 \Rightarrow$

$\Rightarrow \int_0^2 e^t - a - bt\,dt = 0 \Rightarrow \left[e^t - at - \frac{bt^2}{2}\right]_0^2 = 0 \Rightarrow$

$\Rightarrow e^2 - 2a - \frac{4b}{2} - 1 = 0 \Rightarrow \underbrace{e^2 - 2a - 2b = 1}_{2a = e^2 - 2b - 1}$

$\langle e^t - a - bt, t\rangle = 0 \Rightarrow$

$\Rightarrow \int_0^2 te^t - at - bt^2\,dt = 0 \Rightarrow \left[e^t(t-1) - \frac{at^2}{2} - \frac{bt^3}{3}\right]_0^2 = 0 \Rightarrow$

$\Rightarrow e^2 - 2a - \frac{8b}{3} + 1 = 0$

$e^2 - e^2 + 2b + 1 - \frac{8b}{3} + 1 = 0 \Rightarrow \frac{2b}{3} = 2 \Rightarrow b = 3$

$\therefore a = \frac{e^2 - 7}{2} \quad \therefore a + bt = \frac{1}{2}(e^2 - 7) + 3t$ (a)

Q13) $f_1 = x - 1$

$f_2 = x - 2 - \frac{\langle x-1, x-2\rangle}{\|x-1\|^2}(x-1)$

$\langle x-1, x-2\rangle = 6 + 2 + 0 + 0 = 8$

$\|x-1\|^2 = \langle x-1, x-1\rangle = 4 + 1 + 1 = 6$ (b)

$f_2 = x - 2 - \frac{\cancel{8}^4}{\cancel{6}_3}(x-1) = x - 2 - \frac{4x}{3} + \frac{4}{3} = -\frac{x}{3} - \frac{2}{3}$

Q15) $\{1, \text{sen}\,t, \cos t\}$ é ortogonal

$\langle t, 1\rangle = \int_{-\pi}^{\pi} t\,dt = \left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0$

$\langle t, \text{sen}\,t\rangle = \int_{-\pi}^{\pi} t\,\text{sen}\,t\,dt \left[-t\cos t + \text{sen}\,t\right]_{-\pi}^{\pi} =$

$= \pi + \pi = 2\pi$

$\langle t, \cos t\rangle = \int_{-\pi}^{\pi} t\cos t\,dt = \left[t\,\text{sen}\,t + \cos t\right] = 0$

$\|\text{sen}\,t\|^2 = \int_{-\pi}^{\pi} \text{sen}^2 t\,dt = \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos 2t\,dt =$

$= \frac{1}{2}\left[t - 2\,\text{sen}\,t\right]_{-\pi}^{\pi} = \left[\frac{t}{2} - \text{sen}\,t\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$\text{proj}_W t = \frac{2\cancel{\pi}}{\cancel{\pi}} \cdot \text{sen}\,t = 2\,\text{sen}\,t$ (a)

Q16) $\dim U = 2 \therefore \dim U^\perp = 4$ (a)

P_1 de 2015:

Q1) i) $h_1 = v - w_1 \Rightarrow v = h_1 + w_1$

$h_2 = v - w_2 \Rightarrow v = h_2 + w_2 \Rightarrow h_1 = h_2$ e $w_1 = w_2$

ii) Caso contrário $v \in V^\perp$ (b)

iii) Se, e somente se, $\{e_1, \dots, e_n\}$ for uma base ortogonal.

Q2) $\text{Im}\,T = W$

$\text{proj}_W v = 0 \Leftrightarrow \langle v, w_n\rangle = 0, \forall v \neq 0$ e $\forall w_n \neq 0 \in W$

$\Leftrightarrow v \perp W \Leftrightarrow v \in W^\perp \therefore \ker T = W^\perp$

$\ker T^\perp = (W^\perp)^\perp = W = \text{Im}\,T$, $\text{Im}\,T^\perp = W^\perp = \ker T$

Como V tem dimensão finita então $V = W + W^\perp$

$V = \ker T + \ker T^\perp = \text{Im}\,T + \text{Im}\,T^\perp$ (e)

Q3) $\langle (x,y), (1,-1)\rangle = 2x + x - y - y = 0 \Rightarrow$

$\Rightarrow 3x - 2y = 0 \Rightarrow x = \frac{2y}{3}$

$(x,y) = \left(\frac{2y}{3}, y\right) = y\left(\frac{2}{3}, 1\right)$ (a)

$S^\perp = [(2,3)]$

Q4) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{cases} a = a \\ b = c \\ b = c \\ d = d \end{cases}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\dim S = 3 \therefore$ como $M_2(\mathbb{R})$ tem dimensão finita igual a 4 então $\dim S^\perp = 1$ (c)

Q6) $v = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4$

$T(v) = \alpha T(e_1) + \beta T(e_2) + \gamma T(e_3) + \delta T(e_4) \Rightarrow$

$\Rightarrow T(v) = \alpha + \beta a x + \beta x^2 + \gamma + \gamma_x + \gamma_x^2 + \delta b x^2 \Rightarrow$

$\Rightarrow T(v) = (\alpha + \gamma) + x(a\beta + \gamma) + x^2(\beta + \gamma + b\delta)x^2$

$\begin{cases} \alpha + \gamma = 0 \\ a\beta + \gamma = 0 \\ \beta + \gamma + b\delta = 0 \end{cases}$ ∴ para ser injetora $\ker T = 0$

ⓓ

$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & a & 1 & 0 \\ 0 & 1 & 1 & b \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & b \\ 0 & 0 & 1-a & -ab \end{pmatrix}$

$a = 1$ e $b = 0$ para que T seja injetora

$a \neq 1$ e $b \neq 0$ para que T seja sobrejetora

Q7) $\ker A = [1, x]$ ∴ $\dim \ker A = 2$ ⓐ

$\dim \operatorname{Im} P_0$ é necessariamente 1 ∴ $\dim \ker T = 7$

Q8) i) $\langle (a_1, b_1), (a_1, b_1) \rangle = 0 \Rightarrow -a_1^2 + b_1^2 = 0 \Rightarrow$

$\Rightarrow b_1 = \pm a_1$

ii) $\langle (a_1, b_1), (a_1, b_1) \rangle \geq 0 \Rightarrow 3a_1^2 - 2a_1b_1 - 2a_1b_1 - 2b_1^2 = 0$

$\Rightarrow 3a_1^2 - 4a_1b_1 - 2b_1^2 \geq 0$ ⓓ

Para $\langle (1,3), (1,3) \rangle = 3 - 12 - 18 = -27$

iii) $\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 a_2 + 2a_1 b_2 - a_2 b_1 + b_1 b_2$

$\neq$

$\langle (a_2, b_2), (a_1, b_1) \rangle = a_2 a_1 + 2a_2 b_1 - a_1 b_2 + b_2 b_1$

Q9) i) $33 = \dim \operatorname{Im} T + \dim \ker T \Rightarrow \dim \ker T = 33 - \dim \operatorname{Im} T$

$10 \dim \ker T - 5 \dim \operatorname{Im} T = -15 \Rightarrow$

$\Rightarrow 330 - 10 \dim \operatorname{Im} T - 5 \dim \operatorname{Im} T = -15 \Rightarrow$

$\Rightarrow \dim \operatorname{Im} T = \dfrac{330 + 15}{15} \Rightarrow \dim \operatorname{Im} T = 23$ ∴

$\dim \ker T = 10$ ⓓ

ii) $33 = 2 \dim \operatorname{Im} T$ ∴ $\dim \operatorname{Im} T \notin \mathbb{N}$

iii) V tem dimensão finita então $V = U + U^\perp \Rightarrow$

$\dim V = \dim U + \dim U^\perp \Rightarrow \dim V = \dim \operatorname{Im} T + \dim \ker T$

então pode ser que $V = \ker T + \operatorname{Im} T$

Q10) $T(1,0,0) = T(2(1,1,1) - T(0,1,2) - T(1,1,0)) =$

$= 2T(1,1,1) - T(0,1,2) - T(1,1,0) = (1, -2, -1)$

$T(0,1,0) = T(1,1,0) - T(1,0,0) =$

$= (0, 3, 1)$ ⓐ

Q11) i) $\|v\| = \sqrt{\langle v, v \rangle}$

$\|w\| = \|\lambda v\| = \sqrt{\lambda \langle v, v \rangle}$

$\langle v, w \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2 = K \|v\| \dfrac{\|w\|}{|\lambda|}$

$|\langle v, w \rangle| = \|v\| \cdot \|w\|$ $\lambda > 0$ ou $\lambda < 0$

ii) $v = \lambda w$

$\|v + w\| = \sqrt{\langle v + w, v + w \rangle} =$

$= \sqrt{\langle v, v \rangle^2 + 2\langle v, w \rangle + \langle w, w \rangle^2} =$

$= \sqrt{\lambda^4 \langle w, w \rangle + 2\lambda \langle w, w \rangle + \langle w, w \rangle} \neq \|v\| + \|w\|$

iii) Considerando o produto interno canonico

$a = \left(\dfrac{1}{\sqrt{a_1}}, \dfrac{1}{\sqrt{a_2}}, \dfrac{1}{\sqrt{a_3}}, \dfrac{1}{\sqrt{a_4}} \right)$

$b = \left(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4} \right)$ ⓓ

$\langle a, b \rangle = 1 + 1 + 1 + 1 = 4$

$\|a\| = \sqrt{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4}}$ $\|b\| = \sqrt{a_1 + a_2 + a_3 + a_4}$

$\langle a, b \rangle \leq \|a\| \, \|b\| \Rightarrow \langle a, b \rangle^2 \leq \|a\|^2 \, \|b\|^2$

$\Rightarrow 16 \leq \left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4} \right)(a_1 + a_2 + a_3 + a_4)$

Q15) $\begin{cases} a + b + c = 0 \\ a - b + 5c = 0 \\ -a - 2b + c = 0 \end{cases}$ ∴ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ -1 & -2 & 1 \end{pmatrix} \rightsquigarrow$

$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $b = 2c$, $a - 2c + c$, $-3c$

$\dim \ker T = 1$ ∴ $\dim \operatorname{Im} T = 2$ $-3c + 2cx + c$

ⓑ

Base de $P_2 = \{1, x, x^2\}$ $c(-3 + 2x + x \ldots)$

$T(1) = (1, 1, -1)$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ -1 & -2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{pmatrix}$

$T(x) = (1, -1, -2)$

$T(x^2) = (1, 5, 1)$ $\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$