Lista 03 - Cálculo I - 2016

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① Aplicações integral definida

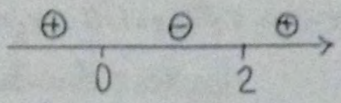
1- Intersecções $f(x)$ e $g(x)$:

$$f(x)=g(x) \Leftrightarrow \frac{1}{1+x^2}=1-\frac{x^2}{5} \Leftrightarrow 5=5+5x^2-x^2-x^4 \Leftrightarrow x^4-4x^2=0$$

$$\therefore x^2(x^2-4)=0 \rightarrow \text{Raízes: } 0, 2, -2$$

Como x varia de 0 a 3, serão usadas apenas as raízes 0 e 2. Para $h(x)=f(x)-g(x)$

$$h(x)=\frac{1}{1+x^2}-1+\frac{x^2}{5}$$



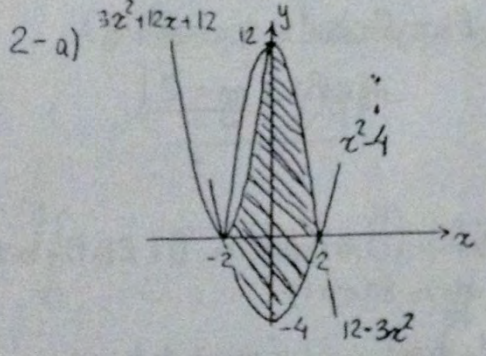
Para o cálculo da área, faz-se: $S=\int_0^2 -h(x)\,dx+\int_2^3 h(x)\,dx$

$$S=\int_0^2\left(1-x^2/5-1/(1+x^2)\right)dx+\int_2^3\left(\frac{1}{1+x^2}-1+\frac{x^2}{5}\right)dx$$

$$S=\left[x-x^3/15-\operatorname{arctg} x\right]_0^2+\left[\operatorname{arctg} x-x+x^3/15\right]_2^3 = \begin{matrix}2-8/15-\operatorname{arctg}2\\+\operatorname{arctg}3-3+27/15\\-(\operatorname{arctg}2-2+8/15)\end{matrix}$$

$$\therefore \underline{S=\operatorname{arctg}3-2\operatorname{arctg}2+26/15}$$

2- a)



$$S=\int_{-2}^{0}\left[(3x^2+12x+12)-(x^2-4)\right]dx+\int_0^2\left[(12-3x^2)-(x^2-4)\right]dx$$

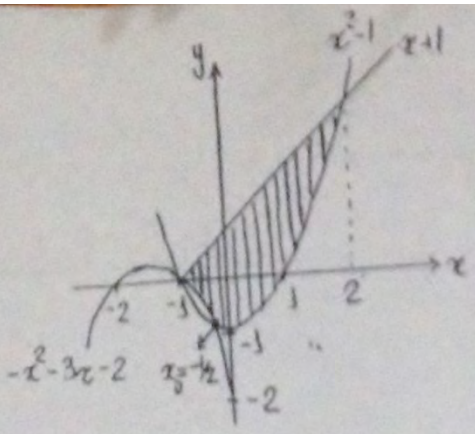
$$S=\int_{-2}^{0}(2x^2+12x+16)\,dx+\int_0^2(-4x^2+16)\,dx$$

$$S=\left[\frac{2x^3}{3}+6x^2+16x\right]_{-2}^{0}+\left[-\frac{4x^3}{3}+16x\right]_0^2$$

$$S=-\left(-16/3+24-32\right)+\left(-32/3+32\right)$$

$$\therefore \underline{S=104/3}$$

b)



$$S=\int_{-1}^{-1/2}[(x+1)-(-x^2-3x-2)]dx+\int_{-1/2}^{2}[(x+1)-(x^2-1)]dx$$

$$S=\int_{-1}^{-1/2}(x^2+4x+3)dx+\int_{-1/2}^{2}(-x^2+x+2)dx$$

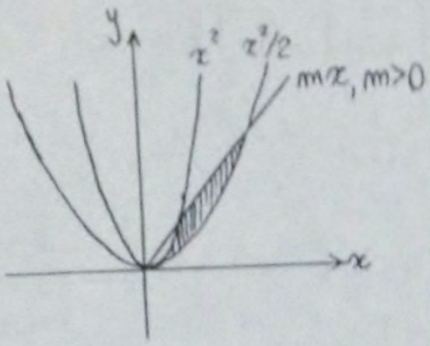
$$S=\left[\frac{x^3}{3}+2x^2+3x\right]_{-1}^{-1/2}+\left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1/2}^{2}$$

$$S=\left[\left(-\frac{1}{24}+2\cdot\frac{1}{4}-\frac{3}{2}\right)-\left(-\frac{1}{3}+2-3\right)+\left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{24}+\frac{1}{8}-1\right)\right]$$

$\therefore S=107/24$

OBS.: intersecção entre parábolas $\rightarrow x^2-1=-x^2-3x-2 \Leftrightarrow 2x^2+3x+1=0 \therefore x_0=-1/2$ ou $x_1=-1$

3-



Intersecções:

$x^2=mx \Leftrightarrow x=0$ ou $x=m$

$x^2/2=mx \Leftrightarrow x=0$ ou $x=2m$

Assim:

$$S=\int_0^m(x^2-x^2/2)dx+\int_m^{2m}(mx-x^2/2)dx=4$$

$$S=\left[\frac{x^3}{3}-\frac{x^3}{6}\right]_0^m+\left[\frac{mx^2}{2}-\frac{x^3}{6}\right]_m^{2m}$$

$\therefore S=m^3/6+4m^3/2-8m^3/6-m^3/2+m^3/6=3/6\,m^3$. Logo, $(3/6)m^3=4 \Leftrightarrow m^3=8$

$m\in\mathbb{R} \therefore m=2$

4- $y=x(x^2-1)$ → raízes: 0, 1 e -1

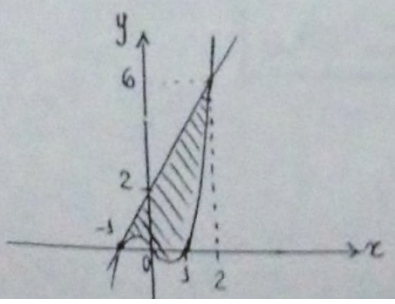
$\frac{dy}{dx}=3x^2-1 \Rightarrow y'(-1)=3-1=2 \therefore$ reta tg $=2x+b$, contém $(-1,0)$. Assim: $0=2(-1)+b \therefore b=2$

$\therefore$ equação da reta $=g(x)=2(x+1)$

Intersecções $y(x)$ e $g(x)$: $x^3-x=2x+2 \Leftrightarrow x^3-3x-2=h(x)$; -1 é raiz de $h(x)$:

$h(x)=(x+1)\underbrace{(x^3-3x-2)}_{(x+1)}=(x+1)\underbrace{(x^2-x-2)}_{\text{raízes: 2 e -1}}=(x+1)^2(x-2)$ ∴ são pontos de intersecção $(-1,0)$ e $(2,6)$

Logo:

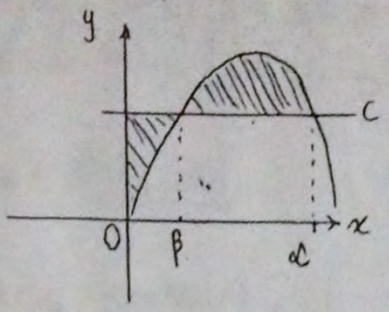


$$\therefore S=\int_{-1}^{2}[(2x+2)-(x^3-x)]dx=\left[x^2+2x-\frac{x^4}{4}+\frac{x^2}{2}\right]_{-1}^{2}$$

$$S=4+4-4+2-(1-2-1/4+1/2)$$

$\therefore S=27/4$

5 - Supondo-se que haja intersecção entre as funções em $x_0 = \beta$ e $x_1 = \alpha$:



Assim: $C = 2x - 3x^3 \Leftrightarrow 3x^3 - 2x + C = 0$

α e β são raízes, pois (α, C) e (β, C) são pontos de intersecção:

$3\alpha^3 - 2\alpha + C = 0 \rightarrow$ (I)

$3\beta^3 - 2\beta + C = 0 \rightarrow$ não usada

Ambas as áreas são equivalentes:

$$\int_0^\beta (C - 2x + 3x^3)\,dx = \int_\beta^\alpha (2x - 3x^3 - C)\,dx \Leftrightarrow \int_0^\beta (C - 2x + 3x^3)\,dx - \int_\beta^\alpha (2x - 3x^3 - C)\,dx = 0$$

$$\int_0^\beta (C - 2x + 3x^3)\,dx + \int_\beta^\alpha (C - 2x + 3x^3)\,dx = \int_0^\alpha (C - 2x + 3x^3)\,dx = 0$$

Desenvolvendo a integral:

$$Cx - x^2 + 3\frac{x^4}{4}\Big|_0^\alpha = 0 \Leftrightarrow \alpha C - \alpha^2 + 3\frac{\alpha^4}{4} = 0 \quad \therefore C = \alpha - \frac{3\alpha^3}{4} \rightarrow \text{(II)}$$

Substituindo (I) em (II):

$$3\alpha^2 - 2\alpha + \alpha - \frac{3\alpha^3}{4} = 0 \Leftrightarrow \frac{9}{4}\alpha^3 = \alpha \quad \therefore \alpha = 0,\ 2/3 \text{ ou } -2/3$$

$\alpha = 0 \Rightarrow C = 0$

$\alpha = 2/3 \Rightarrow C = 2/3 - (3/4)(8/27) = 4/9$

$\alpha = -2/3 \Rightarrow d = -2/3 - (3/4)(-8/27) = -4/9$

A intersecção ocorre no 1º quadrante, o que implica $C > 0$

$$\therefore C = 4/9$$

6 - $$\int_0^1 (x + \sqrt{1 - x^2})\,dx = \int_0^1 \sqrt{1 - x^2}\,dx + \int_0^1 x\,dx = \int_0^{\pi/2} |\cos\theta|\cos\theta\,d\theta + \int_0^1 x\,dx = \int_0^{\pi/2} \cos^2\theta\,d\theta + \int_0^1 x\,dx$$

$x = \operatorname{sen}\theta \Rightarrow dx = \cos\theta\,d\theta$

$x = 1 \rightarrow \theta = \pi/2$

$x = 0 \rightarrow \theta = 0$

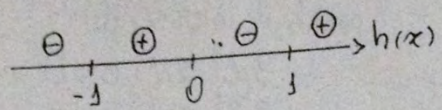
$\sqrt{1 - x^2} = \sqrt{1 - \operatorname{sen}^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$

$$\left[\frac{\theta}{2} + \frac{\operatorname{sen} 2\theta}{2}\right]_0^{\pi/2} + \left[\frac{x^2}{2}\right]_2^1 = \frac{\pi}{4} + \frac{1}{2}$$

7 - Intersecções $f(x)$ e $g(x)$:

$h(x) = f(x) - g(x) = x^3 - 2x + 1 + x - 1 = x^3 - x = x(x^2-1) \rightarrow$ raízes: $0, 1, -1$

Para $-1 \leq x \leq 1$

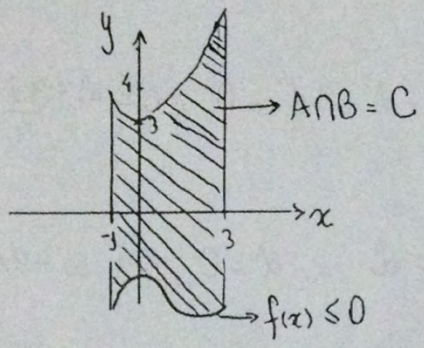


$$\hookrightarrow S = \int_{-1}^{0} h(x)\,dx + \int_{0}^{1} -h(x)\,dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$S = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$\therefore S = 1/2$$

8-



$$S_C = \int_{-1}^{3} [(x^2+3) - f(x)]\,dx = 23$$

$$S_C = \int_{-1}^{3} (x^2+3)\,dx - \int_{-1}^{3} f(x)\,dx = 23$$

$$\left[x^3/3 + 3x\right]_{-1}^{3} - 23 = \int_{-1}^{3} f(x)\,dx$$

$$9 + 9 + 1/3 + 3 - 23 = \int_{-1}^{3} f(x)\,dx$$

$$\therefore \int_{-1}^{3} f(x)\,dx = -5/3$$

(2) Integrais indefinidos

[1] $\int \frac{x^7+x^2+1}{x^2}\,dx = \int \left(x^5+1+\frac{1}{x^2}\right)dx = \underline{\frac{x^6}{6}+x-\frac{1}{x}+K}$

[2] $\int e^{2x}\,dx = \underline{e^{2x}/2+K}$

[3] $\int \cos 7x\,dx = \underline{(\operatorname{sen} 7x)/7+K}$

[4] $\int \operatorname{tg}^2 x\,dx = \int (\sec^2 x - 1)\,dx = \underline{\operatorname{tg} x - x + K}$

[5] $\int \frac{7}{x-2}\,dx = \int \frac{7}{u}\,du = 7\ln|u| + K = \underline{7\ln|x-2|+K}$ → $u = x-2$, $du = dx$

[6] $\int \operatorname{tg}^3 x(\sec^2 x)\,dx = \int u^3\,du = u^4/4 + K = \underline{(\operatorname{tg}^4 x)/4 + K}$ → $u = \operatorname{tg} x$, $du = \sec^2 x\,dx$

[7] $\int (\operatorname{sen}^3 x/\sqrt{\cos x})\,dx = \int \frac{(1-\cos^2 x)\operatorname{sen} x}{\sqrt{\cos x}}\,dx \overset{*}{=} -\int \frac{1-u^2}{\sqrt{u}}\,du = -\int (u^{-1/2}-u^{3/2})\,du$

$= -\left[2u^{1/2} - u^{5/2}(2/5)\right] = \underline{2\sqrt{\cos x}\left(\frac{\cos^2 x}{5}-1\right)+K}$ → * $u = \cos x$, $du = -\operatorname{sen} x\,dx$

[8] $\int \operatorname{tg} x\,dx = \int \frac{\operatorname{sen} x}{\cos x}\,dx = \int -\frac{du}{u} = \underline{-\ln|\cos x|+K}$ → $u = \cos x$, $du = -\operatorname{sen} x\,dx$

[9] $\int \operatorname{tg}^3 x\,dx = \int (\sec^2 x - 1)\operatorname{tg} x\,dx = \int \operatorname{tg} x(\sec^2 x)\,dx - \int \operatorname{tg} x\,dx \overset{*}{=}$

$= \int u\,du - \int \operatorname{tg} x\,dx = \underline{(\operatorname{tg}^2 x)/2 + \ln|\cos x| + K}$ → * $u = \operatorname{tg} x$, $du = \sec^2 x\,dx$

[10] $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \underline{\frac{\ln|1+x^2|}{2}+K}$ → $u = 1+x^2$, $du = 2x\,dx$

[11] $\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x\,dx}{1+x^4} \overset{*}{=} \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\operatorname{arctg} u = \underline{\frac{1}{2}\operatorname{arctg}(x^2)+K}$

* $u = x^2$, $du = 2x\,dx$

[12] $\int \frac{x^2}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx = \int 1.\,dx - \int \frac{1}{1+x^2}\,dx = \underline{x - \operatorname{arctg} x + K}$

[13] $\int x\sqrt{1-x^2}\,dx \overset{*}{=} -\frac{1}{2}\int \sqrt{1+u}\,du = -\frac{1}{2}\cdot\frac{2}{3}\cdot(1+u)^{3/2} = \underline{-\frac{1}{3}\sqrt{(1-x^2)^3}+K}$

* $u = -x^2$, $du = -2x\,dx$

14 $\int \sec x\,dx = \int \sec x\left[(\sec x+\operatorname{tg} x)/(\sec x+\operatorname{tg} x)\right]dx = \int \frac{(\sec^2 x+\sec x\cdot \operatorname{tg} x)\,dx}{(\sec x+\operatorname{tg} x)} \overset{(*)}{=}$

$= \int \frac{du}{u} = \underline{\ln|\sec x+\operatorname{tg} x|+K}$ (*) $u=\sec x+\operatorname{tg} x$, $du=(\sec x\operatorname{tg} x+\sec^2 x)\,dx$

15 $\int \frac{dx}{x\sqrt{1+\ln x}} \overset{(*)}{=} \int \frac{du}{\sqrt{u}} = 2u^{1/2} = \underline{2\sqrt{1+\ln x}+K}$

(*) $u=(1+\ln x)$, $du=(1/x)\,dx$

16 $\int x^2\sqrt[5]{x^3+1}\,dx = \frac{1}{3}\int 3x^2\sqrt[5]{1+x^3}\,dx = \frac{1}{3}\int \sqrt[5]{u}\,du = \frac{1}{3}u^{6/5}\cdot\frac{5}{6} = \underline{\frac{5}{18}\sqrt[5]{(x^3+1)^6}+K}$

↳ $u=1+x^3$, $du=3x^2dx$

17 $\int (4x+8)\,dx/(2x^2+8x+20) = \int \frac{du}{u} = \ln|u| = \underline{\ln|2x^2+8x+20|+K}$

↳ $u=2x^2+8x+20$, $du=(4x+8)\,dx$ OBS.: pode ser escrito sem módulo pela razão do **51**; na dúvida, é mais seguro usar o módulo

18 $\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du = u^{3/2}\cdot\frac{2}{3} = \underline{\frac{2}{3}\sqrt{(\ln x)^3}+K}$

↳ $u=\ln x$, $du=(1/x)dx$

19 $\int \frac{dx}{\operatorname{arcsen} x\sqrt{1-x^2}} = \int \frac{du}{u} = \ln|u|+K = \underline{\ln|\operatorname{arcsen} x|+K}$

↳ $u=\operatorname{arcsen} x$, $du=(1/\sqrt{1-x^2})\,dx$

20 $\int \frac{e^x}{(1+e^x)}\,dx = \int \frac{du}{u} = \ln|u| = \underline{\ln|1+e^x|+K}$

↳ $u=1+e^x$, $du=e^x dx$

21 $\int \frac{\operatorname{sen} 2x}{(1+\cos^2 x)}\,dx = -\int \frac{du}{1+u} = -\ln|1+u| = \underline{-\ln|1+\cos^2 x|+K}$

↳ $u=\cos^2 x$, $du=-2\operatorname{sen} x\cos x\,dx = -\operatorname{sen} 2x\,dx$

22 $\int e^{x^3}x^2\,dx = \frac{1}{3}\int e^{x^3}(3x^2\,dx) = \frac{1}{3}\int e^u\,du = e^u/3 = \underline{e^{x^3}/3+K}$

↳ $u=x^3$, $du=3x^2dx$

23 $\int e^x\sqrt[3]{1+e^x}\,dx = \int \sqrt[3]{1+u}\,du = (1+u)^{4/3}\,\frac{3}{4} = \underline{\sqrt[3]{(1+e^x)^4}\,\frac{3}{4}+K}$

↳ $u=e^x$, $du=e^x dx$

24 $\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx = 2\int \operatorname{sen}\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx = 2\int \operatorname{sen} u\,du = -2\cos u = \underline{-2\cos\sqrt{x}+K}$

↳ $u=\sqrt{x}$, $du=(1/2\sqrt{x})\,dx$

25 $\int \frac{e^{\operatorname{arctg} x}}{(1+x^2)}\,dx = \int e^u\,du = e^u = \underline{e^{\operatorname{arctg} x} + K}$

↳ $u = \operatorname{arctg} x$

$du = [1/(1+x^2)]\,dx$

26 $\int 2x(x+1)^{2010}\,dx = 2\int (u-1)u^{2010}\,du = 2\int (u^{2011} - u^{2010})\,du = 2\frac{u^{2012}}{2012} - 2\frac{u^{2011}}{2011}$

↳ $u = x+1$, $du = dx$

$$= \underline{\frac{2(x+1)^{2012}}{2012} - \frac{2(x+1)^{2011}}{2011} + K}$$

27 $\int \underbrace{x}_{f}\,\underbrace{\operatorname{sen} x}_{g'}\,dx = -x\cos x - \int 1\,(-\cos x)\,dx = \underline{-x\cos x + \operatorname{sen} x + K}$

$f' = 1$, $g = -\cos x$

28 $\int \underbrace{e^x}_{f}\,\underbrace{\cos x}_{g'}\,dx = e^x \operatorname{sen} x - \int \underbrace{\operatorname{sen} x}_{z'}\,\underbrace{e^x}_{h}\,dx = e^x \operatorname{sen} x - \left(-e^x \cos x - \int e^x(-\cos x)\,dx\right)$

$f' = e^x$, $g = \operatorname{sen} x$ $\qquad$ $h' = e^x$, $z = -\cos x$

$= e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x\,dx \Leftrightarrow 2\int e^x \cos x\,dx = e^x(\operatorname{sen} x + \cos x) + K$

$$\therefore \underline{\int e^x \cos x\,dx = (e^x/2)(\operatorname{sen} x + \cos x) + K}$$

29 $\int \underbrace{x^r}_{g'}\,\underbrace{\ln x}_{f}\,dx = \frac{x^{r+1}}{(r+1)}\ln x - \int \frac{x^{r+1}}{(r+1)}\cdot\frac{1}{x}\,dx = \frac{x^{r+1}\ln x}{(r+1)} - \frac{1}{(r+1)}\int x^{r+1-1}\,dx$

$$= \frac{x^{r+1}\ln x}{(r+1)} - \frac{x^{r+1}}{(r+1)^2} + K, \text{ se } r \neq -1$$

Para $r = -1$: $\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{\ln x^2}{2} + K$

$$\therefore \int x^r \ln x\,dx = \begin{cases} \frac{x^{r+1}}{(r+1)}\ln x - \frac{x^{r+1}}{(r+1)^2} + K, & \text{se } r \neq -1 \\ (\ln x^2)/2 + K, & \text{se } r = -1 \end{cases}$$

30 $\int (\ln x)^2\,dx = \int \underbrace{(\ln x)^2}_{f}\cdot\underbrace{1}_{g'}\,dx = x(\ln x)^2 - 2\int \underbrace{\frac{\ln x}{x}}_{r+h}\cdot x\cdot\underbrace{\frac{1}{x}}_{z'}\,dx$

$$= x(\ln x)^2 - 2\left(x\ln x - \int \frac{1}{x}\cdot x\,dx\right) = \underline{x(\ln x)^2 - 2(x\ln x - x) + K}$$

31 $\int x e^{-x}\,dx = \int u e^{u}\,dw = ue^{u} - \int 1\cdot e^{u}\,dw = e^{u}(u-1) = \underline{e^{-x}(-x-1)+K}$

$u=-x \leftarrow f$, $dw=-dx \leftarrow g'$

32 $\int x\,\underbrace{\operatorname{arctg} x}_{f}\,\underbrace{dx}_{g'} = \operatorname{arctg}x\cdot\frac{x^2}{2} - \underbrace{\int \frac{x^2}{2}\cdot\frac{1}{(1+x^2)}\,dx}_{-\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = -\frac{1}{2}\int\left(1-1/(1+x^2)\right)dx = -\frac{x}{2}+\frac{\operatorname{arctg}x}{2}} = \underline{-\frac{x}{2}+\operatorname{arctg}x\,(1/2+x^2/2)+K}$

33 $\int \operatorname{arcsen} x\,dx = \int \underbrace{1}_{g'}\cdot\underbrace{\operatorname{arcsen}x}_{f}\,dx = x\operatorname{arcsen}x - \underbrace{\int x\cdot\left(\frac{1}{\sqrt{1-x^2}}\right)dx}_{\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\frac{u^{1/2}}{(1/2)}} = \underline{x\operatorname{arcsen}x+\sqrt{1-x^2}+K}$

$\hookrightarrow u = 1-x^2,\ du = -2x\,dx$

$= -\sqrt{u} = -\sqrt{1-x^2}$

$\therefore -\int \frac{x}{\sqrt{1-x^2}}\,dx = \sqrt{1-x^2}$

34 $\int \sec^3 x\,dx = \int \underbrace{\sec x}_{f}\,\underbrace{\sec^2 x}_{g'}\,dx = \operatorname{tg}x\cdot\sec x - \int \operatorname{tg}x(\sec x\cdot\operatorname{tg}x)\,dx = \operatorname{tg}x\sec x - \int \operatorname{tg}^2 x\sec x\,dx$

$= \operatorname{tg}x\sec x - \int(\sec^2 x - 1)\cdot\sec x\,dx = \operatorname{tg}x\sec x - \int \sec^3 x\,dx + \int \sec x\,dx$

$\Leftrightarrow 2\int \sec^3 x\,dx = \operatorname{tg}x\sec x + \ln|\sec x+\operatorname{tg}x| \quad \therefore \underline{\int \sec^3 x\,dx = (1/2)(\operatorname{tg}x\sec x + \ln|\sec x+\operatorname{tg}x|)+K}$

35 $\int \cos^2 x\,dx = \int (1/2 + \cos 2x/2)\,dx = \underline{x/2 + \operatorname{sen}2x/4 + K}$

"truque": $\cos^2 x + \operatorname{sen}^2 x = 1 \rightarrow$ (I)

$\cos 2x = \cos(x+x) = \cos^2 x - \operatorname{sen}^2 x \rightarrow$ (II)

para $\int \cos^2 x\,dx \rightarrow$ (I) + (II) $\Rightarrow 2\cos^2 x = 1+\cos 2x \therefore \cos^2 x = 1/2 + \cos 2x/2$

para $\int \operatorname{sen}^2 x\,dx \rightarrow$ (I) − (II) $\Rightarrow 2\operatorname{sen}^2 x = 1-\cos 2x \therefore \operatorname{sen}^2 x = 1/2 - \cos 2x/2$

36 $\int \operatorname{sen}^2 x\cos^3 x\,dx = \int \operatorname{sen}^2 x\,(1-\operatorname{sen}^2 x)\cos x\,dx = \int \operatorname{sen}^2 x\cos x\,dx - \int \operatorname{sen}^4 x\cos x\,dx$

$= \int u^2\,du - \int u^4\,du = u^3/3 - u^5/5 = \underline{\operatorname{sen}^3 x/3 - \operatorname{sen}^5 x/5 + K}$ $\quad u=\operatorname{sen}x,\ du=\cos x\,dx$

37 $\int \operatorname{sen}^2 x\cos^2 x\,dx = \int [1/2 - (1/2)\cos 2x][1/2 + (1/2)\cos 2x]\,dx = (1/4)\int (1-\cos^2 2x)\,dx$

$= (1/4)\int \operatorname{sen}^2 2x\,dx = (1/4)\int [1/2 - (\cos 4x)/2]\,dx$

$= \underline{(1/8)x - \operatorname{sen}4x/32 + K}$

38 $\int \frac{1-\operatorname{sen} x}{\cos x}\,dx = \int \underbrace{\frac{1}{\cos x}}_{\sec x}\,dx - \int \frac{\operatorname{sen} x}{\cos x}\,dx = \ln|\sec x + \operatorname{tg} x| + \ln|\cos x| = \ln|\sec x + \operatorname{tg} x||\cos x|$

↳ $u = \cos x$, $du = -\operatorname{sen} x\,dx \rightsquigarrow -\int -\frac{du}{u} = \ln|u|$

$= \ln|1 + \operatorname{sen} x| + K$

39 $\int \frac{3x^2+4x+5}{(x-1)(x-2)(x-3)}\,dx \overset{\downarrow}{=} \int \frac{6}{(x-1)}\,dx + \int \frac{-25}{(x-2)}\,dx + \int \frac{22}{(x-3)}\,dx = 6\ln|x-1| - 25\ln|x-2| + 22\ln|x-3| + K$

$\frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)} \Rightarrow 3x^2+4x+5 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$x=1 \to A=6$; $x=2 \to B=-25$; $x=3 \to C = 44/2 = 22$

40 $\int \frac{dx}{2(x^2+4x+10)} \overset{\downarrow}{=} \frac{1}{12}\int \frac{du}{(u/\sqrt{6})^2+1} = \frac{\sqrt{6}}{12}\operatorname{arctg}\left(\frac{u}{\sqrt{6}}\right) = \frac{\sqrt{6}}{12}\operatorname{arctg}\left(\frac{x-2}{\sqrt{6}}\right) + K$

$x^2+4x+10 = \underbrace{(x+2)}^{2}+6 = 6\left[\left(\frac{u}{\sqrt{6}}\right)^2 + 1\right]$

$x+2 = u$, $du = dx$

41 $\int \frac{3x^2+4x+5}{(x-1)^2(x-2)}\,dx \overset{\downarrow}{=} \int \left[\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}\right]dx = -22\ln|x-1| + 12/(x-1) + 25\ln|x-2| + K$

$A(x-1)(x-2) + B(x-2) + C(x-1)^2 = 3x^2+4x+5$

$x=1 \to B=-12$ $\qquad x=0 \to 2A+24+25=5$

$x=2 \to C=25$ $\qquad \therefore A=-22$

42 $\int (x^5+x+1)\,dx\,/(x^3-8) = \int \left[x^2 + \frac{(8x^2+x+1)}{x^3-8}\right]dx = \frac{x^3}{3} + \int \left[\frac{A}{(x-2)} + \frac{Bx+C}{(x^2+2x+4)}\right]dx$

↳ divisão de polinômios $\qquad$ ↳ $(x^3-8) = (x-2)(x^2+2x+4)$

$8x^2+x+1 = A(x^2+2x+4) + (Bx+C)(x-2)$

$x=2 \to A = 35/12$; $x=0 \to C = 16/3$; $x=1 \to B = 61/12$

Assim, obtém-se: $\frac{35}{12}\int \frac{1}{x-2}\,dx + \int \frac{\frac{61}{12}x + \frac{16}{3}}{x^2+2x+4} = \frac{35}{12}\ln|x-2| + \frac{1}{12}\underbrace{\int \frac{61x+64}{x^2+2x+4}\,dx}_{\circledast}$

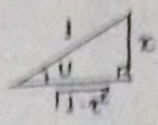
$\circledast \int \frac{61x+64}{\underbrace{(x+1)^2}_{u=x+1}+3}\,dx = \int \frac{61(u-1)+64}{u^2+3}\,du = 61\int \frac{u}{u^2+3}\,du + 3\int \frac{1}{u^2+3}\,du \to \frac{1}{3\left[\left(\frac{u}{\sqrt{3}}\right)^2+1\right]}$

$= \frac{61}{2}\ln(u^2+3) + \frac{3}{\sqrt{3}}\operatorname{arctg}\left(\frac{u}{\sqrt{3}}\right) = \frac{61}{2}\ln(x^2+2x+4) + \underbrace{3}_{\sqrt{3}}\left[\frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right)\right]$

$\therefore \int \frac{x^5+x+1}{x^3-8}\,dx = \frac{x^3}{3} + \frac{35}{12}\ln|x-2| + \frac{61}{24}\ln(x^2+2x+4) + \frac{\sqrt{3}}{12}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + K$

43 $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\operatorname{sen}^2 v}{\cos v}\cos v\,dv = \int \operatorname{sen}^2 v\,dv = \frac{v}{2} + \frac{\operatorname{sen} 2v}{4}$

↳ $v = \operatorname{arcsen} x$, $dx = \cos v\,dv$
$x = \operatorname{sen} v$



$\operatorname{sen} v = x$
$\cos v = \sqrt{1-x^2}$

$= \frac{\operatorname{arcsen} x}{2} - \frac{1}{4}\operatorname{sen}(\operatorname{arcsen} x)\cos(\operatorname{arcsen} x) = \boxed{\frac{\operatorname{arcsen} x}{2} - \frac{x}{2}\sqrt{1-x^2} + K}$

44 $\int x^2\sqrt{1-x^2}\,dx = \int \operatorname{sen}^2 v \cos^2 v\,dv = v/8 - \operatorname{sen} 4v/8 = v/8 - \frac{2\operatorname{sen} 2v \cdot \cos 2v}{32}$

↳ $x = \operatorname{sen} v$, $v = \operatorname{arcsen} x$ ↳ Ex. 37 ↱ como no 43, $\operatorname{sen} v = x \Rightarrow \operatorname{sen}^2 v = x^2$
$dx = \cos v\,dv$ $\cos v = \sqrt{1-x^2} \Rightarrow \cos^2 v = 1-x^2$

$= \frac{v}{8} - \frac{2\operatorname{sen} v \cos v(\cos^2 v - \operatorname{sen}^2 v)}{16} = \boxed{\frac{\operatorname{arcsen} x}{8} - \frac{x\cdot\sqrt{1-x^2}(1-2x^2)}{8} + K}$

45 $\int e^{\sqrt{x}}\,dx = 2\int \underbrace{\frac{e^{\sqrt{x}}}{2\sqrt{x}}}_{g = e^{\sqrt{x}},\ g'} \cdot \underbrace{\sqrt{x}}_{f,\ f' = \frac{1}{2\sqrt{x}}}\,dx = 2\left(\sqrt{x}\,e^{\sqrt{x}} - \underbrace{\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx}_{e^{\sqrt{x}}}\right) = \boxed{2e^{\sqrt{x}}(\sqrt{x}-1) + K}$

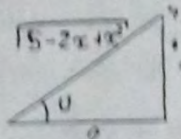
46 $\int \underbrace{\ln(x+\sqrt{1+x^2})}_{f} \cdot \underbrace{1}_{g'}\,dx = x\ln(x+\sqrt{1+x^2}) - \int \frac{x\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}}\,dx$

$f' = \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)/(x+\sqrt{1+x^2})$ $\qquad \int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2}$

$= \boxed{x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + K}$

47 $\int \frac{dx}{\sqrt{5-2x+x^2}} = \int \frac{dx}{\sqrt{4+(x-1)^2}} = \int \frac{dx}{2\sqrt{1+\left(\frac{x-1}{2}\right)^2}} = \int \frac{2\sec^2 v\,dv}{2|\sec v|}\,dv = \int |\sec v|\,dv$

↳ $\operatorname{tg} v = (x-1)/2$, $2\sec^2 v\,dv = dx$



$\sec v = 1/\cos v = (1/2)\sqrt{5-2x+x^2}$
$\operatorname{tg} v = (x-1)/2$

$= \ln|\sec v + \operatorname{tg} v| = \boxed{\ln\left|(1/2)\sqrt{5-2x+x^2} + (x-1)/2\right| + K}$

ou $\ln\left|\sqrt{5-2x+x^2} + (x-1)\right| + \ln 2 + K$ → outra constante C

48 $\int \sqrt{x}\ln x\,dx = \int \frac{x}{\sqrt{x}}\ln x\,dx = 2\int \underbrace{v^2}_{g'} \underbrace{\ln(v^2)}_{f}\,dv = 2\left[(\ln v^2)\frac{v^3}{3} - \underbrace{\int \frac{v^3}{3}\cdot\frac{2}{v}\,dv}_{-\frac{2}{3}\frac{v^3}{3}}\right]$

$v = \sqrt{x}$
$dv = dx/2\sqrt{x}$
$f \to f' = \frac{2v}{v^2} = \frac{2}{v}$

$= \frac{2}{3}v^3(\ln v^2 - 2/3) = \boxed{(2/3)x\sqrt{x}(\ln x - 2/3) + K}$

49 $\int \underbrace{\operatorname{sen}(\ln x)}_{f} \cdot \underbrace{1}_{g'}\, dx = x\operatorname{sen}(\ln x) - \int \underbrace{\cos(\ln x)}_{h} \cdot \underbrace{1}_{z'}\, dx$

$f' = \cos(\ln x)/x$

$-\left(\cos(\ln x)\cdot x - \int -\frac{\operatorname{sen}(\ln x)}{x}\cdot x\, dx\right)$

$\Leftrightarrow 2\int \operatorname{sen}(\ln x)\,dx = x[\operatorname{sen}(\ln x) - \cos(\ln x)]$ $\quad \therefore \int \operatorname{sen}(\ln x)\,dx = (x/2)[\operatorname{sen}(\ln x) - \cos(\ln x)] + K$

50 $\int \frac{x}{x^2-4}\,dx = \int\left[\frac{A}{x+2} + \frac{B}{(x-2)}\right]dx = \frac{1}{2}\int \frac{dx}{x+2} + \frac{1}{2}\int \frac{dx}{x-2} = (1/2)\ln|x^2-4| + K$

↳ $x = A(x-2) + B(x+2)$ $\quad x=2 \to B=1/2$; $x=-2 \to A=1/2$

51 $\int \frac{3x^2+5x+4}{x^3+x^2+x-3}\,dx = \int\left[\frac{A}{(x-1)} + \frac{Bx+C}{x^2+2x+3}\right]dx \underset{\text{(I)}}{=} \int\left[\frac{2}{x-1} + \frac{x+2}{x^2+2x+3}\right]dx = \text{(II)}$

(I) $A(x^2+2x+3) + (Bx+C)(x-1) = 3x^2+5x+4$

$x=1 \to A=2$ $\quad x=2 \to 2.11 + (2B+2) = 26$

$x=0 \to C=2$ $\quad \therefore B=1$

$\text{(II)} = \underbrace{\int \frac{(x+2)}{x^2+2x+3}\,dx} + 2\ln|x-1|$

$\int \frac{(x+2)\,dx}{(x+1)^2+2} = \int \frac{v+1}{v^2+2}\,dv = \int \frac{v}{v^2+2} + \int \frac{dv}{\sqrt{2}\left[\left(\frac{v}{\sqrt{2}}\right)^2+1\right]} = \frac{1}{2}\ln|v^2+2| + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{v}{\sqrt{2}}\right)$

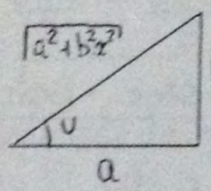
↳ $v = x+1$, $dv = dx$

$\therefore \int \frac{3x^2+5x+4}{x^3+x^2+x-3}\,dx = 2\ln|x-1| + \frac{1}{2}\ln|x^2+2x+3| + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right) + K$

OBS.: O logaritmando da segunda parcela poderia ser escrito sem módulo pois x^2+2x+3 é estritamente positivo (>0) $\forall x \in \mathbb{R}$, porque possui apenas raízes puramente complexas.

52 $\int \sqrt{a^2+b^2x^2}\,dx = a\int \sqrt{1+(bx/a)^2}\,dx = (a^2/b)\int \sec^3 v\,dv = (1/2)(\sec v \operatorname{tg} v + \ln|\sec v + \operatorname{tg} v|)(a^2$

↳ $\operatorname{tg} v = \frac{bx}{a}$, $\sec^2 v\,dv = \frac{b}{a}dx$



$\sec v = \sqrt{a^2+b^2x^2}/a$

$\operatorname{tg} v = bx/a$

$= \left(\frac{\sqrt{a^2+b^2x^2}}{2a}\,\frac{bx}{a} + \frac{1}{2}\ln\left|\frac{\sqrt{a^2+b^2x^2}}{a} + \frac{bx}{a}\right|\right)\frac{a^2}{b}$

$\therefore \int \sqrt{a^2+b^2x^2}\,dx = (1/2)\left[x\sqrt{a^2+b^2x^2} + (a^2/b)\ln\left|(\sqrt{a^2+b^2x^2}+bx)/a\right|\right] + K$

OBS: $\ln\left|\frac{\sqrt{a^2+b^2x^2}+bx}{a}\right| + K = \ln|\sqrt{a^2+b^2x^2}+bx| \underbrace{-\ln a + K}_{\to \text{outra constante } c}$

53 $\int \frac{dx}{\sqrt{a^2+b^2x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1+(bx/a)^2}} = \frac{1}{a}\cdot\frac{a}{b}\int \frac{\sec^2 v}{|\sec v|}\,dv = \frac{1}{b}\ln|\sec v + tgv|$

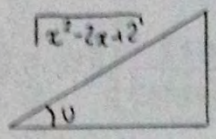
↳ $tgv = bx/a$; $\sec^2 v\,dv = \frac{b}{a}dx$

Ex 52: $\sec v = \sqrt{a^2+b^2x^2}/a$;

$$\therefore \int \frac{dx}{\sqrt{a^2+b^2x^2}} = \frac{1}{b}\ln\left|\frac{\sqrt{a^2+b^2x^2}}{a} + \frac{bx}{a}\right| + K$$

54 $\int \sqrt{x^2-2x+2}\,dx = \int \sqrt{(x-1)^2+1}\,dx = \int \sec^3 v\,dv = (1/2)(\sec v tgv + \ln|\sec v + tgv|$

↳ $tgv = x-1$, $\sec^2 v\,dv = dx$ ↳ Ex 34



$\sec v = \sqrt{x^2-2x+2}$

$$\therefore \int \sqrt{x^2-2x+2}\,dx = \frac{\sqrt{x^2-2x+2}}{2}\cdot(x-1) + \frac{1}{2}\cdot\ln\left|\sqrt{x^2-2x+2} + (x-1)\right| + K$$

55 $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = 2\int \sqrt{1-[(x+1)/2]^2}\,dx = 2\int \sqrt{1-\sin^2 v}\,2\cos v\,dv$

↳ $\frac{x+1}{2} = \sin v$, $dx = 2\cos v\,dv$

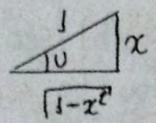
$$4\int \cos^2 v\,dv = 4\left(\frac{v}{2} + \frac{\sin 2v}{4}\right) = 2v + \sin 2v = 2\arcsin\left(\frac{x+1}{2}\right) + \sin\left(2\arcsin\left(\frac{x+1}{2}\right)\right) + K$$

↳ $= 2\sin v\cos v\,(\ldots)$

56 $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{\cos v\,dv}{(1+\sin^2 v)\cos v} \overset{:\cos^3 v}{=\!=\!=} \int \frac{(1/\cos^2 v)\,dv}{\sec^2 v + tg^2 v} = \int \frac{\sec^2 v\,dv}{1+2tg^2 v} = \int \frac{da}{1+2a^2}$

↳ $x = \sin v$, $dv\cos v = dx$ ↳ $a = tgv$, $da = \sec^2 v\,dv$

$$= \frac{\operatorname{arctg}(\sqrt{2}a)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,tgv) = \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\sqrt{2}\,x}{\sqrt{1-x^2}}\right) + K$$



57 $\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int \cos x\,dx - \int \sin^2 x\cos x\,dx = \sin x - (\sin^3 x)/3 + K$

58 $\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx = \int (-1+2v^2-v^4)\,dv$

↳ $v = \cos x$, $dv = -\sin x\,dx$

$$= -v + \frac{2}{3}v^3 - \frac{v^5}{5} + K = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + K$$

59 $\int \frac{\cos^5 x}{\operatorname{sen}^3 x}\,dx = \int \frac{\cos^4 x \cos x\,dx}{\operatorname{sen}^3 x} = \int \frac{(1-\operatorname{sen}^2 x)^2 \cos x\,dx}{\operatorname{sen}^3 x} = \int \frac{(1-v^2)^2}{v^3}\,dv = \int \left(\frac{1}{v^3} - \frac{2}{v} + v\right)dv$

↳ $v = \operatorname{sen} x$, $dv = \cos x\,dx$

$= \frac{-1}{2v^2} - 2\ln|v| + \frac{v^2}{2} = \frac{-1}{2\operatorname{sen}^2 x} - 2\ln|\operatorname{sen} x| + \frac{\operatorname{sen}^2 x}{2} + K$

60 $\int \operatorname{sen}^3\left(\frac{x}{2}\right)\cos^5\left(\frac{x}{2}\right)dx = \int \operatorname{sen}^3 t \cos^5 t\,2\,dt = 2\int \operatorname{sen}^3 t \cos^5 t\,dt = 2\int (1-\cos^2 t)\cos^5 t \operatorname{sen} t\,dt$

↳ $x = 2t$, $dx = 2dt$

$= -2\int (1-v^2)v^5\,dv = -2\int (v^5 - v^7)\,dv = -2\left(v^6/6 - v^8/8\right) = \frac{\cos^8 t}{4} - \frac{\cos^6 t}{3} = (1/4)\cos^8\left(\frac{x}{2}\right) - (1/3)\cos^6\left(\frac{x}{2}\right)$

↳ $v = \cos t$, $dv = -\operatorname{sen} t\,dt$

61 $\int \frac{dx}{\operatorname{sen}^5 x \cos^3 x} = \int \frac{\cos^5 x}{\operatorname{sen}^5 x}\,\frac{1}{\cos^8 x}\,dx = \int \frac{\sec^8 x\,dx}{\operatorname{tg}^5 x} = \int \frac{(1+\operatorname{tg}^2 x)^3 \sec^2 x}{\operatorname{tg}^5 x}\,dx$

$= \int \frac{(1+3\operatorname{tg}^2 x + 3\operatorname{tg}^4 x + \operatorname{tg}^6 x)\sec^2 x\,dx}{\operatorname{tg}^5 x} = \int \operatorname{tg}^{-5} x \sec^2 x\,dx + 3\int \operatorname{tg}^{-3} x \sec^2 x\,dx + 3\int \operatorname{tg}^{-1} x \sec^2 x\,dx + \int \operatorname{tg} x \sec^2 x\,d$

$= -\frac{\operatorname{tg}^{-4} x}{4} - \frac{3\operatorname{tg}^{-2} x}{2} + 3\ln|\operatorname{tg} x| + \frac{\operatorname{tg}^2 x}{2} + K$

62 $\int \operatorname{sen}^4 x\,dx = \int (\operatorname{sen}^2 x)^2\,dx = \int \left(\frac{1}{2} - \cos 2x/2\right)^2 dx = (1/4)\int (1 - 2\cos 2x + \cos^2 2x)\,dx$

$= (1/4)\int \left(1 - 2\cos 2x + 1/2 + \cos 4x/4\right)dx = \frac{3x}{8} - \frac{\operatorname{sen} 2x}{4} + \frac{\operatorname{sen} 4x}{32} + K$

63 $\int \operatorname{sen}^2 x \cos^5 x\,dx = \int v^2 (1-v^2)^2\,dv = \int (v^2 - 2v^4 + v^6)\,dv = v^3/3 - 2v^5/5 + v^7/7$

↳ $v = \operatorname{sen} x$, $dv = \cos x\,dx$

$= \frac{\operatorname{sen}^3 x}{3} - \frac{2\operatorname{sen}^5 x}{5} + \frac{\operatorname{sen}^7 x}{7} + K$

64 $\int \operatorname{sen}^2 x \cos^4 x\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\,dx$

$= \frac{1}{8}\int \left[1 + \cos 2x - \left(\frac{1}{2} + \frac{\cos 4x}{2}\right) - (1 - \operatorname{sen}^2 2x)\cos 2x\right]dx = \frac{1}{8}\int \left(\frac{1}{2} - \frac{\cos 4x}{2} + \operatorname{sen}^2 2x \cos 2x\right)dx$

$\therefore \int \operatorname{sen}^2 x \cos^4 x\,dx = \frac{x}{16} - \frac{\operatorname{sen} 4x}{64} + \frac{\operatorname{sen}^3 2x}{48} + K$

65 $\int \cos^6(3x)\,dx = \int \frac{\cos^6 v}{3}\,dv = \frac{1}{3}\int (1/2 + \cos 2v/2)^3\,dv = \frac{1}{24}\int (1+3\cos(2v)+3\cos^2(2v)+\cos^3($

↳ $v = 3x$, $dv = 3dx$

$= \frac{1}{24}\int [1 + 3\cos(2v) + 3(1/2 + \cos 4v/2) + (1-\sen^2 2v)\cos 2v]\,dv$

$= \frac{1}{24}\int \left[\frac{5}{2} + 4\cos(2v) + \frac{3}{2}\cos(4v) - \sen^2(2v)\cos 2v\right]dv = \frac{1}{24}\left[\frac{5v}{2} + 2\sen(2v) + \frac{3}{8}\sen(4v) - \frac{\sen^3(2v)}{6}\right.$

$= \frac{1}{24}\left[\quad 2\sen(6x) + \frac{5}{2}3x + \frac{3}{8}\sen(12x) - \frac{\sen^3(6x)}{6}\right] = \underline{\frac{5x}{16} + \frac{\sen 6x}{12} + \frac{\sen 12x}{64} - \frac{\sen^3 6x}{144} + K}$

66 $\int \frac{\cos^2 x}{\sen^6 x}\,dx = \int \frac{\cos^2 x}{\sen^2 x}\frac{1}{\sen^4 x}\,dx = \int \cotg^2 x(\cossec^2 x)(\cossec^2 x\,dx)$

$= \int \cotg^2 x(1+\cotg^2 x)\cossec^2 x\,dx = -\int v^2(1+v^2)\,dv = \frac{-v^3}{3} - \frac{v^5}{5} = \underline{-\frac{\cotg^3 x}{3} - \frac{\cotg^5 x}{5} + K}$

↳ $v = \cotg x$, $dv = -\cossec^2 x\,dx$

67 $\int \frac{1}{\sen^2 x\cos^4 x}\,dx = \int \frac{(\sen^2 x + \cos^2 x)}{\sen^2 x\cos^4 x}\,dx = \int\left[\frac{1}{\cos^4 x} + \frac{1}{\sen^2 x\cos^2 x}\right]dx$

$= \int \sec^4 x\,dx + \int \frac{(\sen^2 x + \cos^2 x)}{\sen^2 x\cos^2 x}\,dx = \int \sec^4 x\,dx + \int \sec^2 x\,dx + \int \cossec^2 x\,dx = \int \sec^4 x\,dx + \tg x - \cotg x$

$\int \sec^4 x\,dx = \int \sec^2 x\cdot\sec^2 x = \int (1+\tg^2 x)\sec^2 x\,dx = \int (1+v^2)\,dv = v + v^3/3 = \tg x + \frac{\tg^3 x}{3}$

↳ $v = \tg x$, $dv = \sec^2 x\,dx$

$\therefore \int \frac{1}{\sen^2 x\cos^4 x}\,dx = \underline{2\tg x - \cotg x + \frac{\tg^3 x}{3} + K}$ → aparentemente está correcto, mas diferente do gabarito

68 $\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx \overset{\times\sqrt{1-x}}{=} \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int\left[\frac{1}{\sqrt{1-x^2}} - \frac{2x}{2\sqrt{1-x^2}}\right]dx = \underline{\arcsen x + \sqrt{1-x^2} + K}$

69 $\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{1}{(v^3 - v^2)}\,6v^5\,dv = 6\int \frac{v^5}{v^3 - v^2}\,dv = 6\int \frac{v^3}{v-1}\,dv = 6\int \frac{(v-1)(v^2+v+1)+1}{v-1}$

↳ $x = v^6$, $dv = (1/6)x^{-5/6}\,dx$ ↳ $v^3 \,|\, v-1$ (...)

↳ $v = x^{1/6}$

$= 6\left[\int (v^2+v+1)\,dv + \int \frac{1}{v-1}\,dv\right] = 6\left(\frac{v^3}{3} + \frac{v^2}{2} + v + \ln|v-1|\right) = 2v^3 + 3v^2 + 6v + 6\ln|v-1|$

$\therefore \int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \underline{2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + K}$

70 Zelando por sua própria sanidade, sugiro que nem tente. Coloquei um extra ao final para compensar.

71 $\int \frac{\operatorname{arctg} x}{x^2}\,dx = -\frac{\operatorname{arctg} x}{x} + \underbrace{\int \frac{1}{x(1+x^2)}\,dx}$

↳ $\frac{1}{x} = f' \to f = -\frac{1}{x}$; $\operatorname{arctg} x = g \to g' = \frac{1}{(1+x^2)}$

↳ $\int\left[\frac{A}{x} + \frac{Bx+C}{1+x^2}\right]dx \Rightarrow A(1+x^2) + x(Bx+C) = 1$

$x=0 \to A=1$

$x=1 \to B+C=-1$

$x=-1 \to B-C=-1$ $\Big\}$ $B=-1$, $C=0$

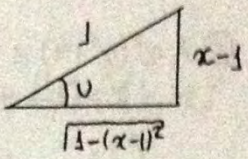
↳ $\int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = \ln|x| - \frac{\ln|1+x^2|}{2}$

$$\therefore \int \frac{\operatorname{arctg} x}{x^2}\,dx = -\frac{\operatorname{arctg} x}{x} + \ln|x| - \frac{\ln|1+x^2|}{2} + K$$

OBS.: $-\ln|1+x^2|/2 = -\ln(|1+x^2|)^{1/2} = -\ln\sqrt{1+x^2}$

72 $\int \frac{x^2}{\sqrt{2x-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx = \int \frac{(\operatorname{sen}^2 v + 2\operatorname{sen} v + 1)}{\cos v}\cos v\,dv = \int \operatorname{sen}^2 v\,dv + 2\int \operatorname{sen} v\,dv + \int 1\,dv$

↳ $x-1 = \operatorname{sen} v$, $dx = \cos v\,dv$



$\cos v = \sqrt{1-(x-1)^2}$, $v = \operatorname{arcsen}(x-1)$

$= \frac{v}{2} - \frac{\operatorname{sen} 2v}{4} - 2\cos v + v = \frac{3}{2}v - \frac{2\operatorname{sen} v \cos v}{4} - 2\cos v$

$$= \frac{3}{2}\operatorname{arcsen}(x-1) - \frac{(x-1)(\sqrt{2x-x^2})}{2} - 2\sqrt{2x-x^2} + K$$

OBS.: $\sqrt{2x-x^2} = a \to -\frac{(x-1)a}{2} - 2a = -\frac{(x-1+4)a}{2} = -\frac{(x+3)\sqrt{2x-x^2}}{2}$

73 $\int \frac{(4x^2-3x+3)}{(x+1)(x^2-2x+2)}\,dx = \int\left[\frac{A}{(x+1)} + \frac{Bx+C}{x^2-2x+2}\right] = \int\left[\frac{2}{x+1} + \frac{2x-1}{x^2-2x+2}\right]dx = 2\ln|x-1| + \int \frac{2x-1}{x^2-2x+2}\,dx$

↳ $A(x^2-2x+2) + (Bx+C)(x+1) = 4x^2-3x+3$

$(\ldots) \therefore A=2,\ B=2,\ C=-1$

$x = v+1 \leftarrow v = x-1$, $dx = dv$

$$\int \frac{2x-1}{x^2-2x+2}\,dx = \int \frac{2x-1}{(x-1)^2+1}\,dx = \int \frac{2(v+1)-1}{v^2+1}\,dv = \int\left[\frac{2v}{v^2+1} + \frac{1}{v^2+1}\right]dv$$

$$= \ln(v^2+1) + \operatorname{arctg} v = \ln(x^2-2x+2) + \operatorname{arctg}(x-1)$$

$$\therefore \int \frac{(4x^2-3x+3)}{(x+1)(x^2-2x+2)}\,dx = 2\ln|x+1| + \ln(x^2-2x+2) + \operatorname{arctg}(x-1) + K$$

↳ saí do módulo pela mesma razão do ex. 51

74 $\int \frac{dx}{1+e^x} = \int \frac{1}{(1+t)}\cdot\frac{1}{t}\,dt = \int\left(\frac{A}{t} + \frac{B}{(t+1)}\right)dt = \int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \ln|t| - \ln|t+1|$

↳ $x = \ln t$, $t = e^x$, $dx = \frac{1}{t}\,dt$

↳ $A(t+1) + Bt = 1$: $t=-1 \to B=-1$; $t=0 \to A=1$

$$\therefore \int \frac{dx}{1+e^x} = \ln|e^x| - \ln|e^x+1| = x - \ln(1+e^x) + K$$

75 $\int \underbrace{\ln(x+1)}_{f} \cdot \underbrace{\frac{1}{x^2}}_{g'} dx = -\frac{\ln(x+1)}{x} + \int \underbrace{\frac{1}{x(x+1)}}_{\int\left[\frac{A}{x}+\frac{B}{x+1}\right]dx} dx$ $\rightarrow f=\frac{1}{x+1}, g=\frac{-1}{x}$

$A(x+1)+Bx=1$

$x=0 \rightarrow A=1$; $x=-1 \rightarrow B=-1$

$= -\frac{\ln(x+1)}{x} + \int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \boxed{-\frac{\ln(x+1)}{x} + \ln|x| - \ln|x+1| + K}$

76 $\int x^5 e^{-x^3} dx = \int x^3 (e^{-x^3} x^2) dx = \int -u e^u (-1/3) du = (1/3)\int \underset{f}{u} \underset{g'}{e^u} du = (1/3)[ue^u - \int e^u du]$

$\hookrightarrow u=-x^3, du=-3x^2 dx$

$= (1/3)(ue^u - e^u) = (1/3)(-x^3 e^{-x^3} - e^{-x^3}) = \boxed{-\frac{e^{-x^3}}{3}\cdot(x^3+1)+K}$

77 $\int \frac{x+1}{x^2(x^2+4)} dx = \int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}\right)dx = \underset{\text{(I)}}{\frac{1}{4}}\int\left(\frac{1}{x}+\frac{1}{x^2}-\frac{x+1}{x^2+4}\right)dx = \frac{\ln|x|}{4} - \frac{1}{4x} \underbrace{-\frac{1}{4}\int\frac{x+1}{x^2+4}dx}_{\text{(II)}}$

(I) $Ax^2(x^2+4)+B(x^2+4)+x^3(Cx+D) = x+1 \Leftrightarrow Ax^3+4Ax+Bx^2+4B+Cx^3+Dx^2 = x+1$

$x=0 \rightarrow 4B=1 \therefore B=1/4$

$x^3(A+C)=0 \therefore A+C=0$

$x^2(B+D)=0 \therefore B+D=0$

$4A=1 \therefore A=1/4$

(II) $= -\frac{1}{4}\cdot\frac{1}{2}\left[\int\frac{2x}{x^2+4}dx + 2\int\frac{1}{x^2+4}dx\right] = -\frac{\ln|x^2+4|}{8} - \operatorname{arctg}\left(\frac{x}{2}\right)\frac{1}{8}$

$\therefore \boxed{\int\frac{x+1}{x^2(x^2+4)}dx = \frac{\ln|x|}{4} - \frac{1}{4x} - \frac{\ln|x^2+4|}{8} - \operatorname{arctg}\left(\frac{x}{2}\right)\frac{1}{8} + K}$

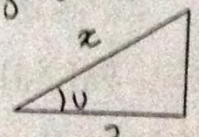
↳ pode sair do módulo pela razão do ex 51

78 (Extra)

$\int \frac{\sqrt{x^2-9}}{x^3} dx = 3\int \frac{\sqrt{(\frac{x}{3})^2-1}}{x^3} dx = 3\int \frac{|tgu|}{27\sec^3 u}\cdot 3\sec u tgu du = \frac{1}{3}\int \frac{tg^2 u}{\sec^2 u} du$

↳ $\sec u = x/3$, $dx = 3\sec u tgu du$

$= \frac{1}{3}\int \operatorname{sen}^2 u du = \frac{1}{3}\left(\frac{u}{2} - \frac{\operatorname{sen} 2u}{4}\right) = \frac{u}{6} - \frac{2\operatorname{sen} u\cos u}{12}$



$\operatorname{sen} u = \sqrt{1-9/x^2}$

$\cos u = 3/x$

$= \frac{1}{6}\operatorname{arcsen}\left(\frac{x}{3}\right) - \frac{\sqrt{1-9/x^2}}{6}\cdot\frac{3}{x}$

$\therefore \boxed{\int\frac{\sqrt{x^2-9}}{x^3}dx = \frac{1}{6}\operatorname{arcsen}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + K}$

③ Outras aplicações da integral definida

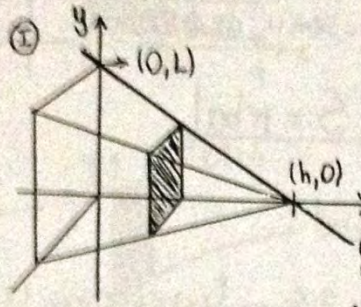
1- $\int_{-1}^{1} x^3 \operatorname{sen}(x^2+1)\,dx \Rightarrow \int x^2 \operatorname{sen}(x^2+1).x\,dx = (1/2)\int \underbrace{v}_{f}\underbrace{\operatorname{sen}(v+1)}_{g'}\,dv$

↳ $v = x^2$, $dv = 2x\,dx$

$= (1/2)\left(-v\cos(v+1) - \int -\cos(v+1)\,dv\right) = (1/2)\left(-x^2\cos(x^2+1) + \operatorname{sen}(x^2+1)\right)$

$\therefore \int_{-1}^{1} x^3 \operatorname{sen}(x^2+1) = \left[-\frac{x^2}{2}\cos(x^2+1) + \frac{\operatorname{sen}(x^2+1)}{2}\right]_{-1}^{1} = -\frac{\cos 2}{2} + \frac{\operatorname{sen}^2}{2} + \frac{\cos 2}{2} - \frac{\operatorname{sen}^2}{2} = \boxed{0}$

Obs.: Nesse caso ainda foi possível primitivar e seguir o caminho acima. Há casos em que isso não é possível, mas já é conhecido que o resultado daria zero. Vide explicação no exercício 5 teste.

2- ① 

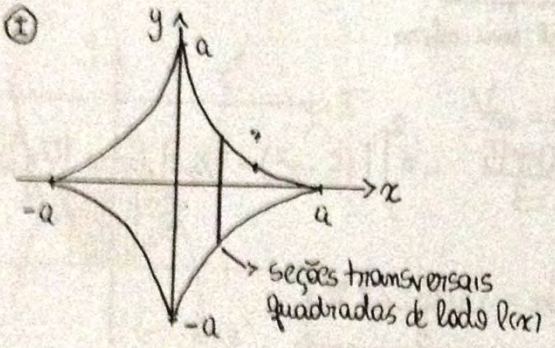
$\therefore y = \frac{L}{h}(h-x) = \ell(x)$

② $S_{\blacksquare}(x) = (\ell(x))^2 = (L/h)^2(h-x)^2$

$V = \int_0^h S_{\blacksquare}(x)\,dx = \frac{L^2}{h^2}\int_0^h (h^2 - 2hx + x^2)\,dx$

$V = \frac{L^2}{h^2}\left(h^2x - hx^2 + \frac{x^3}{3}\right)\Big|_0^h$

③ $\therefore \boxed{V = L^2h/3}$

3- ① 

② $x^{2/3} + y^{2/3} = a^{2/3}$

$\therefore f(x) = y = (a^{2/3} - x^{2/3})^{3/2}$

Pela simetria, $\ell(x) = 2f(x)$

Assim: $A(x) = \ell^2(x) = 4f^2(x)$

$\therefore A(x) = 4(a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)$

③ $V = \int_{-a}^{a} A(x)\,dx = 4\int_{-a}^{a}(a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\,dx$

$= 4\left[a^2x - 3a^{4/3}\cdot\frac{3}{5}x^{5/3} + 3a^{2/3}\cdot x^{7/3}\cdot\frac{3}{7} - \frac{x^3}{3}\right]_{-a}^{a} = 8\left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{a^3}{3}\right)$

$= \frac{8}{105}a^3(105 - 189 + 135 - 35)$

$\therefore \boxed{V = \frac{128}{105}a^3}$

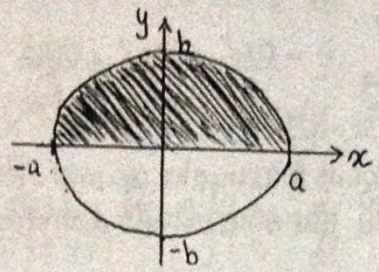
4- $f(x) = \ln(\cos x)$, $0 \le x \le \pi/4$

$C = \int_0^{\pi/4} \sqrt{1 + (f'(x))^2}\, dx$, $f'(x) = -\operatorname{sen}x/\cos x = -\operatorname{tg}x$

Assim: $C = \int_0^{\pi/4} \sqrt{1+\operatorname{tg}^2 x}\, dx = \int_0^{\pi/4} |\sec x|\, dx = [\ln|\sec x + \operatorname{tg}x|]_0^{\pi/4} = \ln(\sqrt{2}+1) - \ln 1$

$\therefore C = \ln(\sqrt{2}+1)$

5-



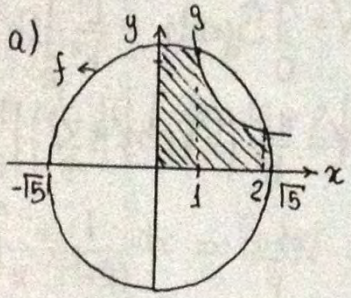
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow f(x) = y = +b\sqrt{1 - \frac{x^2}{a^2}}$

$S_{\text{●}} = \frac{S_0}{2} = b\int_{-a}^{a} \sqrt{1-(x/a)^2}\, dx \Leftrightarrow S = 2ba\int_{-\pi/2}^{\pi/2} \cos^2 v\, dv$

↳ $x/a = \operatorname{sen}v$, $dx/a = \cos v\, dv$

$S = 2ba\left[v/2 + \operatorname{sen}2v/4\right]_{-\pi/2}^{\pi/2} = 2ba \cdot 2\frac{\pi}{4}$ $\quad \therefore S = \pi ba$

6- a)



$xy \le 2 \Rightarrow y \le 2/x$

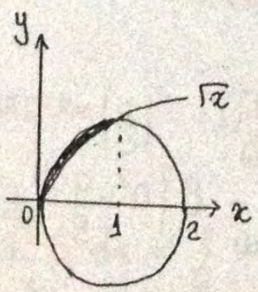
$f(x) = +\sqrt{5-x^2}$, $g(x) = 2/x$. $f(x) = g(x)$ em $(1,2)$ e $(2,1)$

$V_{0x} = \underbrace{\frac{2\pi 5\sqrt{5}}{3}}_{\text{vol. semi-esfera}} - \left[\pi\int_1^2 f^2(x)\, dx - \pi\int_1^2 g^2(x)\, dx\right]$

$= \frac{10\pi\sqrt{5}}{3} - \pi\int_1^2 (f^2(x) - g^2(x))\, dx = \frac{10\pi\sqrt{5}}{3} - \pi\int_1^2 [(5-x^2) - 4/x^2]\, dx = \frac{10\pi\sqrt{5}}{3} - \pi\left[5x - \frac{x^3}{3} + \frac{4}{x}\right]_1^2$

$= \frac{10\pi\sqrt{5}}{3} - \pi\left(3 - \frac{7}{3}\right)$ $\quad \therefore V_{0x} = \frac{(10\sqrt{5}-2)\pi}{3}$

b)



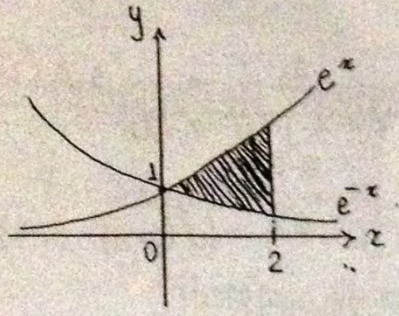
$f(x) = \sqrt{1-x^2}$, $g(x) = \sqrt{x}$. $f(x) = g(x)$ em $x = 0$ ou $x = 1$

$V_{0x} = \pi\int_0^1 \left[\left(\sqrt{1-x^2}\right)^2 - \left(\sqrt{x}\right)^2\right] dx = \pi\int_0^1 (1-x^2-x)\, dx$

$= \pi\left[x - x^3/3 - x^2/2\right]_0^1 = \pi(1 - 1/3 - 1/2)$

$\therefore V_{0x} = \pi/6$

c)



$f(x) = e^x$, $g(x) = e^{-x}$

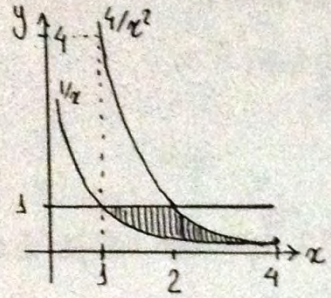
$$V_{0x} = \pi\int_0^2 [f^2(x) - g^2(x)]dx = \pi\int_0^2 (e^{2x} - e^{-2x})dx$$

$$= \left[\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right]_0^2 = \pi\left(\frac{e^4}{2} + \frac{e^{-4}}{2} - \frac{1}{2} - \frac{1}{2}\right)$$

$$\therefore V_{0x} = \pi(e^8 - 2e^4 + 1)/2e^4$$

OBS.: $(e^2 - e^{-2})^2 = (e^4 - 2 + e^{-4}) = (e^8 - 2e^4 + 1)/e^4$

d)



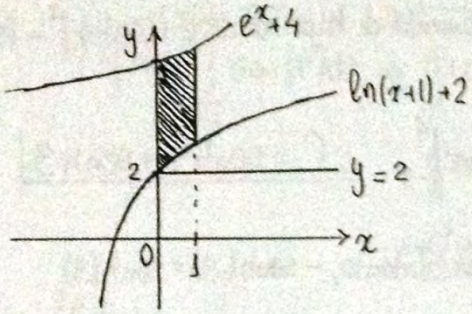
$$V_{0x} = \pi\left[\int_1^2 (1^2 - 1/x^2)\,dx - \int_2^4 (16/x^4 - 1/x^2)dx\right]$$

$$= \pi\left[x + 1/x\right]_1^2 + \pi\left[-\frac{16}{3}\frac{1}{x^3} + \frac{1}{x}\right]_2^4$$

$$= \pi(2 + 1/2 - 1 - 1 - 1/12 + 1/4 + 2/3 - 1/2)$$

$$\therefore V_{0x} = 5\pi/6$$

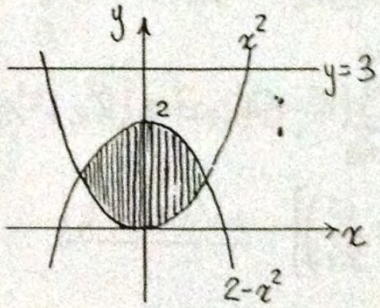
7 -



Rotacionar a região em torno de $y=2$ seria o mesmo que rotacionar $e^x + 4 - 2 = e^x + 2$ e $\ln(x+1) + 2 - 2 = \ln(x+1)$ em torno do eixo Ox. Sendo assim:

$$V_{0x} = \pi\left[\int_0^1 (e^x+2)^2 dx - \int_0^1 \ln^2(x+1)dx\right]$$

8 -



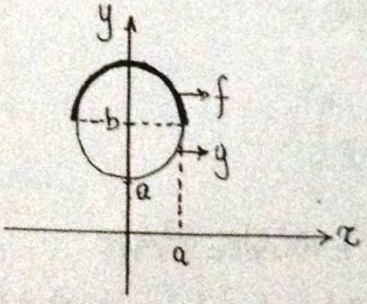
$$V_{0x} = \pi\int_{-1}^1 [(x^2-3)^2 - (-x^2+2-3)^2]dx$$

$$= \pi\int_{-1}^1 (x^4 - 6x^2 + 9 - x^4 - 2x^2 - 1)dx = 8\pi\int_{-1}^1 (1-x^2)dx$$

$$= -8\pi\left[\frac{x^3}{3} - x\right]_{-1}^1 = -8\pi\left(\frac{2}{3} - 2\right) = \frac{32}{3}\pi$$

$$\therefore V_{0x} = 32\pi/3$$

9 -



$f(x) = \sqrt{a^2 - x^2} + b$

$g(x) = -\sqrt{a^2 - x^2} + b$

$$V = \pi\int_{-a}^a [(b + \sqrt{a^2-x^2})^2 - (b - \sqrt{a^2-x^2})^2]dx$$

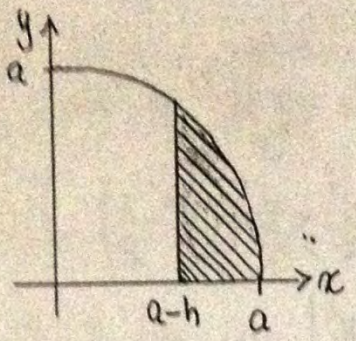
$$= \pi\int_{-a}^a (b^2 + 2b\sqrt{a^2-x^2} + a^2 + x^2 - b^2 + 2\sqrt{a^2-x^2} - a^2 - x^2)dx$$

$$= 4a^2\pi b\int_{-a}^a \sqrt{1-(x/a)^2}\,dx \qquad x/a = \operatorname{sen} u,\ dx/a = \cos u$$

$$V = 4a^2\pi b\int_{-\pi/2}^{\pi/2} \cos^2 u\,du = 4a^2\pi b\left[\frac{u}{2} + \frac{\operatorname{sen} 2u}{4}\right]_{-\pi/2}^{\pi/2} = 4a^2\pi b\frac{\pi}{2}$$

$$\therefore V = (2\pi b)(\pi a^2)$$

OBS.: tente entender a ideia desta resolução ao invés de resolver ao pé-da-letra como no enunciado, talvez seja mais simples

10- 

$$V = \pi \int_{a-h}^{a} \left(\sqrt{a^2-x^2}\right)^2 dx \Leftrightarrow V = \pi \int_{a-h}^{a} (a^2-x^2)\,dx$$

$$= \pi \left[a^2x - x^3/3\right]_{a-h}^{a} = \pi\left[a^3 - a^3/3 - (a^2(a-h) - (a-h)^3/3)\right]$$

$$= \pi\left[\frac{2a^3}{3} - (a-h)\frac{(a^2-a^2+2ah-h^2)}{3}\right]$$

$$= \pi\left[\frac{2a^3}{3} - \left(\frac{2a^3+2a^2h-ah^2-2a^2h-2ah^2+h^3}{3}\right)\right]$$

$$= \pi\left[\frac{2a^3}{3} - \frac{2a^3}{3} + ah^2 - \frac{h^3}{3}\right] \quad \therefore V = \pi h^2(a - h/3)$$

11- $y = \cosh x$, $x \in [-3,4]$. $y = f(x) \Rightarrow f'(x) = \operatorname{senh} x$ (parecidas relações de seno e cosseno de ângulos valem para senos e cossenos hiperbólicos). Logo:

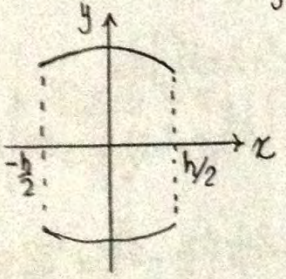
$$C = \int_{-3}^{4} \sqrt{1+(\operatorname{senh} x)^2}\,dx \Rightarrow \text{relação fundamental de hiperbólicas é } (\cosh x)^2 - (\operatorname{senh} x)^2 = 1$$

Assim:

$$C = \int_{-3}^{4} \sqrt{(\cosh x)^2}\,dx = \int_{-3}^{4} \cosh x\,dx = \operatorname{senh} x \Big|_{-3}^{4} \quad \therefore C = \operatorname{senh} 4 + \operatorname{senh} 3$$

OBS.: assim como $\operatorname{sen}\theta$, $\operatorname{senh} x$ também é ímpar. Portanto, $-\operatorname{senh}(-3) = \operatorname{senh}(3)$

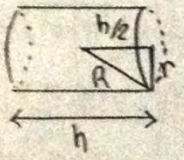
12- Ⓘ Anel deitado, sem furo:



$$V = \pi \int_{-\pi/2}^{\pi/2} \left(\sqrt{R^2-x^2}\right)^2 dx = \pi \int_{-\pi/2}^{\pi/2} (R^2-x^2)\,dx = \pi\left[R^2x - x^3/3\right]_{-h/2}^{h/2}$$

$$= \pi\left[R^2\frac{h}{2} - \frac{h^3}{24} - \left(\frac{hR^2}{2} + \frac{h^3}{24}\right)\right] \quad \therefore V = \pi R^2 h$$

Ⓘ $V_{FURO} = V_{CILINDRO} = \pi\left(R^2 - \frac{h^2}{4}\right).h = \pi\left(R^2h - h^3/4\right)$



$$r^2 = R^2 - \frac{h^2}{4}$$

Ⓘ $V_{ANEL} = V - V_{CILINDRO} =$ Ⓘ − Ⓘ $= \pi R^2 h - \pi R^2 h + (h^3/4)\pi = (h^3/4)\pi$

$\therefore V_{ANEL} = (h^3/4)\,\pi$ Logo, o volume do anel depende de h, mas não de R

④ Aplicações do teorema fundamental do cálculo

1- Efetuando-se mais uma demonstração que uma prova rigorosa de fato, segue:

Para uma primitiva $F(t)$ de $f(t)$, ou seja, $F'(t) = f(t)$, tem-se $\int_{U(x)}^{V(x)} f(t)\,dt = F(V(x)) - F(U(x))$. Assim: $\frac{d}{dx}\int_{U(x)}^{V(x)} f(t)\,dt = \frac{d}{dx}F(V(x)) - \frac{d}{dx}F(U(x))$

↳ regra da cadeia

$$= F'(V(x))\frac{dv}{dx} - F'(U(x))\frac{du}{dx} = \underline{\underline{f(V(x))\frac{dv}{dx} - f(U(x))\frac{du}{dx}}}$$

2- a) $g(x) = \int_{\cos x}^{\operatorname{sen} x} e^{t^2}\,dt \Rightarrow g'(x) = e^{\operatorname{sen}^2 x}\frac{d\operatorname{sen} x}{dx} - e^{\cos^2 x}\cdot\frac{d\cos x}{dx}$

$$\therefore \underline{\underline{g'(x) = e^{\operatorname{sen}^2 x}\cos x + e^{\cos^2 x}\operatorname{sen} x}}$$

b) $g(x) = \int_{\sqrt{x}}^{2\sqrt{x}} \operatorname{sen}(t^2)\,dt \Rightarrow g'(x) = \operatorname{sen}(4x)\frac{d}{dx}2\sqrt{x} - \operatorname{sen} x\cdot\frac{d}{dx}\sqrt{x}$

$$g'(x) = \operatorname{sen} 4x\cdot x^{-1/2} - \operatorname{sen} x\cdot x^{-1/2}\cdot\frac{1}{2}$$

$$\therefore \underline{\underline{g'(x) = x^{-1/2}(\operatorname{sen} 4x - \operatorname{sen} x/2)}}$$

3- $\int_0^2 x\,F(x)\,dx$. Integrando por partes, para $f' = x$ e $g = F(x)$:

$$\int_0^2 x\,F(x)\,dx = \frac{x^2}{2}F(x)\Big|_0^2 - \int_0^2 \frac{x^2}{2}\sqrt{1+x^3}\,dx \qquad \begin{matrix} u = 1+x^3 \\ du = 3x^2\,dx \end{matrix} \qquad \begin{matrix} x=0 \to u=1 \\ x=2 \to u=9 \end{matrix}$$

$$= 2F(2) - 0\cdot F(0) - \int_0^9 \frac{1}{3}\cdot\frac{1}{2}\cdot u^{1/2}\,du$$

$$= 2F(2) - (1/6)(2/3)\,u^{3/2}\Big|_1^9 = 2F(2) - 26/9$$

$$\therefore \underline{\underline{\int_0^2 x\,F(x)\,dx = 2F(2) - 26/9}}$$

4- $\operatorname{sen}(x-t) = \operatorname{sen}x\cos t - \operatorname{sen}t\cos x$. Assim:

$$y(x) = \int_0^x [\operatorname{sen}x\cos t\, f(t) - \operatorname{sen}t\cos x\, f(t)]\,dt = \operatorname{sen}x\int_0^x \cos t\, f(t)\,dt - \cos x\int_0^x \operatorname{sen}t\, f(t)\,dt$$

$$y'(x) = \left[\cos x\int_0^x \cos t\, f(t)\,dt + \operatorname{sen}x\cos x\, f(x)\right] + \left[\operatorname{sen}x\int_0^x \operatorname{sen}t\, f(t)\,dt - \cos x\operatorname{sen}x\, f(x)\right]$$

(I) $\therefore y'(x) = \cos x\int_0^x \cos t\, f(t)\,dt + \operatorname{sen}x\int_0^x \operatorname{sen}t\, f(t)\,dt$. De tal modo:

$$y''(x) = \left[-\operatorname{sen}x\int_0^x \cos t\, f(t)\,dt + \cos x.\cos x\, f(x)\right] + \left[\cos x\int_0^x \operatorname{sen}t\, f(t)\,dt + \operatorname{sen}x.\operatorname{sen}x\, f(x)\right]$$

$$y''(x) = -\underbrace{\left[\operatorname{sen}x\int_0^x \cos t\, f(t)\,dt - \cos x\int_0^x \operatorname{sen}t\, f(t)\,dt\right]}_{y(x)} + f(x)\underbrace{(\operatorname{sen}^2 x + \cos^2 x)}_{1}$$

$y''(x) = -y(x) + f(x)$ $\therefore$ $\underline{y''(x) + y(x) = f(x)\ \forall x \in I}$!

De (I): $y'(0) = \cos 0.\underbrace{\int_0^0 \cos t\, f(t)\,dt}_{0} + \operatorname{sen}0.\underbrace{\int_0^0 \operatorname{sen}t\, f(t)\,dt}_{0} = 0\ \forall x \in I$

Ainda, $y(0) = \operatorname{sen}0\underbrace{\int_0^0 \cos t\, f(t)\,dt}_{0} - \cos 0\underbrace{\int_0^0 \operatorname{sen}t\, f(t)\,dt}_{0} = 0\ \forall x \in I$

$\underline{\therefore y(0) = y'(0) = 0\ \forall x \in I}$!

5- $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \cos(t^2)\,dt}{\int_0^x e^{-t^2}\,dt}$:

Ambos os extremos superiores de integração das integrais definidas tendem a zero. Como o extremo inferior é zero (valor fixo), pode-se inferir que o intervalo de integração tende a zero e, assim, ambos os resultados das integrais tendem a zero.

Sendo assim, por serem funções contínuas, é possíver aplicar a regra de L'Hospital:

$$\lim_{x\to 0} \frac{\int_0^{x^2} \cos(t^2)\,dt}{\int_0^x e^{-t^2}\,dt} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\cos(x^2).2x - \cancelto{0}{\cos(x^2)\frac{d0}{dx}}}{e^{-x^2}(1) - \cancelto{0}{e^{-x^2}\frac{d0}{dx}}} = \lim_{x\to 0} \frac{\cos(x^2)2x}{e^{-x^2}} = \frac{0}{1} = \underline{0}!$$

6- $f'(x) = \frac{1}{\left(\frac{1}{x}\right)^2+1}\left(\frac{-1}{x^2}\right) + \frac{1}{x^2+1} = \frac{(-1/x^2)}{(1+x^2)(1/x^2)} + \frac{1}{x^2+1} = 0 \;\forall x>0$

Como $f'(x) = 0 \;\forall x>0$, então $f(x)$ é constante em $(0,\infty)$. Para $x\to\infty$:

$$\lim_{x\to\infty} f(x) = \underbrace{\lim_{a\to+\infty} \operatorname{arctg}(t)\Big|_0^{1/a}}_{0} + \lim_{b\to+\infty} \operatorname{arctg}(t)\Big|_0^{b}$$

$$\therefore \lim_{x\to\infty} f(x) = \lim_{b\to+\infty} \operatorname{arctg}(b) - \operatorname{arctg}0 = \underline{\pi/2}$$

7- $f(x) = \int_0^x e^{(x^2-t^2)/2}\,dt = e^{x^2/2}\int_0^x e^{-t^2/2}\,dt$. Assim.

$$f'(x) = \underbrace{xe^{x^2/2}\int_0^x e^{-t^2/2}\,dt}_{f(x)} + \underbrace{e^{x^2/2}\cdot e^{-x^2/2}}_{1} = xf(x)+1$$

$$\therefore f'(x) - xf(x) = 1 \;\forall x\in\mathbb{R}$$

8- a) $F(x) = \int_1^x \sqrt{t^3-1}\,dt \Rightarrow F'(x) = \sqrt{x^3-1}$

$$C = \int_1^4 \sqrt{1+(F'(x))^2}\,dx = \int_1^4 \sqrt{1+x^3-1}\,dx = \int_1^4 x^{3/2}\,dx = (2/5)\,x^{5/2}\Big|_1^4$$

$$= \frac{2}{5}(4^2 - 4^{1/2}) - \frac{2}{5} = \frac{62}{5} \qquad \therefore C = 62/5$$

(gab. errado)

b) $\lim_{x\to2} \frac{F(x^3)-F(8)}{\operatorname{sen}(x-2)} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to2} \frac{(F(x^3))'}{\cos(x-2)} = \lim_{x\to2} \frac{3x^2F'(x^3)}{\cos(x-2)}$

$$= \lim_{x\to2} \frac{3x^2\sqrt{x^9-1}}{\cos(x-2)} = 12\sqrt{2^9-1} = 12\sqrt{511}$$

$$\therefore \lim_{x\to2} \frac{F(x^3)-F(8)}{\operatorname{sen}(x-2)} = 12\sqrt{511}$$

9- (I) Primeiramente,

$$\left|e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n\right)\right| < \frac{|x|^{n+1}}{(n+1)!}$$

Assim, para todo $x \geq 0$:

$$\left|e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n\right)\right| \leq e^x \frac{x^{n+1}}{(n+1)!}$$

Substituindo x por x^2, pelo fato de $x^2 \geq 0 \ \forall x$, tem-se:

$$\left|e^{x^2} - \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \ldots + \frac{1}{n!}x^{2n}\right)\right| \leq e^{x^2} \frac{x^{2n+2}}{(n+1)!}$$

(II) Para aproximar integrais definidas por Taylor é necessário saber o seguinte:

Sendo f e g duas funções contínuas em um intervalo $[a,b]$ no qual $0 \leq f(x) \leq g(x)$ então $0 \leq \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

De tal modo, supondo f contínua, então $|f|$ também é em $[a,b]$. Assim:

$$-|f(x)| \leq f(x) \leq |f(x)| \Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

Ou seja, $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Aplicando (I) e (II) ao exercício, obtém-se, sabendo que $e^{x^2} \leq e < 3$ para $x \in [0,1]$:

$$\left|e^{x^2} - \left(1 + x^2 + \frac{1}{2}x^4 + \ldots + \frac{1}{n!}x^{2n}\right)\right| \leq 3 \cdot \frac{x^{2n+2}}{(n+1)!}$$

$$\left|\int_0^1 e^{x^2}\,dx - \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4 + \ldots + \frac{1}{n!}x^{2n}\right)dx\right| \leq \int_0^1 \left|e^{x^2} - \left(1 + x^2 + \frac{1}{2}x^4 + \ldots + \frac{1}{n!}x^{2n}\right)\right|dx$$

$$\leq \int_0^1 \frac{3x^{2n+2}}{(n+1)!}\,dx$$

$$\int_0^1 \frac{3x^{2n+2}}{(n+1)!} = \frac{1}{(n+1)!}\,\frac{3x^{2n+2+1}}{(2n+3)}\Bigg|_0^1 = \frac{3}{(2n+3)(n+1)!}$$

Portanto:

$$\left|\int_0^1 e^{x^2}dx - \int_0^1 \left(1+x^2+\frac{1}{2}x^4+\ldots+\frac{1}{n!}x^{2n}\right)dx\right| \leq \frac{3}{(2n+3)(n+1)!}$$

Chutando, para $n=7$, tem-se que:

$$\frac{3}{(2n+3)(n+1)!} < 10^{-5}$$; ou seja, para $\underline{n=7}$, obtém-se o erro inferior a 10^{-5}

OBS.: essa resolução é apresentada no Guidorizzi

10 - Para mostrar que $\left|\int_0^1 \cos(x^2)dx - \left(1-\frac{1}{5.2!}+\frac{1}{9.4!}-\frac{1}{13.6!}\right)\right| \leq \frac{1}{15.7!}$:

Sejam $f(x)=\cos x$ e $x_0=0 \Rightarrow f'(x)=-\operatorname{sen}x$, $f''(x)=-\cos x$, $f'''(x)=\operatorname{sen}x$
$\ldots f^{(7)}(x)=\operatorname{sen}x$

Para $n=6 \Rightarrow P_6(x) = 1 - x^2/2! + x^4/4! - x^6/6!$

$\cos x \approx P_6(x)$, $\cos(x^2) \approx P_6(x^2) = 1 - x^4/2! + x^8/4! - x^{12}/6!$

$$\left|\int_0^1 \cos x^2 - \int_0^1 P_6(x^2)dx\right| = \left|\int_0^1 (\cos(x^2)-P_6(x^2))dx\right| \leq \int_0^1 |\cos(x^2)-P_6(x^2)|dx$$

$$\int_0^1 |\cos(x^2)-P_6(x^2)|dx = \int_0^1 |E_6(x^2)|dx = \int_0^1 \left|\frac{\operatorname{sen}\bar{x}}{7!}x^{14}\right|dx$$

$$|\operatorname{sen}\theta| \leq 1 \Rightarrow \int_0^1 \left|\frac{\operatorname{sen}\bar{x}}{7!}x^{14}\right|dx \leq \int_0^1 \frac{1\cdot x^{14}}{7!}dx = \frac{x^{15}}{15}\frac{1}{7!}\Bigg|_0^1 = \frac{1}{15.7!}$$

Ainda, $\int_0^1 P_6(x)dx = \int_0^1 (1 - x^4/2! + x^8/4! - x^{12}/6!)dx = \left[x - \frac{x^5}{5.2!} + \frac{x^9}{9.4!} - \frac{x^{13}}{13.6!}\right]_0^1$

$$\therefore \int_0^1 P_6(x)dx = 1 - \frac{1}{5.2!} + \frac{1}{9.4!} - \frac{1}{13.6!}$$

Logo, como $\left|\int_0^1 (\cos(x^2)-P_6(x^2))dx\right| \leq \int_0^1 |E_6(x^2)|dx$, então

$$\underline{\left|\int_0^1 \cos(x^2)dx - \left(1-\frac{1}{5.2!}+\frac{1}{9.4!}-\frac{1}{13.6!}\right)\right| \leq \frac{1}{15.7!}}$$

OBS.: Sim, a 9 e a 10 são difíceis, mas são necessárias. Em prova, é preciso ter pelo menos um dos dois modelos em mente para resolver a questão.

⑤ Miscelânea

1- De Física:

$$W = \int_a^b F(x)\,dx$$

Vale ressaltar, entretanto, que existe uma fórmula mais geral para o cálculo de trabalho, especialmente o de forças e deslocamentos de mais de uma variável. Sem aprofundar muito, a fórmula é:

$$\mathcal{C} = \int_\gamma \vec{F}.d\vec{r}$$, ou seja, o trabalho é o somatório dos produtos escalares entre o vetor força ($\vec{F}$) e o deslocamento infinitesimal (vetor composto pela multiplicação entre o deslocamento escalar infinitesimal e um ou mais versores) ao longo de uma trajetória γ.

No caso de uma força variável que atua na direção de x positiva, movendo-se o corpo também nessa direção, ou seja, em uma trajetória $\gamma(x) = (x, 0)$, $x \in [a, b]$:

$$\mathcal{C} = \int_\gamma \vec{F}\,d\vec{r} = \int_a^b (F(x).\hat{\imath}).(dx.\hat{\imath}) = \int_a^b F(x).dx$$

OBS.: para alterar o sentido/direção da força ou deslocamento, são alterados os sinais dos versores ou eles próprios (ex.: $\vec{F}(x) = F(x).\hat{\jmath}$ para uma força vertical no sentido positivo de y, $d\vec{r} = dx(-\hat{\imath})$ para sentido negativo do x etc).
A razão de se efetuar um produto escalar é que feito isso, restará apenas a componente tangencial da força, que é a que realiza trabalho ($\vec{F}_{radial}.d\vec{r} = 0$).

2- $$W = \int_{x_1}^{x_2} \underbrace{\left(m\frac{dv}{dt}\right)}_{\hookrightarrow 2^a\ L.\ Newton}.dx = \int_{x_1}^{x_2} m\frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v(x)=\frac{dx}{dt}}.dx = \int_{x_1}^{x_2} m\,v(x).\frac{dv}{dx}\,dx$$

Pelo teorema da mudança de variável:

$$\int_{v_1}^{v_2} f(v)\,dv = \int_{x_1}^{x_2} f(v(x))\frac{dv(x)}{dx}\,dx$$. Assim:

$$\int_{x_1}^{x_2} m\,v(x)\frac{dv}{dx}\,dx = \int_{v_1}^{v_2} m\,v\,dv = m\frac{v^2}{2}\Big|_{v_1}^{v_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \Delta E_c$$

$$\therefore W = \Delta E_c$$

3- (I) $m(x'(t)) = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{m_0 c}{\sqrt{c^2-v^2}} = m_0 c\,(c^2-(x'(t))^2)^{-1/2}$

$m'(x'(t)) = m_0 c\,(-1/2)(c^2-v^2)^{-3/2}(-2(x'(t)))(x''(t))$

$= m_0 c\, x'(t)\,(c^2-v^2)^{-3/2}\, x''(t)$

↳ $v = (x'(t))$

(II) $\int_{x_0}^{x_1} f(x)\,dx = \int_{t_0}^{t_1} f(x(t))\,x'(t)\,dt = \int_{t_0}^{t_1} \underbrace{(m(x'(t))\,x'(t))'}_{\text{(III)}}\,x'(t)\,dt$

↳ t. mudança de variável

(III) $(m(x'(t))\,x'(t))' = \underbrace{m'(x'(t))}_{\text{(I)}}.x'(t) + m(x'(t))\,x''(t)$

$= \frac{m_0 c\, x''(t)(x'(t))^2}{(\sqrt{c^2-x'^2(t)})^3} + \frac{m_0 c}{(c^2-x'^2(t))^{1/2}}.x''(t)$

$= \frac{m_0 c\, x''(t)(x'(t))^2 + m_0 c\, x''(t)\,[c^2-(x'(t))^2]}{[c^2-(x'(t))^2]^{3/2}} = \frac{m_0 c^3 x''(t)}{[c^2-(x'(t))^2]^{3/2}}$

(IV) Substituindo (III) em (II):

$\int_{t_0}^{t_1} c^2 \cdot \frac{m_0 c\, x''(t)}{[c^2-(x'(t))^2]^{3/2}}\, x'(t)\,dt = \int_{t_0}^{t_1} \frac{dm(t)}{dt}\,dt = c^2\, m(t)\Big|_{t_0}^{t_1} = c^2(m(t_1)-m(t_0$

$\frac{dm(t)}{dt}$

$= \Delta m c^2$

$\therefore \int_{x_0}^{x_1} f(x)\,dx = \mathcal{E} = \Delta E = \Delta m c^2$ ∎

4- Por Leibnitz: $G'(x) = x\int_0^x e^{v^2}dv$ ∎

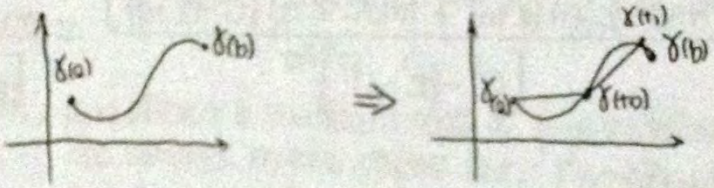
$G''(x) = 1.\int_0^x e^{v^2}dv + xe^{x^2}$ (regra do produto) $= \int_0^x e^{v^2}dv + xe^{x^2}$ ∎

5- A única forma viável de resolução desse exercício necessita de parametrização da curva.

No Cálculo I, são ensinadas as funções de uma a uma variável real ($\mathbb{R}\to\mathbb{R}$). Aos Cálculos II e III, todo o conhecimento será generalizado para funções de mais de uma variável ($\mathbb{R}\to\mathbb{R}^2$, $\mathbb{R}\to\mathbb{R}^3$, $\mathbb{R}^2\to\mathbb{R}$ etc). No começo de Cálculo II, ensina-se a parametrização de curvas. Por exemplo, a função $\mathbb{R}\to\mathbb{R}$ $y=x$ pode ser escrito como $\gamma_1(t)=(t,t)$, $t\in\mathbb{R}$, ou seja, uma função $\mathbb{R}\to\mathbb{R}^2$. Um caso comum é a parametrização de circunferências e elipses. Para $x^2+y^2=1$, $\gamma_2(t)=(x(t),y(t))=(\cos t,\operatorname{sen} t)$ em $t\in[0,2\pi[$ gera uma circunferência centrada na origem, de raio 1, percorrida uma vez no sentido anti-horário. Para elipses: $(x/a)^2+(y/b)^2=1$, $\gamma_3(t)=(a\cos t, b\operatorname{sen} t)$ para $t\in[0,2\pi[$ é uma forma de parametrizá-la.

Ainda, também será visto que a derivada de uma curva dada de forma paramétrica é a derivada das suas componentes. De tal modo, $\gamma_1'(t)=(1,1)$, $\gamma_2'(t)=(-\operatorname{sen} t,\cos t$ $\gamma_3'(t)=(-a\operatorname{sen} t, b\cos t)$. Ou seja, $\gamma'(t)=\left(\frac{dx}{dt}(t),\frac{dy}{dt}(t)\right)$. Além disso, será visto que $\gamma'(t)$ é um vetor, possuindo, assim, $\|\gamma'(t)\|=\sqrt{\left(\frac{dx}{dt}(t)\right)^2+\left(\frac{dy}{dt}(t)\right)^2}$.

No espaço, a curva $\gamma(t)$ é representada por:

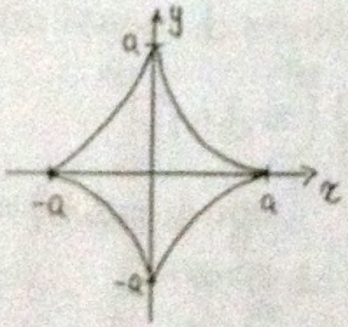


Essa segunda figura é para determinar o caso geral de comprimento de curvas.

A bem grosso modo, o comprimento (norma) do deslocamento infinitesimal ($\gamma'(t)$) é igual à relação anteriormente apresentada. Sendo assim, o comprimento de uma curva pode ser dado pelo somatório desses vários comprimentos, desde um ponto inicial "a" até um final "b". Em outras palavras:

$$L(\gamma)=\int_a^b \|\gamma'(t)\|\,dt$$

No caso do exercício, para o gráfico:



será parametrizada, a curva, em função de t para, em seguida, calcular um quarto do perímetro (não dela inteira porque a curva não é diferenciável nos bicos). Por fim, obtém-se o comprimento total desejado

Assim: $a^{2/3} = y^{2/3} + x^{2/3} \Leftrightarrow 1 = \left(\frac{y}{a}\right)^{2/3} + \left(\frac{x}{a}\right)^{2/3} = \left[\left(\frac{y}{a}\right)^{1/3}\right]^2 + \left[\left(\frac{x}{a}\right)^{1/3}\right]^2$

$\gamma(t) = (a\,\mathrm{sen}^3 t, a\cos^3 t)$. Para percorrer (1/4) da "elipse" $\Rightarrow 0 \le t \le \pi/2$

$\gamma'(t) = (a \cdot 3\,\mathrm{sen}^2 t \cos t, a \cdot 3\cos^2 t(-\mathrm{sen}\, t)) \Rightarrow \|\gamma'(t)\| = \sqrt{(a\,3\,\mathrm{sen}^2 t \cos t)^2 + (a\,3\cos^2 t(-\mathrm{sen}\, t))^2}$

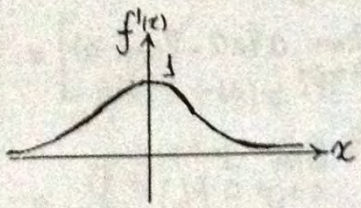
$= \sqrt{9a^2 \mathrm{sen}^2 t \cos^2 t (\mathrm{sen}^2 t + \cos^2 t)}$

$\therefore \|\gamma'(t)\| = 3a\,\mathrm{sen}\, t \cos t$

$\therefore \frac{L(\gamma)}{4} = \int_0^{\frac{\pi}{2}} \|\gamma'(t)\|\,dt = \int_0^{\pi/2} 3a\,\mathrm{sen}\, t \cos t\,dt = 3a \int_0^{\pi/2} \frac{\mathrm{sen}\, 2t}{2}\,dt = 3a\left[\frac{-\cos 2t}{2}\right]_0^{\pi/2} = \frac{3a}{2}$

$\therefore L(\gamma) = 6a$

6- a) $f'(x) = \frac{1}{\sqrt{1+x^4}} \ge 1 \ \forall x \in \mathbb{R}$ $\therefore f$ é estritamente crescente



Para uma função ímpar, segue que $g(x) = -g(-x)$

Assim, $g'(x) = -g'(-x).(-1) = g'(-x)$

Pelo gráfico ao lado, $f'(x) = f'(-x)$ $\therefore f$ é ímpar

b) $0 \le \frac{1}{\sqrt{1+t^4}} \le \frac{1}{t^2} \Rightarrow \int_1^x 0\,dt \le \int_1^x \frac{1}{\sqrt{1+t^4}}\,dt \le \int_1^x \frac{1}{t^2}\,dt$

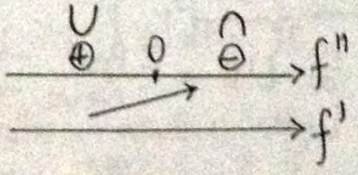
$f(x) = \int_0^x \frac{1}{\sqrt{1+t^4}}\,dt = \underbrace{\int_0^1 \frac{1}{\sqrt{1+t^4}}\,dt}_{f(1)} + \int_1^x \frac{1}{\sqrt{1+t^4}}\,dt \quad \therefore \int_1^x \frac{1}{\sqrt{1+t^4}}\,dt = f(x) - f(1)$

$\int_1^x \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_1^x = 1 - \frac{1}{x}$

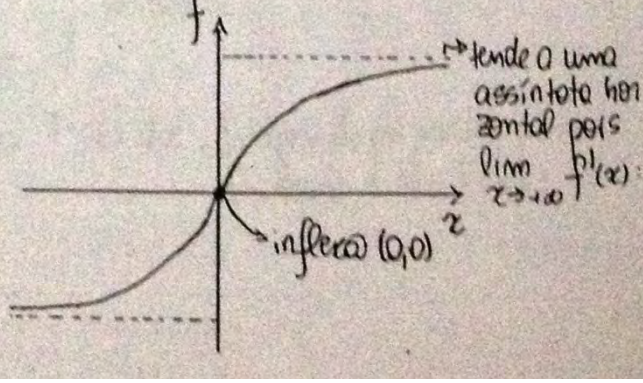
$\therefore 0 \le f(x) - f(1) \le 1 - 1/x$

$\therefore f(x) \le f(1) + 1 - 1/x \quad \forall x \ge 1$

d) $f''(x) = (1/\sqrt{1+x^2})' = (-1/2)(1+x^2)^{-3/2}\,2x = \frac{-x}{(1+x^2)^{3/2}}$



Esboço de f:



OBS.: não lembro o item ©, mas deixo o caminho que acredito ser o da solução. Após algumas tentativas, fica evidente que não é possível primitivar $\int \frac{1}{1+t^4}dt$ de modo viável. Sendo assim, a solução não é por integral imprópria. Possivelmente, ela está relacionada com a existência clara de uma assíntota horizontal quando $x \to \infty$ ou com o fato da existência de $\lim_{x\to\infty} f'(x)$. Se for pela primeira justificativa, talvez seja algo como encontrar m e n tais que $m = \lim_{x\to+\infty} \frac{f(x)}{x}$ e $n = \lim_{x\to+\infty} [f(x) - mx]$, ficando a cargo do leitor descobrir...

7 - (Itens @ e © têm erro de gabarito)

a) $\int_0^1 \ln x\,dx \Rightarrow x\ln x - \int x\frac{1}{x}dx = \int \ln x\,dx \therefore \int \ln x\,dx = x(\ln x - 1)$

↳ $f' = \ln x$, $g = 1$

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+} x(\ln x - 1)\Big|_a^1 = 1(\ln 1 - 1) - \lim_{a\to 0^+}(\ln a - 1)a = \underline{-1}$$

$$-\left[\lim_{a\to 0^+} a\ln a - \lim_{a\to 0^+}(-a)\right] = -\lim_{a\to 0^+}\frac{\ln a}{\frac{1}{a}}$$

$$\overset{\infty/\infty}{\underset{L'H}{=}} \lim_{a\to 0^+} \frac{1}{a}\left(\frac{-1}{a^2}\right) = 0$$

b) $\int_1^{+\infty} \frac{1}{1+x^2}dx = \lim_{z\to+\infty} \operatorname{arctg}(x)\Big|_1^z = \lim_{z\to+\infty} \operatorname{arctg}(z) - \operatorname{arctg}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\frac{\pi}{4}}$

c) $\int_0^{+\infty} x e^{-x}dx \Rightarrow \int e^{-x}\,x\,dx = -e^{-x}\cdot x - \int -e^{-x}dx = e^{-x}(-1-x)$

$f \quad g'$

$$\int_0^{+\infty} x e^{-x}dx = \lim_{c\to+\infty} e^{-x}(-1-x)\Big|_0^c = \lim_{c\to+\infty} e^{-c}(1-c) - e^0(-1-0) = \underline{1}$$

$$\underbrace{\lim_{c\to+\infty} e^{-c}}_{\to 0} - \underbrace{\lim_{c\to+\infty} \frac{c}{\frac{1}{e^{-c}}}}_{\frac{\infty}{\infty}\ L'H}$$

$$\to -\lim_{c\to+\infty} \frac{1}{\frac{-e^{-c}}{e^{-2c}}} = \lim_{c\to+\infty} e^{-2c+c}$$

$$= \lim_{c\to+\infty} e^{-c} = \underline{0}$$

⑥ Testes

1- $\int_a^{x^3} f(t)\,dt = e^{3x} - 2$. Pela regra de Leibnitz: $\frac{d}{dx}\int_a^{x^3} f(t)dt = f(x^3)\frac{dx^3}{dx} - f(a)\cancel{\frac{da}{dx}}^{0}$

$$\therefore \frac{d}{dx}\int_a^{x^3} f(t)dt = f(x^3)\,3x^2$$

Ainda, $\frac{d}{dx}(e^{3x}-2) = 3e^{3x}$. Portanto:

$f(x^3)3x^2 = 3e^{3x} \Leftrightarrow f(x^3) = e^{3x}x^{-2}$. Para $x^3 = t \Rightarrow x = t^{1/3}$

$\therefore f(t) = e^{3t^{1/3}}.(t^{1/3})^{-2}$ $\therefore \underline{f(t) = e^{3\sqrt[3]{t}}\,t^{-2/3}}$

2- Eq reta normal = g(x):

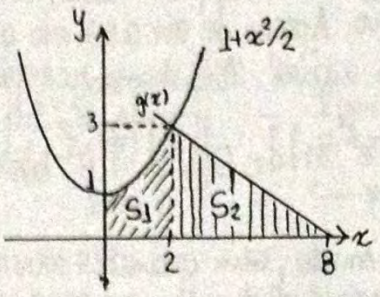
sendo $f(x) = 1 + x^2/2 \Rightarrow f'(x) = x \therefore f'(2) = 2$

$\therefore m_{tg_{(2)}} = 2$ $\therefore m_{normal_{(2)}} = -(1/m_{tg}) = -1/2$

$g(x) = (-1/2)x + b$ → contém o ponto (2,3) $\therefore g(2) = 3 = -(1/2).2 + b$ $\therefore b = 4$

$\therefore g(x) = -x/2 + 4 \Rightarrow g(x_0) = 0 \Leftrightarrow x_0 = 8$

Gráfico:



$V_1 = \pi\int_0^2 (1+x^2/2)^2 dx = \int_0^2 (1+x^2+x^4/4)dx$

$= \pi\left[x + \frac{x^3}{3} + \frac{x^5}{20}\right]_0^2 = \pi(2 + 8/3 + 32/20)$

$\therefore \underline{V_1 = 94/15}$

V_2: volume do cone de raio 3 e altura 6 $= \pi.3^2.6/3 = \underline{18\pi}$

$V_T = V_1 + V_2$ $\therefore \underline{V_T = 364\pi/15}$

3- Pela regra de Leibnitz: $\frac{d}{dx}\int_1^{\operatorname{sen}x} e^{t^2}dt = f(\operatorname{sen}x).\frac{d}{dx}\operatorname{sen}x - f(1)\cancel{\frac{d1}{dx}}^{0} = e^{\operatorname{sen}^2 x}\cos x$

($f(t) = e^{t^2}$)

$\therefore \int_1^{\operatorname{sen}x} e^{t^2}dt = \int e^{\operatorname{sen}^2 x}\cos x\,dx = F(x)$ → integra-se o resultado acima na variável x

Reescrevendo a integral do enunciado, tem-se:

$\int_0^{\pi/2} \cos x \left(\int e^{\operatorname{sen}^2 x}\cos x\,dx\right)dx = \int_0^{\pi/2} \cos x\,F(x)\,dx$. Integrando por partes, para $g' = \cos x$ e $h = F(x)$:

$$\int_0^{\pi/2} \underbrace{\cos x}_{g'}\ \underbrace{F(x)}_{h}\,dx = \underbrace{\left[F(x)\sin x\right]_0^{\pi/2}}_{\text{(I)}} - \underbrace{\int_0^{\pi/2} \underbrace{e^{\sin^2 x}}_{h'}\ \underbrace{\cos x}_{g}\,(\sin x)\,dx}_{\text{(II)}}$$

$$\text{(I)} = F(\pi/2)\sin\pi/2 - F(0).\,0 = F(\pi/2) = \int_1^{\sin\frac{\pi}{2}} e^{t^2}dt = \int_1^1 e^{t^2}dt = \underline{0}$$

$$\text{(II)} = \left[-\frac{e^{\sin^2 x}}{2}\right]_0^{\pi/2} = -\frac{e^{\sin^2(\pi/2)}}{2} - \left(-\frac{e^0}{2}\right) = -\frac{e^1}{2} + \frac{1}{2} = \underline{(-1/2)(e-1)}$$

$$\therefore \int_0^{\pi/2}\cos x \int_1^{\sin x} e^{t^2}dt\,dx = \underline{(-1/2)(e-1)}$$

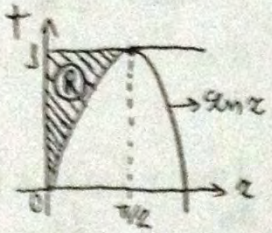
OBS: $\frac{d}{dx}\left(-\frac{e}{2}^{\sin^2 x}\right) = -\frac{e}{2}^{\sin^2 x}\cdot(2\sin x)(\cos x) = -e^{\sin^2 x}\sin x\cos x$

3- Solução alternativa

Em vista da excentricidade da resolução acima apresentado, acredito que seja valida apresentar o seguinte caminho alternativo. Apesar de ser assunto abordado em Cálculo III logo de início, ele torna a solução mais viável. Para tanto, reescreve-se a integral primeiramente

$$\int_0^{\pi/2}\cos x \int_1^{\sin x} e^{t^2}dt\,dx = \int_0^{\frac{\pi}{2}} \cos x\left(-\int_{\sin x}^1 e^{t^2}dt\right)dx = -\int_0^{\frac{\pi}{2}}\int_{\sin x}^1 e^{t^2}\cos x\,dt\,dx$$

Até aqui, reescreveu a integral de outro modo, sem grandes novidades. Vale ressaltar que $\cos x$ "entra" na integral calculada na variável t pois, no caso, é como se fosse uma constante em relação a t. Desenhando a região de integração:



Na região R, x varia de 0 a $\pi/2$ e t varia da função $\sin x$ até 1, ou seja, $0 \le x \le \pi/2$ e $\sin x \le t \le 1$

No caso da integral que se deseja calcular, como a primeira variável a ser integrada é t, por não conseguir encontrar $\int e^{t^2}dt$, foi adotada a estranha solução anterior.

A grande ideia, assim, é uma técnica muito utilizada para casos desse tipo, em que não é possível primitivar naquela variável. O que é feito, portanto, é começar integrando na outra variável, como mostrado abaixo. Entretanto, é necessário trocar os extremos de integração, meu tavelmente, mantendo-se a mesma região R de integração, mas reescrita.

Dizer que $0 \le x \le \pi/2$ e $\sin x \le t \le 1$ é o mesmo que $0 \le x \le \arcsin t$ e $0 \le t \le 1$.

Ou seja: $-\int_0^{\pi/2}\int_{\sin x}^1 e^{t^2}\cos x\,dt\,dx = -\int_0^1\int_0^{\arcsin t} e^{t^2}\cos x\,dx\,dt \Rightarrow -\int_0^1 e^{t^2}\sin x\Big|_0^{\arcsin t}dt$

$$= -\int_0^1 e^{t^2}(\sin(\arcsin t) - \sin 0)\,dt = -\int_0^1 e^{t^2}.t\,dt = -\frac{e^{t^2}}{2}\Big|_0^1 = -\frac{e}{2} - \left(-\frac{e^0}{2}\right) = (-1/2)(e-1)$$

4- Técnica importante de integração: somar e subtrair números/funções do nominador (ou seja, soma-se um 0, mantendo, portanto, o que havia anteriormente), de tal forma que seja possível separar frações, simplicá-las com o denominador e, enfim, obter duas parcelas facilmente integráveis:

$$\frac{e^{ax}-e^{bx}}{(1+e^{ax})(1+e^{bx})} = \frac{1+e^{ax}-1-e^{bx}}{(1+e^{ax})(1+e^{bx})} = \frac{(1+e^{ax})-(1+e^{bx})}{(1+e^{ax})(1+e^{bx})}$$

$$= \frac{(1+e^{ax})}{(1+e^{ax})(1+e^{bx})} - \frac{(1+e^{bx})}{(1+e^{ax})(1+e^{bx})} = \frac{1}{(1+e^{bx})} - \frac{1}{(1+e^{ax})}$$

$$= \frac{1+e^{bx}-e^{bx}}{(1+e^{bx})} - \frac{(1+e^{ax}-e^{ax})}{(1+e^{ax})} = 1 - \frac{e^{bx}}{1+e^{bx}} - 1 + \frac{e^{ax}}{(1+e^{ax})}$$

$$\therefore \frac{e^{ax}-e^{bx}}{(1+e^{ax})(1+e^{bx})} = \frac{e^{ax}}{(1+e^{ax})} - \frac{e^{bx}}{(1+e^{bx})}$$

Reescrevendo a integral inicial:

$$\int_0^\infty \left[\frac{e^{ax}}{(1+e^{ax})} - \frac{e^{bx}}{(1+e^{bx})}\right]dx = \left[\frac{\ln(1+e^{ax})}{a} - \frac{\ln(1+e^{bx})}{b}\right]_0^\infty$$

$$= \left[\ln\left[\frac{(1+e^{ax})^{1/a}}{(1+e^{bx})^{1/b}}\right]\right]_0^\infty = \underbrace{\lim_{z\to\infty} \ln\frac{(1+e^{az})^{1/a}}{(1+e^{bz})^{1/b}}}_{\text{I}} \underbrace{- \ln\frac{(1+e^{a.0})^{1/a}}{(1+e^{b.0})^{1/b}}}_{\text{II}}$$

I Para $e^z = t \Rightarrow \text{I} = \lim_{t\to\infty} \ln\frac{(1+t^a)^{1/a}}{(1+t^b)^{1/b}} = \lim_{t\to\infty} \ln(g(t))$

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{\sqrt[a]{t^a\left(\frac{1}{t^a}+1\right)}}{\sqrt[b]{t^b\left(\frac{1}{t^b}+1\right)}} = \lim_{t\to\infty} \underbrace{\frac{(t^a)^{1/a}}{(t^b)^{1/b}}}_{=1,\ \text{pois } t\geq 0} \cdot \frac{\left(\overset{0}{\frac{1}{t^a}}+1\right)^{1/a}}{\left(\underset{0}{\frac{1}{t^b}}+1\right)^{1/b}} = \lim_{t\to\infty} 1^{\left(\frac{1}{a}-\frac{1}{b}\right)} = \underline{1}$$

$$\therefore \lim_{t\to\infty} \ln(g(t)) = \ln 1 = 0$$

$$\text{II} \quad -\ln\left(\frac{2^{1/a}}{2^{1/b}}\right) = -\ln 2^{\left(\frac{1}{a}-\frac{1}{b}\right)} = -\ln 2^{\left(\frac{b-a}{ab}\right)} = \ln 2^{\left(\frac{a-b}{ab}\right)} = \left(\frac{a-b}{ab}\right)\ln 2$$

$$\therefore \text{I}+\text{II} = 0 + \left(\frac{a-b}{ab}\right)\ln 2 = \underline{\left(\frac{a-b}{ab}\right)\ln 2}$$

Posso ter errado, mas acredito que o gabarito é alternativa e, não d

5 - Esse exercício merece uma pausa para discussão. Estranho não encontrar/encontrar com muita dificuldade uma primitiva para a segunda parcela da soma? Pois é.

Mais à frente, no curso de Álgebra linear, será apresentado o conceito de "produto interno de um espaço vetorial". Conforme talvez já tenha sido dito, vetores não são apenas os de V^3. Eles podem ser matrizes, funções etc. Sendo assim, diversos conceitos aplicados ao V^3, se bem definidos, podem ser extendidos aos demais espaços vetoriais. Um deles é o de perpendicularidade. Dois vetores são ditos perpendiculares se o produto interno do e.v. ao qual pertencem, aplicado a eles, resultar em zero. No V^3, o produto interno é o produto escalar. No caso de funções contínuas, $f(x)$ e $g(x)$ serão perpendiculares se $\int_{-a}^{a} f(x)\,g(x)\,dx = 0$ (integral definida em intervalo simétrico). O caso mais comum de funções perpendiculares são as pares com as ímpares, como o cosseno com seno, polinômios de grau par com de grau ímpar (faça $\int_{-\pi}^{\pi} \operatorname{sen}x \cos x\,dx$ ou $\int_{-1}^{1} x^2 \cdot x\,dx$ e veja o resultado).

Essa digressão enorme é para explicar como é rápida a solução desse exercício. Na segunda parcela da soma, observa-se o produto de três funções: $\sqrt{1+t^2}$ (par, mais fácil de ver desenhando gráfico), $\cos^3 t$ (par, pois é função par elevada a grau ímpar, o que mantém a paridade original de $\cos t$) e $\operatorname{sen}^3 t$ (ímpar). Pelo fato de a integração ocorrer em intervalo simétrico $[-\pi/4, \pi/4]$:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos t\,dt + \underbrace{\int_{-\pi/4}^{\pi/4} \underbrace{\sqrt{1+t^2}}_{\text{par}}\ \underbrace{\operatorname{sen}^3 t}_{\text{ímpar}}\ \underbrace{\cos^3 t}_{\text{par}}\,dt}_{=0} = \int_{-\pi/4}^{\pi/4} \cos t\,dt = \operatorname{sen}t\Big|_{-\pi/4}^{\pi/4} = \frac{\sqrt{2}}{2} - \left(\frac{-\sqrt{2}}{2}\right)$$

$$\therefore \int_{-\pi/4}^{\pi/4} \cos t\,dt = \sqrt{2}$$

OBS.: uma forma de lembrar a diferença entre funções pares e ímpares é memorizar que as pares são "espelhadas" em relação ao eixo y; O produto interno de funções $\int_{-a}^{a} f(x)\,g(x)\,dx$ é um tipo de produto interno para funções contínuas, podem existir outros definidos em outros intervalos não simétricos.

6 -

$$\underbrace{\int_0^{\pi/2} \frac{\operatorname{sen}^3 t}{\operatorname{sen}^3 t + \cos^3 t}\,dt - \int_0^{\pi/2} \frac{\cos^3 t}{\operatorname{sen}^3 t + \cos^3 t}\,dt}_{(*)}$$

Mudança de variável: $t = \pi/2 - a$ $\quad t=0 \Rightarrow a = \pi/2$

$\therefore dt = -da$ $\quad t = \pi/2 \Rightarrow a = 0$

$\cos t = \cos(\pi/2 - a) = \cos\pi/2 \cos a + \operatorname{sen}\pi/2 \operatorname{sen} a = \operatorname{sen} a$

$\operatorname{sen} t = \operatorname{sen}(\pi/2 - a) = \operatorname{sen}\pi/2 \cos a - \operatorname{sen} a \cos\pi/2 = \cos a$

Assim: $(*) = -\int_{\pi/2}^{0} \frac{(\operatorname{sen}(a))^3 (-da)}{(\operatorname{sen}(a))^3 + (\cos(a))^3} = -\int_0^{\pi/2} \frac{\operatorname{sen}^3(a)\,da}{\operatorname{sen}^3(a) + \cos^3(a)}$

$\therefore \int_0^{\pi/2} \frac{\operatorname{sen}^3 t\,dt}{\operatorname{sen}^3 t + \cos^3 t} - \int_0^{\pi/2} \frac{\operatorname{sen}^3 a\,da}{\operatorname{sen}^3 a + \cos^3 a} = 0$, pelo fato de ambas as integrais serem de uma mesma função, uma aplicada a t e uma aplicada para a

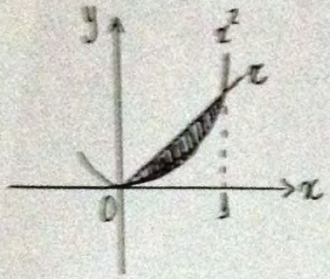
7- $\int \frac{1}{x}\frac{1}{\ln x}dx = \ln(\ln x)$

↳ primitiva imediata

$$\int_{e^{-3}}^{e^{-2}} \frac{1}{x\ln x}dx = \ln(\ln x)\Big|_{e^{-3}}^{e^{-2}} = \ln(\ln e^{-2}) - \ln(\ln e^{-3}) = \ln\left(\frac{\ln e^{-2}}{\ln e^{-3}}\right) = \ln\left(\frac{-2}{-3}\right)$$

$$\therefore \int_{e^{-3}}^{e^{-2}} \frac{1}{x\ln x} = \underline{\ln(2/3)}$$

8-



Rotação em Oy: cilindros infinitesimais de altura dy que varia de $y_1(x) = x^2$ até $y_2(x) = x$ e raio dx que varia de 0 até 1

Assim.

$$V_{oy} = \int_0^1 \int_{x^2}^{x} 2\pi\, dy\, dx = \int_0^1 2\pi(x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \cdot 2\pi$$

$$\therefore \underline{V_{oy} = \pi/3}$$

9- Pela regra de Leibnitz, para $f(t) = e^{t^2}$:

$$\frac{d}{dx}\int_{x^3}^{x^4} e^{t^2}dt = f(x^4)\frac{d}{dx}x^4 - f(x^3)\frac{d}{dx}x^3 = e^{x^8}4x^3 - e^{x^6}3x^2$$

$$\therefore \frac{d}{dx}\int_{x^3}^{x^4} e^{t^2}dt = x^2e^{x^6}(4xe^{x^8-x^6} - 3)$$

//

Considerações finais.

É de suma importância que as partes ①, ③, ④ e ⑥ tentem ser resolvidas por completo. Para a parte ②, há exercícios que recomendaria uma atenção especial, sendo eles: 8, 12, 14, 17, 28, 29, 34, 35, 37, 42, 45 ao 47, 53, 54 ao 56, 60 ao 78 (excluindo o 70). Ainda, recomendo a resolução da parte ⑤ como uma forma de aprofundamento após a resolução dos demais.

Por fim, aos que interessarem, deixo no verso a demonstração da integral de Riemann de um modo compacto e espero que mais inteligível em relação ao encontrado nos livros. Ela é de extrema importância ao Cálculo e será retomada mais à frente no curso de Cálculo III, expandida às funções de mais de uma variável.

Bons estudos,

Danilo Ap. C. Castillo