GABARITO Sub CIII

MAT2455 - Cálculo III - POLI
Prova Substitutiva - 03/07/2007 2007

Turma B

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Turma

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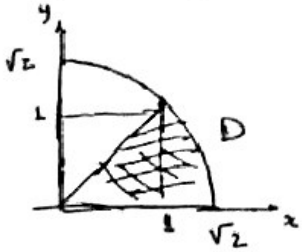
1	
2	
3	
Total	

1 (a) (1,5 pontos) Calcule $\int_0^1 \int_y^{\sqrt{2-y^2}} y e^{\sqrt{x^2+y^2}} dx\, dy$

(b) (2,0 pontos) Calcule $\iint_S |y| dS$, onde S é a porção do cone $z = \sqrt{x^2+y^2}$ compreendida entre os planos $z = 1$ e $3z = x + 7$.

$$\int_0^1 \int_y^{\sqrt{2-y^2}} y e^{\sqrt{x^2+y^2}} dx\, dy = \iint_D y e^{\sqrt{x^2+y^2}} dx\, dy$$



com a mudança: $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$, JACOBIANO $= r$, $0 \le \theta \le \frac{\pi}{4}$, $0 \le r \le \sqrt{2}$

fica $= \int_0^{\pi/4} \int_0^{\sqrt{2}} r\,\text{sen}\,\theta\, e^r\, r\, dr\, d\theta = \left(\int_0^{\pi/4} \text{sen}\,\theta\, d\theta\right) \underbrace{\left(\int_0^{\sqrt{2}} r^2 e^r dr\right)}_{\text{(POR PARTES, 2 VEZES} \to)} =$

$= \left[-\cos\theta\right]_0^{\pi/4} \left[r^2 e^r - 2re^r + 2e^r\right]_0^{\sqrt{2}} = \left(1 - \frac{\sqrt{2}}{2}\right)\left(4e^{\sqrt{2}} - 2\sqrt{2}\,e^{\sqrt{2}} - 2\right)$

$= \dots = (6 - 4\sqrt{2})e^{\sqrt{2}} - (2 - \sqrt{2})$

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Como F conservativo, tomo ϕ potencial com $c=0$

Assim $\int_\gamma \vec{F}\,dr = \phi(\gamma f) - \phi(\gamma i) = \phi(-5,-50) - \phi(-1,30) =$

$= -25e^{-5} - (-e^{3}) = e^{3} - 25e^{-5}$

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8) Por Teo de Stokes, seja $\gamma = \{(x,y,1) / x^2+3y^2=1\}$ orientada no sentido horário e S a superfície dada, orientada por $\vec{N}$. Logo γ recebe a orient horária de $\vec{N}$. Assim

$$\iint \operatorname{rot}\vec{F}\cdot\vec{N}\,d\sigma = \int_\gamma \vec{F}\,dr$$

$\gamma(t) = \left(\cos t, \frac{\operatorname{sen} t}{\sqrt{3}}, 1\right)$, $-\pi \le t \le \pi$; $\gamma' = \left(-\operatorname{sen} t, \frac{\cos t}{\sqrt{3}}, 0\right)$

$F(\gamma(t)) = \left(\frac{\operatorname{sen} t}{\sqrt{3}} e^{-1}, \cos t \ln 3, -\operatorname{arctg}\left(\cos^2 t + \frac{\operatorname{sen}^2 t}{3}\right)\right)$

Logo

$$\int_\gamma \vec{F}\,dr = \int_{-\pi}^{\pi} \left(\frac{\operatorname{sen} t}{\sqrt{3}} e^{-1}, \cos t \ln 3, -\operatorname{arctg}\left(\cos^2 t + \frac{\operatorname{sen}^2 t}{3}\right)\right)\cdot\left(-\operatorname{sen} t, \frac{\cos t}{\sqrt{3}}, 0\right) dt$$

$$= \int_{-\pi}^{\pi} -\frac{\operatorname{sen}^2 t}{\sqrt{3}} e^{-1} + \cos^2 t \frac{\ln 3}{\sqrt{3}}\,dt = \frac{1}{\sqrt{3}}\int_{-\pi}^{\pi} -e^{-1}\left(\frac{1-\cos 2t}{2}\right) + \ln 3\left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{1}{2\sqrt{3}}\left[-e^{-1}\left(t - \frac{\operatorname{sen} 2t}{2}\right) + \ln 3\left(t + \frac{\operatorname{sen} 2t}{2}\right)\right]_{-\pi}^{\pi} =$$

$$= \frac{2\pi}{2\sqrt{3}}\left(\ln 3 - e^{-1}\right) = \frac{\pi}{\sqrt{3}}\left(\ln 3 - e^{-1}\right)$$

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2. (a) (2,0 pontos) Sejam

$$\vec{F}(x,y,z) = \left(\frac{xz^2}{1+x^2} - 2xe^y\right)\vec{i} + (2y\,\text{sen}\,z - x^2e^y)\,\vec{j} + (z\ln(1+x^2) + y^2\cos z)\,\vec{k}$$

e γ a parte da interseção do elipsóide $x^2 + y^2 + 9z^2 = 50$ com o plano $x - 3y + 7z = 10$ com $z \geq 0$, supondo que o ponto inicial de γ está no 3º quadrante do plano xy e o final no 2º

(i) Decida se $\vec{F}$ é um campo conservativo.

(ii) Calcule $\int_\gamma \vec{F} \cdot d\vec{r}$

(b) (2,0 pontos) Calcule $\iint_S \text{rot}\vec{F} \cdot \vec{N}\, dS$, onde

$$\vec{F}(x,y,z) = ye^{-z}\vec{i} + x\ln(1+2z^2)\vec{j} - \text{arctg}(x^2+y^2)\vec{k}$$

e S é a porção do elipsóide $x^2 + 3y^2 + 4z^2 = 5$ que fica na região $z \geq 1$, com a normal $\vec{N}$ tal que $\vec{N} \cdot \vec{k} \geq 0$.

a) i) $\text{dom}\,\vec{F} = \mathbb{R}^3$; $\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{xz^2}{1+x^2} - 2xe^y & 2y\,\text{sen}\,z - x^2e^y & z\ln(1+x^2) + y^2\cos z \end{vmatrix} =$

$= \left(2y\cos z - 2y\cos z,\ \frac{2zx}{1+x^2} - \frac{2xz}{1+x^2},\ -2xe^y + 2xe^y\right) = \vec{0}$

Como $\text{dom}\,\vec{F} = \mathbb{R}^3$ (simplesmente conexo) e $\text{rot}\,\vec{F} = \vec{0} \Rightarrow \vec{F}$ conservativo

Calculamos um potencial para $\vec{F}$.

$\frac{\partial\phi}{\partial x} = \frac{xz^2}{1+x^2} - 2xe^y$; ② $\frac{\partial\phi}{\partial y} = 2y\,\text{sen}\,z - x^2e^y$, ③ $\frac{\partial\phi}{\partial z} = z\ln(1+x^2) + y^2\cos z$

$\Rightarrow \phi = -x^2e^y + \frac{z^2}{2}\ln(1+x^2) + \alpha(y,z)$. Em ②

$= -x^2e^y + \frac{\partial\alpha}{\partial y} = 2y\,\text{sen}\,z - x^2e^y \Rightarrow \frac{\partial\alpha}{\partial y} = 2y\,\text{sen}\,z \Rightarrow \alpha = y^2\,\text{sen}\,z + \beta(z)$

$\phi = -x^2e^y + \frac{z^2}{2}\ln(1+x^2) + y^2\,\text{sen}\,z + \beta(z)$. Em ③

$= z\ln(1+x^2) + y^2\cos z + \beta'(z) = z\ln(1+x^2) + y^2\cos z \Rightarrow \beta = c \Rightarrow$

$\phi = -x^2e^y + \frac{z^2}{2}\ln(1+x^2) + y^2\,\text{sen}\,z + c$.

Terminamos pi e pf de γ: Eles satisfazem

$z = 0$, $x^2+y^2+9z^2 = 50$, $x - 3y + 7z = 10$ ou

$z = 0$ $x^2+y^2 = 50$, $x - 3y = 10 \Rightarrow 100 + 9y^2 + 60y + y^2 = 50$

$10y^2 + 60y + 50 = 0 \Leftrightarrow y^2 + 6y + 5 = 0 \Rightarrow y = -6 \pm \sqrt{36-20}\,/2 = \frac{-6\pm 4}{2} =$

$-3 \pm 2 \nearrow -1 \rightarrow x = 7$ $(-1,7) = pf$

$\searrow -5 \rightarrow x = -5$ $(-5,5) = pi$

3. (2,5 pontos) Seja $\vec{F}(x,y,z) = xyz\,\vec{i} + x^2y\,\vec{j} + \left(3 + z - \frac{yz^2}{2}\right)\vec{k}$ e seja S a superfície $x^2+y^2+z^2=1$, $z \geq 0$, sendo $\vec{n}$ a normal com componente $\vec{k} \geq 0$. Calcule $\iint_S \vec{F}\cdot\vec{n}\,dS$

Por Gauss: $\operatorname{dom}\vec{F} = \mathbb{R}^3$; $\operatorname{div}\vec{F} = yz + x^2 + 1 - yz = x^2+1$

Consideramos $T = \{(x,y,0) / \ x^2+y^2 \leq 1\}$ e

$R = \{(x,y,z) / \ x^2+y^2+z^2 \leq 1,\ z \geq 0\}$. Portanto $\partial R = S \cup T$.

Por Gauss $\iiint_R \operatorname{div}\vec{F}\,dx\,dy\,dz = \iint_{S\cup T} \vec{F}\cdot\vec{n_e}\,d\sigma$

1. $\iiint_R \operatorname{div}\vec{F}\,dx\,dy\,dz = \int_0^1\int_0^{\pi/2}\int_0^{2\pi} \rho^2 \operatorname{sen}\varphi\,(\rho^2\cos^2\theta\operatorname{sen}^2\varphi + 1)\,d\theta\,d\varphi\,d\rho =$

$= \int_0^1\int_0^{\pi/2}\int_0^{2\pi}\left(\rho^4\operatorname{sen}^3\varphi\,\frac{(1+\cos 2\theta)}{2} + \rho^2\operatorname{sen}\varphi\right)d\theta\,d\varphi\,d\rho =$

$= \pi\int_0^1\int_0^{\pi/2} \rho^4\operatorname{sen}^3\varphi + 2\rho^2\operatorname{sen}\varphi\,d\varphi\,d\rho = \pi\int_0^{\pi/2} \frac{\rho^5}{5}\operatorname{sen}^3\varphi + \frac{2\rho^3}{3}\operatorname{sen}\varphi\Big|_0^1 d\varphi$

$= \pi\int_0^{\pi/2} \frac{1}{5}\operatorname{sen}\varphi(1-\cos^2\varphi) + \frac{2}{3}\operatorname{sen}\varphi\,d\varphi = \pi\left[-\frac{\cos\varphi}{5} + \frac{\cos^3\varphi}{15} - \frac{2}{3}\cos\varphi\right]_0^{\pi/2}$

$= \pi\left[\frac{1}{5} - \frac{1}{15} + \frac{2}{3}\right] = \pi\left[\frac{2}{15} + \frac{2}{3}\right] = \frac{2}{3}\pi\left[\frac{1}{5} + 1\right] = \frac{2\pi}{\cancel{3}}\,\frac{\cancel{6}^{2}}{5} = \frac{4\pi}{5}$

2. $\iint_T \vec{F}\cdot\vec{n_e}\,d\sigma = ?$ Obs que $\vec{n_e} = -\vec{k}$ em T. Logo

$\vec{F}\cdot\vec{n_e} = \vec{F}\cdot(-\vec{k}) = -\left(3 + z - \frac{yz^2}{2}\right)$. Em T temos $z = 0 \Rightarrow$

$\vec{F}\cdot\vec{n_e} = -3$ em T. Assim

$\iint_T \vec{F}\cdot\vec{n_e}\,d\sigma = \iint_T -3\,d\sigma = -3\pi$

Logo $\iint_S \vec{F}\cdot\vec{n_e} = \iiint_R \operatorname{div}\vec{F}\,dx\,dy\,dz - \iint_T \vec{F}\cdot\vec{n_e}\,d\sigma = \frac{4\pi}{5} + 3\pi = \frac{19}{5}\pi$

Por último observamos que $\vec{n_e} = \vec{n}$ em $S \Rightarrow$

$\iint \vec{F}\cdot\vec{n}\,d\sigma = \frac{19}{5}\pi$